An introduction to Harmonic Analysis

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Abstract

The note is mainly for personal record, if you want to read it, please be careful. This lecture was given by Prof. Jenn-Nan Wang in National Taiwan University, during February to June 2016.

1 Introduction and Motivation

From Lebesgue's differentiation theorem. Let $f \in L^1_{loc}(\mathbb{R}^n)$, then

$$\lim_{r\to 0} \frac{1}{|B_r(x)|}\int_{B_r(x)} f(y)dy = f(x) \text{ a.e.},$$

where $B_r(x) = \{y : |y-x| \leq r\}$ and $|B_r(x)|$ is the Lebesgue measure of $B_r(x)$. Instead of taking limit, we study

$$\begin{aligned} Mf(x) :&= \sup_{r>0} \frac{1}{|B_r(x)|}\int_{B_r(x)} |f(y)|dy \\ &= \sup_{r>0} \frac{1}{|B_r(0)|}\int_{B_r(0)} |f(x-y)|dy, \end{aligned}$$

which is called the Hardy-Littlewood maximal function and M is the Hardy-Littlewood maximal operator.

1.1 Basic properties for maximal functions

Problem 1.1. Boundedness of M.

Theorem 1.2. *For $1 \leq p \leq \infty$, we have*
(a) $Mf(x)$ is finite a.e. for all $f \in L^p(\mathbb{R}^n)$.
(b) Mf is weak $(1,1)$, i.e., $\forall \alpha > 0$, $\exists A = A(n)$ such that

$$|\{x : Mf(x) > \alpha\}| \leq \frac{A}{\alpha}\|f\|_{L^1}.$$

(c) M is strong (p,p) for $1 < p \leq \infty$, i.e., $\exists C_p$ such that

$$\|Mf\|_{L^p} \leq C_p\|f\|_{L^p}.$$

Remark 1.3. (a) is a consequence from (b) and (c).

Why do we study $|\{x \in \mathbb{R}^n : Mf(x) > \alpha\}|$ (the distribution function of Mf).

Proposition 1.4. *For $1 \le p < \infty$, if $f \in L^p(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n} |f|^p dx = p \int_0^\infty \alpha^{p-1} |\{x \in \mathbb{R}^n : |f(x)| > \alpha\}| d\alpha. \tag{1}$$

Notation: $\lambda_f(\alpha) = |\{x \in \mathbb{R}^n : |f(x)| > \alpha\}|$.
Derivation of (**??**): We can see it by using Tonelli's theorem. Moreover, integration by parts will give

$$p \int_0^\infty \alpha^{p-1} |\{x \in \mathbb{R}^n : |f(x)| > \alpha\}| d\alpha = -\int_0^\infty \alpha^p d\lambda_f(\alpha).$$

Remark 1.5. Chebyshev's inequality: $\lambda_f(\alpha) \le \frac{1}{\alpha}\|f\|_{L^1}$. If $f \in L^1(\mathbb{R}^n)$, then $\lambda_f(\alpha)$ is finite. Even for nonintegrable function, we still can estimate $\lambda_f(\alpha)$.

Example 1.6. Let $f(x) = \frac{1}{|x|^n}$ in $\mathbb{R}^n$, then $\lambda_f(\alpha) = \frac{c_n}{\alpha}$ for some constant c_n.

Remark 1.7. Strong $(1,1)$ implies weak $(1,1)$. $\|Tf\|_1 \le C_1\|f\|_1$ implies

$$|\{|Tf| > \alpha\}| \le \frac{1}{\alpha}\|Tf\|_1 \le \frac{C_1}{\alpha}\|f\|_1.$$

The Hardy-Littlewood maximal operator M is never strong $(1,1)$. For example, $f(x) = \chi_{B_1(0)} \in L^1(\mathbb{R}^n)$, but $Mf(x) \approx \frac{1}{|x|^n}$.

Note that $Mf \notin L^1$, but we still can bound $\lambda_{M_f}(\alpha)$. The proof of weak type $(1,1)$ for Mf, i.e., $\forall \alpha > 0$, $f \in L^1(\mathbb{R}^n)$,

$$|\{x \in \mathbb{R}^n : Mf(x) > \alpha\}| \le \frac{A}{\alpha}\|f\|_1,$$

where $A = A(n)$. Need a covering lemma.

Lemma 1.8. *(Vitali's covering lemma) Let E be a measurable subset in $\mathbb{R}^n$. Assume that E is covered by a family of balls$\{B_j\}$ of bounded diameters. Then we can find a sequence of disjoint balls B_1, B_2, $\cdots$ from $\{B_j\}'s$ such that*

$$\sum_k |B_k| \ge C_n|E|, \text{ with } C_n \le \frac{1}{5^n}, \tag{2}$$

and $E \subset \cup_k B_k^*$ where $B_k^* = 5B_k$.

Proof. Let B_1 be the ball chosen from $\{B_j\}$ such that

$$\text{diam}B_1 \ge \frac{1}{2}\sup\{\text{diam}B_j\}.$$

Assume that we have chosen $B_1, B_2, \cdots, B_k$. Then B_{k+1} is chosen that

$$\text{diam}B_{k+1} \ge \frac{1}{2}\sup\{\text{diam}B_j,\ B_j \cap B_i = \emptyset,\ \forall i = 1, 2, \cdots, k\}.$$

So we have chosen a sequence of balls $B_1, B_2, \cdots$ disjoint balls.

Now, if $\sum_k |B_k| = \infty$, (**??**) holds automatically. So we consider the case $\sum_k |B_k| < \infty$, which implies $\text{diam}B_k \to 0$ as $k \to \infty$. Let B_j be any ball which is not chosen. Since $\text{diam}B_k \to 0$, there exists a first k such that

$$\text{diam}B_{k+1} \leq \frac{1}{2}\text{diam}B_j.$$

Claim: B_j must intersect some balls of $B_1, B_2, \cdots, B_k$. If $B_j \cap B_i = \emptyset$ $\forall j = 1, \cdots, k$, then B_j would have been chosen, since $\text{diam}B_j \geq 2\text{diam}B_{k+1}$. Therefore, $\exists$ smallest k_0 with $1 \leq k_0 \leq k$ such that $B_j \cap B_{k_0} \neq \emptyset$. Then we have $\frac{1}{2}\text{diam}B_j \leq \text{diam}B_{k_0}$, which implies $B_j \subset 5B_{k_0}$. □

(b) Proof of weak $(1,1)$. Let $E_\alpha = \{x \in \mathbb{R}^n : Mf(x) > \alpha\}$. For $x \in E_\alpha$, i.e., $Mf(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)|dy > \alpha$, $\exists B_r(x)$ such that $\frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)|dy > \alpha \Leftrightarrow |B_r(x)| \leq \frac{1}{\alpha} \int_{B_r(x)} |f(y)|dy$. So $E_\alpha \subset \cup_{x \in E_\alpha} B_r(x)$. From the covering lemma, $\exists B_1, B_2, \cdots$ such that $\sum_k |B_k| \geq C|E_\alpha|$. Therefore,

$$\begin{aligned} |E_\alpha| &\leq C \sum_k |B_k| \leq \frac{C}{\alpha} \sum_k \int_{B_k} |f(y)|dy \\ &\leq \frac{C}{\alpha} \|f\|_{L^1(\mathbb{R}^n)}. \end{aligned}$$

(c) Proof of strong (p,p), $1 < p \leq \infty$. The proof is in fact a special case of Marcinkiewicz interpolation theorem.

Known fact: M is weak $(1,1)$ and M is strong (∞, ∞), i.e., $\|Mf\|_\infty \leq \|f\|_\infty$ for a.e. x. By Marcinkiewicz, we have M is strong (p,p) for $1 < p \leq \infty$. Let $f \in L^p$, $1 < p < \infty$. $\forall \alpha > 0$, define

$$f_1(x) = \begin{cases} f(x), & \text{if } |f(x)| > \frac{\alpha}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

Then $f = f_1(x) + f_2(x)$, where $|f_2(x)| \leq \frac{\alpha}{2}$ for all x. Note that $|Mf_2(x)| \leq \frac{\alpha}{2}$ for all x. Also, M is subadditive, i.e.,

$$Mf \leq Mf_1 + Mf_2 \leq Mf_1 + \frac{\alpha}{2}$$

and

$$|E_\alpha| = |\{x \in \mathbb{R}^n : Mf(x) > \alpha\}| \leq |\{x \in \mathbb{R}^n : Mf_1(x) > \frac{\alpha}{2}\}|.$$

By the weak $(1,1)$ of M, we have

$$|E_\alpha| \leq |\{x \in \mathbb{R}^n : Mf_1(x) > \frac{\alpha}{2}\}| \leq \frac{2A}{\alpha}\|f_1\|_{L^1}.$$

Now,

$$
\begin{aligned}
\|Mf\|_p^p &= p\int_0^\infty \alpha^{p-1}|E_\alpha|d\alpha \\
&\le 2pA\int_0^\infty \alpha^{p-2}\int_{\{|f(x)|>\frac{\alpha}{2}\}}|f(x)|dx \\
&\le 2pA\int_{\mathbb{R}^n}|f(x)|\int_0^{2|f(x)|}\alpha^{p-2}d\alpha dx \\
&= \frac{2^p pA}{p-1}\int_{\mathbb{R}^n}|f(x)|^p dx.
\end{aligned}
$$

1.2 Proof of Lebesuge's differentiation theorem

$f \in L^1_{loc}(\mathbb{R}^n)$ implies $\lim_{r\to 0} ⨍_{B_r(x)} f(y)dy = f(x)$ a.e.. We can assume $f \in L^1(\mathbb{R}^n)$. Denote

$$f_r(x) = ⨍_{B_r(x)} f(y)dy,$$

then we claim that $\|f_r - f\|_{L^1} \to 0$ as $r \to 0$.

Proof. Prove it is true for $f \in C_0(\mathbb{R}^n)$. For $f \in L^1$, by density argument since $C_0(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$.
Goal. For a.e. x, $f_r(x) \to f(x)$ as $r \to 0$.

We have shown that $f_{r_j}(x) \to f(x)$ a.e. for some $r_j \to 0$. We only need to show that $\lim_{r\to 0} f_r(x)$ exists. Denote

$$\Omega f(x) := |\limsup_{r\to 0} f_r(x) - \liminf_{r\to 0} f_r(x)|.$$

It suffices to prove that $\forall \epsilon > 0$, $|\{\Omega f(x) > \epsilon\}| = 0$.

Note that $f \in C_0(\mathbb{R}^n)$, then $\Omega f(x) = 0$ for all x. For any $f \in L^1(\mathbb{R}^n)$, $\exists h \in C_0(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$ with $\|g\|_1$ is as small as we wish such that $f = h + g$. Now,

$$|\{\Omega f(x) > \epsilon\}| \le |\Omega h(x) > \frac{\epsilon}{2}\}| + |\Omega g(x) > \frac{\epsilon}{2}\}| = |\Omega g(x) > \frac{\epsilon}{2}\}|.$$

Recall $\Omega g(x) = |\limsup_{r\to 0} g_r(x) - \liminf_{r\to 0} g_r(x)|$ will imply $\Omega g(x) \le 2Mg(x)$. Therefore,

$$|\{\Omega g(x) > \frac{\epsilon}{2}\}| \ge |\{Mg(x) > \frac{\epsilon}{4}\}| \ge \frac{4A}{\epsilon}\|g\|_{L^1}$$

and $||g||_{L^1}$ can be small as we wish, which completes the proof. □

Remark 1.9. The proof of Lebesgue's differentiation theorem only uses the weak type $(1, 1)$ of the maximal functions.

1.3 Marcinkiewicz interpolation theorem for $L^P(\mathbb{R}^n)$ $(1 \le p \le \infty)$

Definition 1.10. For $f \in L^p$, an operator T is called strong (p, q), $1 \le q \le \infty$ if

$$\|Tf\|_q \le C_{p,q}\|f\|_p.$$

T is called weak (p, q) for $q < \infty$ if $\forall \alpha > 0$,

$$|\{|Tf(x)| > \alpha\}| \leq \left(\frac{A\|f\|_p}{\alpha} \right)^q.$$

T is weak (p, ∞) if T is strong (p, ∞).

Proposition 1.11. *T is strong (p, q) will imply that T is weak (p, q).*

Proof. We have $\|Tf\|_q^q \leq C\|f\|_p^q$ and

$$\begin{aligned} \|Tf\|_q^q = \int_{\mathbb{R}^n} |Tf|^q dx &= \int_{\{|Tf(x)| \leq \alpha\}} |Tf|^q + \int_{\{|Tf(x)| > \alpha\}} |Tf|^q \\ &\geq \alpha^q |\{|Tf(x)| > \alpha\}. \end{aligned}$$

Then we are done. □

1.4 Marcinkiewcz Interpolation Theorem for L^p

Definition 1.12. $f \in L^{p_1} + L^{p_2}$ iff $f = f_1 + f_2$, $f_1 \in L^{p_1}$ and $f_2 \in L^{p_2}$.

Remark 1.13. Assume $p_1 < p_2$. Then $f \in L^p \Rightarrow f \in L^{p_1} + L^{p_2}$, $p_1 \leq p \leq p_2$.

Given any $\gamma > 0$. Define

$$f_1(x) = \begin{cases} f(x) & |f(x)| > \gamma \\ 0 & \text{otherwise } (|f(x)| \leq \gamma) \end{cases}$$

$$f_2(x) = \begin{cases} 0 & |f(x)| > \gamma \\ f(x) & \text{otherwise } (|f(x)| \leq \gamma) \end{cases}$$

Then

$$\begin{aligned} \int |f_1(x)|^{p_1} dx = \int_{\{|f(x)| > \gamma\}} |f(x)|^{p_1} dx &= \int_{\{|f(x)| > \gamma\}} |f(x)|^p |f(x)|^{p_1 - p} dx \\ &\leq \gamma^{p_1 - p} \int_{\mathbb{R}^n} |f(x)|^p dx < \infty, \end{aligned}$$

$$\begin{aligned} \int |f_2(x)|^{p_2} dx = \int_{\{|f(x)| \leq \gamma\}} |f(x)|^{p_2} dx &= \int_{\{|f(x)| \leq \gamma\}} |f(x)|^p |f(x)|^{p_2 - p} dx \\ &\leq \gamma^{p_2 - p} \int_{\mathbb{R}^n} |f(x)|^p dx < \infty. \end{aligned}$$

Remark 1.14. The level γ is arbitrary.

Theorem 1.15. *For $1 \leq r \leq \infty$. Assume that the operator T is sublinear, i.e.*

$$|T(f + g)(x)| \leq |Tf(x)| + |Tg(x)|;$$

moreover, T is weak(1,1) and weak(r,r). Then T is strong(p,p) for $1 < p < r$, i.e. $\exists\ A_{p,r} > 0$ s.t.

Remark 1.16. For $f \in L^p$, $1 < p < r$, we can write $f = f_1 + f_2$, $f_1 \in L^1$ and $f_2 \in L^r$. Since

- T is weak(1,1) i.e.

$$|\{|Tf_1(x)| > \alpha\}| \leq \frac{A_1}{\alpha}\|f_1\|_1$$

- T is weak(r,r) i.e.

$$|\{|Tf_2(x)| > \alpha\}| \leq (\frac{A_r}{\alpha}\|f_2\|_r)^r$$

Proof. Recall that

$$\int |Tf|^p dx = p\int_0^\infty \alpha^{p-1}\lambda_{Tf}(\alpha)d\alpha$$

Need to estimate $\lambda_{Tf}(\alpha)$

$$\lambda_{Tf}(\alpha) = |\{|Tf(x)| > \alpha\}| \leq |\{|Tf_1(x)| > \alpha/2\}| + |\{|Tf_2(x)| > \alpha/2\}|$$

The inequality is from T is sublinear, $|Tf(x)| \leq |Tf_1(x)| + |Tf_2(x)|$. To determine $f_1(x)$ and $f_2(x)$, we choose of the level $\gamma = \alpha$.

$$\text{weak(1,1)} \Rightarrow |\{|Tf_1(x)| > \alpha/2\}| \leq \frac{2A_1}{\alpha}\|f_1\|_1 = \frac{2A_1}{\alpha}\int_{\{|f(x)|>\alpha\}} |f(x)|dx$$

$$\text{weak(r,r)} \Rightarrow |\{|Tf_2(x)| > \alpha/2\}| \leq (\frac{2A_r}{\alpha})^r\|f_2\|_r^r = \frac{2^rA_r^r}{\alpha^r}\int_{\{|f(x)|\leq\alpha\}} |f(x)|^r dx$$

that is

$$\lambda_{Tf}(\alpha) \leq \frac{2A_1}{\alpha}\int_{\{|f(x)|>\alpha\}} |f(x)|dx + \frac{2^rA_r^r}{\alpha^r}\int_{\{|f(x)|\leq\alpha\}} |f(x)|^r dx$$

Hence

$$\begin{aligned}
\int |Tf|^p dx &= p\int_0^\infty \alpha^{p-1}\lambda_{Tf}(\alpha)d\alpha \\
&\leq p\int_0^\infty \alpha^{p-1}\cdot\frac{2A_1}{\alpha}\int_{\{|f(x)|>\alpha\}} |f(x)|dxd\alpha \\
&\quad +p\int_0^\infty \alpha^{p-1}\cdot\frac{2^rA_r^r}{\alpha^r}\int_{\{|f(x)|\leq\alpha\}} |f(x)|^r dxd\alpha \\
&:= \mathrm{I} + \mathrm{II}
\end{aligned}$$

$$\begin{aligned}
\mathrm{I} &= 2pA_1\int_0^{infty} \alpha^{p-2}\int_{\{|f(x)|>\alpha\}} |f(x)|dxd\alpha \\
&= 2pA_1\int_{\mathbb{R}^n} |f(x)|\int_0^{|f(x)|} \alpha^{p-2}d\alpha dx \\
&= \frac{2pA_1}{p-1}\int_{\mathbb{R}^n} |f(x)|^p dx
\end{aligned}$$

$$\begin{aligned}
\text{II} &= 2^r p A_r^r \int_0^{infty} \alpha^{p-r-1} \int_{\{|f(x)| \leq \alpha\}} |f(x)|^r dx d\alpha \\
&= 2^r p A_r^r \int_{\mathbb{R}^n} |f(x)|^r \int_{|f(x)|}^{\infty} \alpha^{p-r-1} d\alpha dx \\
&= \frac{2^r p A_r^r}{r-p} \int_{\mathbb{R}^n} |f(x)|^r |f(x)|^{p-r} dx \\
&= \frac{2^r p A_r^r}{r-p} \int_{\mathbb{R}^n} |f(x)|^p dx
\end{aligned}$$

Thus we obtain

$$\|Tf\|_p \leq (\frac{2pA_1}{p-1} + \frac{2^r p A_r^r}{r-p})^{1/p} \|f\|_p$$

for $A_{p,r} = (\frac{2pA_1}{p-1} + \frac{2^r p A_r^r}{r-p})^{1/p}$. □

Here we proved the case of $r < \infty$.
The case $r = \infty$ can be proved by the same argument for Mf.

Remark 1.17. $A_{p,r} \to \infty$ as $p \to 1+$ and $p \to r-$

Exercise 1.18. What happens to $A_{p,r}$ if T is either strong(1,1) or strong(r,r) ?

1.5 Lebesgue differentiation theorem

Recall the Lebesgue differentiation theorem. If $f \in L^1_{loc}(\mathbb{R}^n)$, then

$$\lim_{r \to 0} \fint_{B_r(x)} f(y) dy = f(x) \text{ a.e..} \tag{3}$$

Can we replace $B_r(x)$ by other family of measurable sets ?

For example, is

$$\lim_{|B| \to 0, x \in B} \fint_B f(y) dy = f(x) \text{ a.e. ?}$$

Definition 1.19. (Regular family) A family $\mathcal{F}$ of measurable sets is called regular if $\exists c > 0$ such that $\forall S \in \mathcal{F}$, $\exists B$ (centered at the origin) satisfying $S \subset B$ and $|S| \geq c|B|$.

Example 1.20. 1. $\mathcal{F} = \{$balls containing 0$\}$.

2. The family of cubes whose distance from the origin is bounded by a constant multiplier of their diameters.

So let $\mathcal{F}$ be regular, we can define the maximal operator associated with $\mathcal{F}$ by

$$M_{\mathcal{F}} f(x) = \sup_{S \in \mathcal{F}} \fint_S |f(x-y)| dy.$$

Observe that

$$\fint_S |f(x-y)| dy \leq \frac{1}{c} \fint_B |f(x-y)| dy \leq c^{-1} Mf(x).$$

So

$$\lim_{S\in\mathcal{F},|S|\to 0} ⨍_S f(x-y)dy = f(x) \text{ a.e..}$$

Problem 1.21. We know that $\exists E \subset \mathbb{R}^n$ such that $|E^c| = 0$ and

$$\lim_{S\in\mathcal{F},|S|\to 0} ⨍_S f(x-y)dy = f(x)\ \forall x \in E,$$

where E^c is the exceptional set. The set E or E^c depends on $\mathcal{F}$. Can we find a set E such that $|E^c| = 0$ and

$$\lim_{S\in\mathcal{F},|S|\to 0} ⨍_S f(x-y)dy = f(x)\ \forall x \in E\ (E \text{ depends on } f)$$

independent of what regular family $\mathcal{F}$ is ?

Definition 1.22. (Lebesgue set) The Lebesgue set $\mathcal{L}$ of a function f is defined as $x \in \mathbb{R}^n$ and

$$\lim_{r\to 0} ⨍_{B_r(x)} |f(y) - f(x)|dy = 0$$

or

$$\lim_{r\to 0} ⨍_{B_r(0)} |f(x-y) - f(x)dy = 0. \tag{4}$$

Remark 1.23. Note that (**??**) is stronger than (**??**).

Lemma 1.24. *(**??**) holds almost everywhere.*

Proof. For any $c \in \mathbb{R}$, we know that $\exists E_c$ (exceptional set), $|E_c| = 0$ such that

$$\lim_{r\to 0} ⨍_{B_r(x)} |f(y) - c|dy \to |f(x) - c|\ \forall x \notin E_c.$$

In particular, if $c \in \mathbb{Q}$ (rational) and $E = \cup_{c\in\mathbb{Q}} E_c$ ($|E| = 0$), then

$$\lim_{r\to 0} ⨍_{B_r(x)} |f(y) - q|dy \to |f(x) - q|\ \forall x \notin E.$$

Since $\mathbb{Q}$ is dense in $\mathbb{R}$ and take $c = f(x)$, then we are done. □

So $\mathcal{L} = E^c$. Now, let $\mathcal{F}$ be regular, then for $x \notin E$,

$$\begin{aligned} |⨍_S f(x-y)dy - f(x)| &= |⨍_S [f(x-y) - f(x)]dy \\ &\le ⨍_S |f(x-y) - f(x)|dy \\ &\le c^{-1} ⨍_{B_r(0)} |f(x-y) - f(y)|dy. \end{aligned}$$

Then as $|S| \to 0$ ($\Rightarrow |B_r(0)| \to 0$) will imply

$$⨍_S f(x-y)dy \to f(x)\ \forall x \notin E.$$

Definition 1.25. Let E be a measurable set, $x \in \mathbb{R}^n$ is called a point of density of E if

$$\lim_{r\to 0} \frac{|E \cap B_r(x)|}{|B_r(x)|} = 1.$$

Theorem 1.26. *Almost every point of E is a point of density of itself.*

Proof. Let $f(x) = \chi_E(x) \in L^1_{loc}(\mathbb{R}^n)$, the characteristic function of E. Then

$$\frac{|E \cap B_r(x)|}{|B_r(x)|} = \fint_{B_r(x)} f(y)dy \to f(x) = \chi_E(x) \text{ a.e..}$$

□

Remark 1.27. Almost all point of E^c are not points of density of E.

1.6 Approximation of the identity

Let $\phi \in L^1(\mathbb{R}^n)$ with $\int \phi dx = 1$. Consider $\phi_t(x) = t^{-n}\phi(\frac{x}{t})$, then $\forall g \in \mathcal{S}(\mathbb{R}^n)$ (Schwartz class), i.e., $\forall \alpha, \beta \in \mathbb{Z}^n$, $\sup_{x\in\mathbb{R}^n} |x^\beta D^\alpha g| < \infty$. We can show that

$$\int_{\mathbb{R}^n} \phi_t(x)g(x)dx \to g(0) \text{ as } t \to 0+, \tag{5}$$

$\phi_t \to \delta$ in the sense of distribution.

Proof. (Proof of (**??**))

$$\begin{aligned}
\int_{\mathbb{R}^n} \phi_t(x)g(x)dx &= t^{-n} \int_{\mathbb{R}^n} \phi(\frac{x}{t})g(x)dx \\
&= t^{-n} \int_{\mathbb{R}^n} [\phi(\frac{x}{t})g(x) - g(0)]dx + g(0) \\
&= \int_{\mathbb{R}^n} [\phi(x)(g(tx) - g(0))]dx + g(0) \\
&\to g(0)
\end{aligned}$$

as $t \to 0+$ by using the Lebesgue dominated convergence theorem. □

In other words, for $g \in S$, we have

$$\lim_{t\to 0}(\phi_t * g)(x) = g(x) \ \forall x \in \mathbb{R}^n \text{ (pointwise convergence).}$$

ϕ_t is called the approximation of the identity.

Theorem 1.28. *For $1 \le p < \infty$, we have*

$$\|\phi_t * f - f\|_p \to 0 \text{ as } t \to 0,$$

for all $f \in L^p(\mathbb{R}^n)$. For $p = \infty$, we have

$$\|\phi_t * f - f\|_\infty \to 0 \text{ as } t \to 0,$$

for all $f \in C_0(\mathbb{R}^n)$ (continuous functions vanishing at ∞).

Proof. Exercise. □

From this theorem, we know that $\phi_t * f \to f$ in L^p for all $f \in L^p$. How about pointwise convergence of $\phi_t * f$? The theorem implies that there exists a subsequence $\{t_k\}$, $t_k \to 0$ such that $\phi_{t_k} * f \to f(x)$ for a.e. x. If we can show that $\lim_{t\to 0} \phi_t * f(x)$ exists almost everywhere, then $\lim_{t\to 0} \phi_t * f(x) = f(x)$ a.e..

1.7 Relations between weak (p, q) bound and pointwise convergence

Let (X, μ) be a measure space and $\{T_t\}$ be a family of linear operators on $L^p(X, \mu)$. Define the maximal operator

$$T^* f(x) = \sup_{t>0} |T_t f(x)|.$$

If T^* is weak (p, q), then the set

$$S = \{f \in L^p(X, \mu) : \lim_{t\to t_0} T_t f(x) = f(x) \text{ a.e.}\}$$

is closed in in $L^p(X, \mu)$.

Proof. To prove that S is closed, we let $\{f_k\} \subset L^p(X, \mu)$ with $\lim_{t\to t_0} T_t f_k(x) = f_k(x)$ a.e. and $f_k \to f$ in $L^p(X, \mu)$. Need to show $f \in S$. In other words, we need to show

$$\mu(\{x \in \mathbb{R}^n : \limsup_{t\to t_0} |T_t f(x) - f(x)| > 0\}) = 0$$

or we can prove

$$\sum_{k=1}^{\infty} \mu(\{x \in \mathbb{R}^n : \limsup_{t\to t_0} |T_t f(x) - f(x)| > \frac{1}{k}\}) = 0.$$

It suffices to prove $\forall\lambda > 0$, $\mu(\{x \in \mathbb{R}^n : \limsup_{t\to t_0} |T_t f(x) - f(x)| > \lambda\}) = 0$.

$$\begin{aligned}
&\mu(\{x \in \mathbb{R}^n : \limsup_{t\to t_0} |T_t f(x) - f(x)| > \lambda\}) \\
=&\mu(\{x \in \mathbb{R}^n : \limsup_{t\to t_0} |T_t(f - f_k) + T f_k - f_k + f_k - f| > \lambda\}) \\
\le&\mu(\{x \in \mathbb{R}^n : \limsup_{t\to t_0} |T_t(f - f_k)| > \frac{\lambda}{2}\}) + \mu(\{x \in \mathbb{R}^n : |f_k - f| > \frac{\lambda}{2}\}) \\
\le&\mu(\{x \in \mathbb{R}^n : T^*(f_k - f)| > \frac{\lambda}{2}\}) + \mu(\{x \in \mathbb{R}^n : |f_k - f| > \frac{\lambda}{2}\}) \\
\le&\left(\frac{2A}{\lambda}\|f - f_k\|_p\right)^q + \left(\frac{2}{\lambda}\|f_k - f\|_p\right)^p \to 0 \text{ as } k \to \infty.
\end{aligned}$$

□

Remark 1.29. Under the same condition, we can show that

$$S = \{f \in L^p : \lim_{t\to t_0} T_t f(x) \text{ exists a.e.}\}$$

is closed in $L^p(X, \mu)$. The proof is left as an exercise.

Remark 1.30. Let ϕ_k be an approximation of the identity, then $\phi_t * f(x) \to f(x)$ as $t \to 0$ for all $f \in \mathcal{S}$. Also, S is closed in $L^P(\mathbb{R}^n)$. $\mathcal{S} \subset S \subset L^p(\mathbb{R}^n)$ will imply $S = L^p(\mathbb{R}^n)$.

1.8 Discuss the pointwise convergence

$\phi_t * f(x) \to f(x)$ a.e. as $t \to 0$. It suffices to prove that $\sup_{t>0} |\phi_t * f|$ is weakly bounded.

Proposition 1.31. *Let $\phi(x) = \phi(|x|)$ be radial, positive and decreasing in $|x|$. Assume ϕ is integrable. Then*

$$\sup_{t>0} |\phi_t * f| \le \|\phi\|_{L^1} Mf(x).$$

Proof. Let us consider the case where $\phi = \sum_j a_j \chi_{B_j}(x)$, $a_j > 0$. Then

$$\phi * f(x) = \sum_j a_j |B_j| \cdot \frac{1}{|B_j|} \chi_{B_j} * f.$$

Then

$$|\phi * f(x)| \le \|\phi\|_{L^1} Mf(x).$$

We obtain the similar estimate for $\phi_t = t^{-n}\phi(\frac{x}{t})$, and by the limiting process, which finishes the proof. $\square$

Corollary 1.32. *If $|\phi(x)| \le \psi(|x|)$, where ψ satisfies the condition in Proposition 1.31. Then $\sup_{t>0} |\phi_t * f|$ is weak $(1,1)$ and strong (p,p), where $1 < p \le \infty$.*

2 Fourier transform in $L^p(\mathbb{R}^n)$

Definition 2.1. If $f \in L^1(\mathbb{R}^n)$, then we define the Fourier transform

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx.$$

Fact 2.2. *1. $\mathcal{F}(f) := \widehat{f}$, $\mathcal{F} : L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$ (continuous) and $|\widehat{f}(\xi)| \le \|f\|_{L^1}$.*

2. Riemann-Lebesgue lemma: $\lim_{|\xi| \to \infty} |\widehat{f}(\xi)| = 0$.

Recall that $\mathcal{S}$ is the Schwartz space, then $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ ($\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx$) and $\mathcal{F}^{-1} : \mathcal{S} \to \mathcal{S}$ ($f(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi$). Also, $\|\widehat{f}\|_2 = \|f\|_2$, for all $f \in \mathcal{S}$ (Plancheral theorem). Since $\mathcal{S}$ is dense in $L^2(\mathbb{R}^n)$, i.e., for all $f \in L^2(\mathbb{R}^n)$, there exists $\{f_k\} \in \mathcal{S}$ such that $f_k \to f$ in $L^2(\mathbb{R}^n)$. Then we can define $\widehat{f}(\xi) = \lim_{k\to\infty} \widehat{f_k}(\xi)$ in $L^2(\mathbb{R}^n)$. Also,

$$\widehat{f}(\xi) = \lim_{R\to\infty} \int_{|x|<R} f(x) e^{-2\pi i x \cdot \xi} dx \text{ and } f(x) = \lim_{R\to\infty} \int_{|\xi|<R} \widehat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi.$$

The limit is in the L^2 sense. So we have $\mathcal{F} : L^1 \to L^\infty$ and $\mathcal{F} : L^2 \to L^2$. Now, for $1 < p < 2$ and $f \in L^p$, we can write $f = f_1 + f_2$, where $f_1 \in L^1$, $f_2 \in L^2$. Define $\widehat{f} = \mathcal{F} f = \widehat{f_1} + \widehat{f_2} \in L^\infty + L^2$.

Theorem 2.3. *(Riesz-Thorin interpolation theorem) Let $1 \le p_0, p_1, q_0, q_1 \le \infty$. Assume that T is a linear operator from $L^{p_0} + L^{p_1}$ to $L^{q_0} + L^{q_1}$ satisfying*

$$\|Tf\|_{q_0} \le M_0 \|f\|_{p_0} \text{ and } \|Tf\|_{q_1} \le M_1 \|f\|_{p_1}.$$

Then for $\theta \in (0,1)$*, define* $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$*. Then* T *is a bounded operator from* L^p *to* L^q *and*

$$\|Tf\|_q \leq M_0^{1-\theta} M_1^{\theta} \|f\|_p.$$

Proof. LOL □

Theorem 2.4. *(Hausdorff-Young inequality) Let* $1 \leq p \leq 2$*, then* $\|\widehat{f}\|_{p'} \leq \|f\|_p$*, where* $\frac{1}{p} + \frac{1}{p'} = 1$*.*

Proof. $\mathcal{F} : L^1 \to L^\infty$, $\|\mathcal{F}f\|_\infty \leq \|f\|_1$ $(M_0 = 1)$ and $\mathcal{F} : L^2 \to L^2$, $\|\mathcal{F}f\|_2 = \|f\|_2$ $(M_1 = 1)$. For $1 < p < 2$, $\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{2} = 1 - \frac{\theta}{2}$, $\frac{1}{q} = \frac{1-\theta}{\infty} + \frac{\theta}{2} = \frac{\theta}{2}$. □

Remark 2.5. For $1 \leq p \leq 2$, $f \in L^p$, $\widehat{f}$ is a classical function. Now, for $p > 2$, we define the Fourier transform $\widehat{f}$ as a tempered distribution. Recall that $\mathcal{S}$ is the Schwartz space. The tempered distribution $\mathcal{S}'$ is the continuous linear functional on $\mathcal{S}$, i.e., $T \in \mathcal{S}'$,

$$|\langle T, \varphi \rangle| \leq C \|\varphi\|_{\mathcal{S}}, \ \forall \varphi \in \mathcal{S}.$$

For example, $\langle \delta, \varphi \rangle = \varphi(0)$.

Definition 2.6. $T \in \mathcal{S}'$, we define $\widehat{T}$ as

$$\left\langle \widehat{T}, \varphi \right\rangle = \langle T, \widehat{\varphi} \rangle.$$

We can define $\widehat{f}$ if $f \in L^p$ for $p > 2$ since $L^p \subset \mathcal{S}$, but $\widehat{f}$ may not be a classical function.

Theorem 2.7. *(Young's inequality) Let* $1 \leq p, q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$*. Given* $f \in L^p$, $g \in L^q$*, then* $f * g \in L^r$*, where* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$ *and*

$$\|f * g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}.$$

Proof. Let $f \in L^p$ and define the linear operator $T_f(g) = f * g$. Observe that

$$T_f : L^1 \to L^p \text{ with } \|T_f g\|_{L^p} \leq \|f\|_{L^p} \|g\|_{L^1}.$$

From the Minkowski's integral inequality

$$\begin{aligned} \|T_f g\|_p &= (\int | \int f(x-y) g(y) dy|^p dx)^{\frac{1}{p}} \\ &\leq \|f\|_p \|g\|_1. \end{aligned}$$

In addition, $T_f : L^{p'} \to L^1$ $(\frac{1}{p} + \frac{1}{p'} = 1)$. Note that

$$|(f * g)(x)| \leq \|f\|_p \|g\|_{p'},$$

which means $\|T_f g\|_\infty \leq \|f\|_p \|g\|_{p'}$. Then for $\theta \in (0,1)$, $\frac{1}{r} = \frac{1-\theta}{1} + \frac{\theta}{p'}$, $\frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{\infty} = \frac{1-\theta}{p}$. Thus,

$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

Calculate θ in terms of p and q, then we can find $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$. □

Now, we'd like to prove the Riesz-Thorin theorem.

Theorem 2.8. *(Hadamard-Phragmen-Lindelof theorem) Let $S = \{\theta + i\tau : \theta \in [0,1], \tau \in \mathbb{R}\}$. Assume $F(z)$ is bounded continuous on S and analytic (or holomorphic) in the interior of S. If $|F(i\tau)| \leq M_0$, $|F(1+i\tau)| \leq M_1$, then*

$$|F(\theta + i\tau)| \leq M_0^{1-\theta} M_1^{\theta} \text{ for } 1 < \theta < 1.$$

Proof. We want to construct a new function from $F(z)$ such that the new function decays to zero as $|\tau| \to \infty$. So we define

$$F_\epsilon(z) = e^{\epsilon z^2 + \lambda z} F(z),$$

where $\epsilon > 0$ and $\lambda \in \mathbb{R}$ (will be determined later).

We only need to check

$$\begin{aligned}|F_\epsilon(\theta + i\tau)| &= |e^{\epsilon(\theta+i\tau)^2 + \lambda(\theta+i\tau)} F(\theta + i\tau)| \\ &= |e^{\epsilon(\theta^2 - \tau^2 + 2i\theta\tau + \lambda\theta + i\lambda\tau} F(z)| \\ &\leq e^{\epsilon(\theta^2 - \tau^2 + \lambda\theta)} |F(z)| \to 0\end{aligned}$$

as $|\tau| \to \infty$. Next,

$$|F_\epsilon(i\tau)| = |e^{\epsilon(i\tau)^2 + \lambda(i\tau)} F(i\tau)| \leq |F(i\tau)| \leq M_0$$

and

$$\begin{aligned}|F_\epsilon(1 + i\tau)| &= |e^{\epsilon(1+i\tau)^2 + \lambda(1+i\tau)} F(1 + i\tau)| \\ &\leq e^{\epsilon(1-\tau^2) + \lambda} |F(1 + i\tau)| \\ &\leq e^{\epsilon + \lambda} M_1.\end{aligned}$$

So by the maximum principle,

$$|F_\epsilon(\theta + i\tau)| \leq \max(M_0, e^{\epsilon + \lambda} M_1),$$

$$\begin{aligned}|F_\epsilon(\theta + i\tau)| &= |e^{\epsilon(\theta+i\tau)^2 + \lambda(\theta+i\tau)} F(\theta + i\tau)| \\ &= e^{\epsilon(\theta^2 - \tau^2) + \lambda\theta} |F(\theta + i\tau)|.\end{aligned}$$

So $|F(\theta + i\tau)| \leq e^{-\epsilon(\theta^2 - \tau^2)} e^{-\lambda\theta} \max(M_0, e^{\epsilon+\lambda} M_1)$. Let $\epsilon \to 0$, then

$$|F(\theta + i\tau)| \leq \max(e^{-\lambda\theta} M_0, e^{\lambda(1-\theta)} M_1) = \max(\rho^{-\theta} M_0, \rho^{1-\theta} M_1),$$

where $\rho = e^\lambda > 0$. We now choose ρ such that $\rho^{-\theta} M_0 = \rho^{1-\theta} M_1$, or $\rho = \frac{M_0}{M_1}$ and

$$|F(\theta + i\tau)| \leq M_0^{1-\theta} M_1^{\theta}.$$

□

2.1 Proof of Riesz-Thorin interpolation theorem

We need to show that $T : L^{p_\theta} \to L^{q_\theta}$, where $\dfrac{1}{p_\theta} = \dfrac{1-\theta}{p_0} + \dfrac{\theta}{p_1}$, $\dfrac{1}{q_\theta} = \dfrac{1-\theta}{q_0} + \dfrac{\theta}{q_1}$ and

$$\|Tf\|_{q_\theta} \le M_0^{1-\theta} M_1^{\theta} \|f\|_{p_\theta}.$$

By the duality argument, it suffices to prove

$$|\langle Tf, g\rangle| \le M_0^{1-\theta} M_1^{\theta} \|f\|_{p_\theta} \|g\|_{q'_\theta}, \ \forall f \in L^{p_\theta}, g \in L^{q'_\theta}.$$

Without loss of generality, we can choose $\|f\|_{p_\theta} = \|g\|_{q'_\theta} = 1$ and show $|\langle Tf, g\rangle| \le M_0^{1-\theta} M_1^{\theta}$.

Observe that $\dfrac{1}{q'_\theta} = \dfrac{1-\theta}{q'_0} + \dfrac{\theta}{q'_1}$. Define

$$\frac{1}{p_z} = \frac{1-z}{p_0} + \frac{z}{p_1} \text{ and } \frac{1}{q'_z} = \frac{1-z}{q'_0} + \frac{z}{q'_1}, \ z \in \mathbb{C}.$$

Set

$$\Phi(x,z) = |f(x)|^{\frac{p_\theta}{p_z}-1} f(x) \text{ and } \Psi(x,z) = |g(x)|^{\frac{q'_\theta}{q'_z}-1} g(x)$$

(we normally define $\dfrac{f(x)}{|f(x)|} = 0$ if $f(x) = 0$). Consider

$$F(z) = \langle T\Phi(x,z), \Psi(x,z)\rangle = \int T\Phi(x,z)\Psi(x,z)dx.$$

To proceed, we consider f and g are simple functions, i.e., $f(x) = \sum a_j \chi_{E_j}$ and $g(x) = \sum b_k \chi_{F_k}$, where $\{E_j\}$ and $\{F_k\}$ have finite measures. Here $a_j, b_k \in \mathbb{C}$. We can write $a_j = |a_j| e^{i\theta_j}$, $b_k = |b_k| e^{i\eta_k}$.

Hence,

$$\begin{aligned} F(z) &= \sum_j \sum_k |a_j|^{\frac{p_\theta}{p_z}-1} |a_j| e^{i\theta_j} |b_k|^{\frac{q'_\theta}{q'_z}-1} |b_k| e^{i\eta_k} \left\langle T\chi_{E_j}, \chi_{F_k}\right\rangle \\ &= \sum_j \sum_k |a_j|^{\frac{p_\theta}{p_z}} |a_j| |b_k|^{\frac{q'_\theta}{q'_z}} |b_k| e^{i(\theta_j+\eta_k)} \left\langle T\chi_{E_j}, \chi_{F_k}\right\rangle. \end{aligned}$$

We then know that $F(z)$ satisfies the conditions in Hadamard et al's theorem. Now, we compute

$$\begin{aligned} |F(i\tau)| &\le \|T\Phi(\cdot, i\tau)\|_{q_0} \|\Psi(\cdot, i\tau)\|_{q'_0} \\ &\le M_0 \|\Phi\|_{p_0} \|\Psi\|_{q'_0} \end{aligned}$$

and

$$\|\Phi(\cdot, i\tau)\|_{p_0} = 1 \text{ and } \|\Psi(\cdot, i\tau)\|_{q'_0} = 1,$$

which implies $|F(i\tau)| \le M_0$.

On the other hand, we can show that $|F(1+i\tau)| \le M_1 \|\Phi(\cdot, 1+i\tau)\|_{p_1} \|\Psi(\cdot, 1+i\tau)\|_{q'_1}$ and $\|\Phi(\cdot, 1+i\tau)\|_{p_1} = \|\Psi(\cdot, 1+i\tau)\|_{q'_1} = 1$ implies $|F(1+i\tau)| \le M_1$. Bt the three-lines theorem (Hadamard et al), we have $|F(\theta+i\tau)| \le M_0^{1-\theta} M_1^{\theta}$. In particular, $|F(\theta + i0)| \leq M_0^{1-\theta} M_1^{\theta}$. For $z = \theta + i0$, we have $\Phi(\cdot, \theta) = f(x)$ and $\Psi(\cdot, \theta) = g(x)$. Therefore,

$$F(\theta) = \langle T\Phi(\cdot, \theta), \Psi(\cdot, \theta) \rangle = \langle Tf, g \rangle .$$

So

$$| \langle Tf, g \rangle | \leq M_0^{1-\theta} M_1^{\theta}$$

and

$$| \langle Tf, g \rangle | \leq M_0^{1-\theta} M_1^{\theta} \|f\|_{p_\theta} \|g\|_{q'_\theta}.$$

In the final step, we approximate f, g by simple functions.

2.2 Summability of Fourier integral

Problem 2.9. Does

$$\lim_{R \to \infty} \int_{B_R} \widehat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi = f(x) \text{ ?}$$

where $B_R = \{R_x : x \in B$ B is an open convex neighborhood of $0\}$. In what sense ? in L^p or pointwise almost everywhere ? It is true in L^2, if

$$\lim_{R \to \infty} \int_{B_R} \widehat{f}(\xi) e^{2\pi i x \cdot \xi} dx = f \text{ in } L^2.$$

Is it true for $p \neq 2$?

Define an (linear) operator

$$(S_R f)^\wedge = \chi_{B_R} \widehat{f}(\xi).$$

The problem is equivalent to

$$\lim_{R \to \infty} S_R f = f$$

in L^p or pointwise a.e..

Theorem 2.10. *For $p \in (1, \infty)$, we have*

$$\lim_{R \to \infty} S_R f = f \text{ in } L^p$$

is equivalent to $\exists C = C(p) > 0$ such that

$$\|S_R f\|_p \leq C_p \|f\|_p.$$

Proof. Exercise. Later, we will prove this when $n = 1$ (related to the Hilbert transform). □

We introduce the Cesaro summability in the following. Define

$$\sigma_R f(x) = \frac{1}{R} \int_0^R S_t f dt.$$

For $n = 1$, $B = (-1, 1)$, we can write $S_R f = D_r * f$, where $D_R = \int_{-R}^{R} e^{2\pi i x\cdot\xi} d\xi = \dfrac{\sin(2\pi R x)}{\pi x}$ is the Dirichlet kernel. Next, we can write $\sigma_R f = F_R * f$, where

$$F_R(x) = \frac{1}{R}\int_0^R D_t dt = \frac{1}{R}\int_0^R \frac{\sin(2\pi t x)}{\pi x} dt$$
$$= \frac{\sin^2(\pi R x)}{R(\pi x)^2}.$$

Note that for $R = 1$, $F_1(x) = \dfrac{\sin^2(\pi x)}{(\pi x)^2}$, $F_R(x) = RF(Rx)$ $(t = \dfrac{1}{R})$. We can see that

$$|F_1(x)| \le \min\{1, (\pi x)^2\} \text{ (integrable).}$$

Corollary 2.11. *We have*

$$\lim_{R\to\infty} \sigma_R f = f \text{ in } L^p(\mathbb{R}) \text{ for } 1 \le p < \infty,$$

$$\lim_{R\to\infty} \sigma_R f = f \text{ in } L^\infty \text{ if } f \in C_0(\mathbb{R}),$$

and

$$\lim_{R\to\infty} \sigma_R f = f \text{ a.e.}.$$

2.3 Other summability methods

1. Abel-Poisson method
 Consider

$$u(x,t) = \int_{\mathbb{R}^n} e^{-2\pi t|\xi|}\widehat{f}(\xi)e^{2\pi x\cdot\xi} d\xi.$$

We can check that u is harmonic for $t > 0$, i.e.,

$$\Delta u = 0 \text{ in } \mathbb{R}^n_+ = \{(x,t)|t > 0\}.$$

Impose the boundary condition $u(x, 0) = f(x)$ (in suitable sense) and $\lim_{t\to 0+} u(x,t) = f(x)$. We can express

$$u(x,t) = P_t * f(x),$$

where $\widehat{P_t}(\xi) = e^{-2\pi t|\xi|}$.
Claim: (Exercise, in Stein-Weiss' book)

$$P_t(x) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \frac{t}{(t^2 + |x|^2)^{\frac{n+1}{2}}} \text{ (Poisson kernel).}$$

So for $P_1 = \dfrac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \dfrac{1}{(1 + |x|^2)^{\frac{n+1}{2}}}$, radially symmetric, decreasing, integrable.

Corollary 2.12. $\lim_{t\to 0+} P_t * f = f$ *in* L^p*, pointwise.*

2. Gauss-Weierstrass method
 Consider

$$w(x,t) = \int e^{-\pi t^2|\xi|^2}\widehat{f}(\xi)e^{2\pi i x\cdot\xi} d\xi,$$

and $\lim_{t\to 0} u(x,t) = f(x)$? We can write $w(x,t) = W_t * f$, where $\widehat{w_t}(\xi) = e^{-\pi t^2|\xi|^2}$, which implies

$$w_t(x) = t^{-n} e^{-\pi|x|^2/t^2} \text{ (Heat kernel, exercise).}$$

Let $\widetilde{w}(x,t) = w(x, \sqrt{4\pi t})$, then

$$\begin{cases} \partial_t \widetilde{w} - \Delta \widetilde{w} = 0 & \text{in } x \in \mathbb{R}^n, t > 0, \\ \widetilde{w}(x,0) = f(x). \end{cases}$$

For $t = 1$, $w_1(x) = e^{-\pi|x|^2}$ radially symmetric, decreasing, integrable.

Corollary 2.13. *We have*

$$\lim_{t\to 0} w(x,t) = f(x) \text{ in } L^p, \text{ a.e.}.$$

3 Calderón-Zygmund decomposition

Let $f \in L^1(\mathbb{R}^n)$ and $f \geq 0$. Given any $\alpha > 0$, we have

1. $\mathbb{R}^n = \Omega \cup F$ $(\Omega \cap F = \emptyset)$,
2. On F (good set), $f(x) \leq \alpha$ a.e.,
3. $\Omega = \cup_k Q_k$, where $\{Q_k\}$'s are non-overlapping cubes, then

$$\alpha < \frac{1}{|Q_k|} \int_{Q_k} f \leq 2^n \alpha.$$

Proof. We partition $\mathbb{R}^n$ into cubes with same diameter. Since $f \in L^1(\mathbb{R}^n)$, we can find a large enough partition cube Q s.t.

$$\int_Q f \leq \alpha |Q| \quad (\int_Q f \leq \int_{\mathbb{R}^n} \leq \alpha |Q|)$$

Next, we divide Q into Q' whose side is half of that of Q. Namely, Q is partitioned into 2^n subcubes. There are only two cases

$$\frac{1}{|Q'|} \int_{Q'} f \leq \alpha \quad \text{or} \quad \frac{1}{|Q'|} \int_{Q'} f > \alpha$$

For Q' satisfying

$$\alpha < \frac{1}{|Q'|} \int_{Q'} f$$

we put it into Ω. To check the other half of (iii), we note that

$$\frac{1}{|Q'|} \int_{Q'} \leq \frac{|Q|}{|Q'|} \frac{1}{|Q|} \int_Q f \leq 2^n \alpha$$

Now for the case

$$\frac{1}{|Q'|} \int_{Q'} f \leq \alpha$$

we repeat the process, partition such Q' into 2^n subcubes Q''. There are two cases:

$$\frac{1}{|Q''|} \int_{Q''} f \leq \alpha \quad \text{or} \quad \frac{1}{|Q''|} \int_{Q''} f > \alpha$$

For Q'' with

$$\alpha < \frac{1}{|Q''|}\int_{Q''} f \leq \frac{|Q'|}{|Q''|}\frac{1}{|Q'|}\int_{Q'} f \leq 2^n\alpha$$

Therefore, we find $\Omega = \bigcup_k Q_k$ satisfying (iii). Now let $F = \mathbb{R}^n - \Omega$, then by Lebesgue Differential Theorem

$$f(x) \leq \alpha \text{ a.e.}$$

□

Corollary 3.1. *$f,\ \alpha,\ F,\ \Omega$ are given as above. $\exists\ A,\ B$ (depending on n) s.t.* *[(i)]*

1. $|\Omega| \leq \frac{A}{\alpha}\|f\|_1$
2. $\forall\ Q_k \in \Omega$
$$\frac{1}{|Q_k|}\int_{Q_k} f \leq B\alpha$$

In fact, from the proof above, $A = 1$, $B = 2^n$.

Proof.

$$|\Omega| = |\bigcup_k Q_k| = \sum_k |Q_k| \leq \sum_k \frac{1}{\alpha}\int_{Q_k} f = \frac{1}{\alpha}\int_{\cup_k Q_k} f \leq \frac{1}{\alpha}\|f\|_1$$

□

Question: What are F and Ω? Is $F = \{x \in \mathbb{R}^n : f(x) \leq \alpha\}$?

3.1 Another proof of Calderón-Zygmund decomposition

For any open set $\Omega \subset \mathbb{R}^n$, we can write $\Omega = \bigcup_k Q_k$, where $\{Q_k\}$ are non-overlapping cubes. Here we need to construct cubes with some geometric restrictions.

Theorem 3.2. *Let F be a (non-empty) closed set in $\mathbb{R}^n$. Denote $\Omega = F^c$ (open). Then there exists a collection of cubes $\mathcal{F} = \{Q_1, Q_2, \cdots\}$ satisfying [(i)]*
(1) $\Omega = \bigcup_k Q_k$
(2) $\{Q_k\}$ are non-overlapping
(3) $\exists\ c_1,\ c_2$ (independent of F) s.t.

$$c_1\text{diam}(Q_k) \leq \text{dist}(Q_k, F) \leq c_2\text{diam}(Q_k).$$

In fact, we can choose $c_1 = 1,\ c_2 = 4$.
We now use Whitney's theorem to re-prove Calderón-Zygmund corollary

Proof. Let $f \in L^1(\mathbb{R}^n)$, then the Hardy-Littlewood maximal function

$$Mf(x) = \sup_{r>0}\frac{1}{|B_r(x)|}\int_{B_r(x)} f dy$$

Mf is lower semicontinuous. Prove Mf is lower semi-continuous (exercise). Then

$$F := \{Mf(x) \leq \alpha\} \text{ (closed)}$$
$$\Omega := \{Mf(x) > \alpha\} \text{ (open)} \quad = \bigcup_k Q_k$$

$\{Q_k\}$ is constructed in Whitney's decomposition.

To prove (i)

$$|\Omega| = |\{Mf(x) > \alpha\}| \leq \frac{A}{\alpha}\|f\|_1$$

Since M is weak(1,1). □

Remark 3.3. Here $A = 5^n$

Proof. To prove (ii), Given $Q_k \subset \Omega$. Pick $p_k \in F$ s.t.

$$\text{dist}(p_k, Q_k) = \text{dist}(Q_k, F)$$

We now pick $B_{r_k}(p_k)$ be the smallest ball containing Q_k as the interior. Since $p_k \in F$

$$\alpha \geq Mf(p_k) \geq \frac{1}{|B_{r_k}(p_k)|}\int_{B_{r_k}(p_k)} f \geq \frac{|Q_k|}{|B_{r_k}(p_k)|}\frac{1}{|Q_k|}\int_{Q_k} f \geq \frac{1}{B}\frac{1}{|Q_k|}\int_{Q_k} f$$

where B depends only on n. □

3.2 Proof of Whitney's Theorem

Proof. We partition $\mathbb{R}^n$ into cubes with integer coordinates (lattice) M_0. For $k \in \mathbb{Z}$, we denote $M_k = 2^{-k}M_0$. Note that for each cube in M_k, its diameter is $\sqrt{n}2^{-k}$. Next, we construct a series of layers

$$\Omega_k = \{x \in \mathbb{R}^n : 2\sqrt{n}2^{-k} < \text{dist}(x, F) \leq 2\sqrt{n}2^{-k+2}\} \subset \Omega$$

Then $\Omega = \bigcup_k \Omega_k$. Now we choose

$$\mathcal{F}_0 = \bigcup_k \{Q \in M_k : Q \cap \Omega_k \neq \emptyset\}$$

Note that if $Q \in \mathcal{F}_0$, $Q \subset \Omega$. In fact

$$\Omega = \bigcup_{Q \in \mathcal{F}_0} Q$$

Claim: For $Q \in \mathcal{F}_0$, $\text{diam}(Q) \leq \text{dist}(Q, F) \leq 4\text{diam}(Q)$

Proof. Since $Q \in \mathcal{F}_0$, $\exists\ x \in Q \bigcap \Omega_k$ for some k

$$\text{dist}(Q, F) \leq \text{dist}(x, F) \leq 2\sqrt{n}2^{-k+1} = 4\sqrt{n}2^{-k} = 4\text{diam}(Q)$$

Next, $\text{dist}(Q, F) + \text{diam}(Q) \geq \text{dist}(x, F) \geq 2\sqrt{n}2^{-k}$, then

$$\text{dist}(Q, F) \geq 2\sqrt{n}2^{-k} - \text{diam}(Q) = \text{diam}(Q)$$

□

So we obtain that all cubes in $\mathcal{F}_0$ satisfy (iii), i.e.

$$\text{diam}(Q) \leq \text{dist}(Q, F) \leq 4\text{diam}(Q)$$

Now the question is that there are not non-overlapping. Observe that if $Q_1 \in M_{k_1}$, $Q_2 \in M_{k_2}$ and $Q_1 \cap Q_2 \neq \emptyset$, then

$$Q_1 \subset Q_2 \quad \text{if } k_1 > k_2$$

Also, if $Q \subset Q'$ and $Q,\ Q' \in \mathcal{F}_0$ then

$$\text{diam}(Q') \leq \text{dist}(Q', F) \leq \text{dist}(Q, F) \leq 4\text{diam}(Q)$$

For any $Q \in \mathcal{F}_0$, we can find the maximal cube $\tilde{Q} \in \mathcal{F}_0$ s.t. $Q \subset \tilde{Q}$. Finally,

$$\Omega = \bigcup_k Q_k$$

where $Q_k \in \mathbb{F}_0$ and maximal cube, $\{Q_k\}$: non-overlapping. □

3.3 Dyadic maximal function

In $\mathbb{R}^n$, let $\widetilde{Q_0}$ be the set of cubes (with lattices coordinates) which are congruent to $[0,1)^n$. Let $\widetilde{Q_k}$ be the set cubes formed by dilation $2^{-k}\widetilde{Q_0}$, $k \in \mathbb{Z}$. Note that for any $x \in \mathbb{R}^n$, x lies in a unique cube for each k. On each level ($k \in \mathbb{Z}$), cubes are disjoint. If two cubes from different k's intersect, then one is contained in other completely.

Let $f \in L^1_{loc}(\mathbb{R}^n)$, define

$$E_* f(x) = \sum_{Q \in Q_k} \left(\fint_Q f dx \right) \chi_Q(x).$$

In other words, for $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$, then there exists only $Q \in \widetilde{Q_k}$ with $x \in Q$. $E_k f(x) = \fint_Q f dx$.

Definition 3.4. For $f \in L^1_{loc}(\mathbb{R}^n)$, we define the dyadic maximal function

$$M_d f(x) = \sup_k E_k[|f|](x) = \sup_{\substack{x \in Q \\ Q \subset \widetilde{Q_k}}} \fint_Q |f|.$$

Lemma 3.5. *For $f \in L^1(\mathbb{R}^n)$, then*

$$\lim_{k \to -\infty} E_k f(x) = 0.$$

Proof. Observe that

$$E_k[|f|](x) = \fint_Q |f| \leq \frac{1}{|Q|} \|f\|_1 \to 0 \text{ as } k \to -\infty.$$

□

Theorem 3.6. *Let $f \in L^1(\mathbb{R}^n)$, $\lambda > 0$, there exists a collection of disjoint dyadic cubes $\{\mathcal{Q}_j\}$ such that*

$$\{x \in \mathbb{R}^n : M_d f(x) > \lambda\} = \cup_j \mathcal{Q}_j$$

and

$$\lambda \leq \fint_{\mathcal{Q}_j} |f| \leq 2^n \lambda.$$

Corollary 3.7. *(a) M_d is weak $(1,1)$.*

(b) Lebesgue differentiation theorem

$$\lim_{k\to\infty} E_k f(x) = f(x) \ a.e..$$

Proof. (b) follows from (a) (exercise). For (a),

$$\begin{aligned}|\{x \in \mathbb{R}^n : M_d f(x) > \lambda\}| = |\cup_j \mathcal{Q}_j| = \sum_j |\mathcal{Q}_j| \\ \leq \frac{1}{\lambda} \sum_j \int_{\mathcal{Q}_j} |f| \leq \frac{1}{\lambda} \|f\|_1.\end{aligned}$$

□

Proof. (Proof of Theorem) Let

$$E_\lambda = \{x \in \mathbb{R}^n : M_d f(x) > \lambda\} = \cup_k \{x \in \mathbb{R}^n : E_k[|f|](x) > \lambda\}.$$

Now if $x \in E_\lambda$, $\exists x \in Q_k \in \widetilde{Q_k}$ such that $\fint_{Q_k} |f| > \lambda$. Note that $Q_k \subset E_\lambda$. By lemma, there must exist a largest $Q_{k^*} \supset Q_k$ such that $E_{k^*}[|f|](x) > \lambda$ $(k^* \leq k)$. For any $x \in E_\lambda$, there exists a unique cube Q_k such that $E_k[|f|](x) > \lambda$ but $E_{k-1}[|f|](x) \leq \lambda$. So $E_\lambda = \cup_k Q_k$. Next, on each cube Q_k,

$$\lambda < \fint_{Q_k} |f| \leq \frac{|Q_{k-1}|}{|Q_k|} \frac{1}{|Q_{k-1}|} \int_{Q_{k-1}} |f| \leq 2^n \lambda.$$

□

Theorem 3.8. *(Calderón-Zygmund decomposition)*

Let $f \in L^1(\mathbb{R}^n)$, $\lambda > 0$, there exists a collection of disjoint dyadic cubes $\{Q_k\}$ and $g \in L^1(\mathbb{R}^n)$ and $\{b_k\}$ such that $f = g + \sum b_k$, where $\|g\|_{L^\infty} \leq 2^n \lambda$ and $\|g\|_1 \leq \|f\|_1$ (good part), supp$(b_k) \subset Q_k$ and $\int_{Q_k} b_k dx = 0$.

Proof. Let $\{Q_k\}$ be constructed as above. Define

$$b_k = \left(f(x) - \fint_{Q_k} f \right) \chi_{Q_k}(x).$$

So b_k satisfies all conditions. Define

$$g(x) = f(x) - \sum_k b_k(x)$$

Need to show that $\|g\|_\infty \leq 2^n \lambda$, $\|g\|_1 \leq \|f\|_1$. If $x \in \cup_j Q_j$ and note that $\{Q_j\}$ disjoint then $g(x) = \fint_{Q_j} f(x) dx \ \forall \ x \in Q_j$.

For $x \notin \cup_j Q_j$, $f(x) = g(x)$. We know that $M_d f(x) \leq \lambda$ for $x \notin \cup_j Q_j$. Also $|f(x)| \leq M_d f(x)$ a.e., then $|g(x)| \leq \lambda\ \forall\ x \notin \cup_j Q_j$. For $x \in \cup_j Q_j$, $|g(x)| \leq |\fint_{Q_j} f(x)| \leq 2^n \lambda$. Thus $\|g\|_\infty \leq 2^n \lambda$. On the other hand

$$\begin{aligned}
\textstyle\int_{\mathbb{R}^n} |g| &= \textstyle\int_{\cup_j Q_j} |g(x)| + \int_{\mathbb{R}^n - \cup_j Q_j} |g(x)| \\
&\leq \textstyle\int_{\cup_j Q_j} |f(x)| + \int_{\mathbb{R}^n - \cup_j Q_j} |f(x)| \\
&= \|f\|_1.
\end{aligned}$$

□

3.4 Another maximal functions defined by cubes

Definition 3.9. Let $x \in \mathbb{R}^n$ and Q_r be the cube with centered at x and $l(Q_r) = 2r$, then if $f \in L^1_{loc}(\mathbb{R}^n)$, we define

$$M'f(x) = \sup_{r>0} \frac{1}{|Q_r|} \int_{Q_r} |f(y)| dy = \sup_{r>0} \frac{1}{(2r)^n} \int_{Q_r} |f(y)| dy$$

Note that $\exists\ c_1, c_2$ (depends only on n) s.t.

$$c_1 M'f(x) \leq Mf(x) \leq c_2 M'f(x)$$

Here Mf is Hardy-Littlewood maximal function.

Theorem 3.10. *We have that* $\forall\ \lambda > 0$,

$$|\{x \in \mathbb{R}^n : M'f(x) > 4^n \lambda\}| \leq 2^n |\{x \in \mathbb{R}^n : M_d f(x) > \lambda\}|.$$

Proof. Recall that

$$\{x \in \mathbb{R}^n : M_d f(x) > \lambda\} = \cup_j Q_j$$

$\{Q_j\}$: dyadic cubes (disjoint). So it suffices to show that

$$\{x \in \mathbb{R}^n : M'f(x) > 4^n \lambda\} \subset \cup_j 2Q_j$$

(Q_j and $2Q_j$ have the same center). Equivalently, we want to show

$$x \notin \cup_j 2Q_j \Rightarrow M'f(x) \leq 4^n \lambda$$

Let Q be any cube centered at x. Then we know that $\exists\ k \in \mathbb{Z}$ s.t.

$$2^{-(k+1)} \leq l(Q) < 2^{-k}$$

$l(Q)$: the length of side of Q. Observe that Q intersects m cubes in $\widetilde{Q}_k$, where $m \leq 2^n$. We assume Q intersects $R_1, R_2, \cdots, R_m \subset \widetilde{Q}_k$.

Note that none of these cubes $R_1, \cdots, R_m$ is contained in any Q_j. If not, then $x \in 2Q_j$. Hence on each R_i, $i = 1, \cdots, m$, we have

$$\frac{1}{|R_i|} \int_{R_i} |f(x)| \leq \lambda$$

So

$$
\begin{aligned}
⨍_Q |f| &= \textstyle\sum_{i=1}^m \frac{1}{|Q|} \int_{Q\cap R_i} |f| \\
&\le \textstyle\sum_{i=1}^m \frac{|R_i|}{|Q|}\frac{1}{|R_i|} \int_{R_i} |f| \\
&\le \frac{2^{-kn}}{|Q|} \textstyle\sum_{i=1}^m \frac{1}{|R_i|} \int_{R_i} |f| \\
&\le \frac{2^{-kn}}{|Q|} \textstyle\sum_{i=1}^m \lambda \\
&\le \frac{2^{-kn}}{|Q|} 2^n\lambda \\
&\le \frac{2^{-kn}2^n}{2^{-(k+1)n}}\lambda \\
&= 4^n\lambda.
\end{aligned}
$$

Thus, $M'f(x) \le 4^n\lambda$. □

3.5 The Hilbert transform

Consider the mapping H

$$Hf(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(y)}{x-y}dy$$

The definition does not make sense since $\frac{1}{x-y}$ is not locally integrable!

In fact, H is defined by the sense of principle value, i.e.

$$Hf(x) = \lim_{\epsilon\to 0}\frac{1}{\pi}\int_{|x-y|>\epsilon} \frac{f(y}{)} x - ydy = p.v. \int_{-\infty}^{\infty}\frac{f(y)}{x-y}dy$$

To see that the definition makes sense, we let $f \in C_0^1(\mathbb{R})$

$$
\begin{aligned}
\textstyle\int_{|x-y|>\epsilon} \frac{f(y)}{x-y}dy &= \textstyle\int_{x-y} > \epsilon\frac{f(y)}{x-y}dy + \int_{x-y} < -\epsilon\frac{f(y)}{x-y}dy \\
&= \textstyle\int_{|x-y|>\epsilon} \frac{f(y)}{x-y}dy - f(x)\int_{|x-t|>\epsilon}\frac{dy}{x-y} \\
&= \textstyle\int_{|x-y|>\epsilon} \frac{f(y)-f(x)}{x-y}dy \\
&= \textstyle\int_{\epsilon<|x-y|<1} \frac{f(y)-f(x)}{x-y}dy + \int_{|x-y|>1}\frac{f(y)-f(x)}{x-y}dy
\end{aligned}
$$

The second of RHS is finite. Since $f \in C^1(\mathbb{R})$, we have

$$
\begin{aligned}
|\int_{\epsilon<|x-y|<1} \frac{f(y)-f(x)}{x-y}dy| &\le \int_{\epsilon<|x-y|<1} |\frac{f(y)-f(x)}{x-y}|dy \\
&\le \|f'\|_\infty \int_{\epsilon<|x-y|<1} dy \le 2\|f'\|_\infty.
\end{aligned}
$$

This method is called regularization.

Note that the same method does not work for $\frac{1}{|x-y|}$ (no cancellation !)

Remark 3.11. The Hilbert transform of any function (compactly supported) is not always defined pointwise, e.g. if $f = \chi_{[0,1]}$ then $Hf(x) = -\infty$. Check the above example (exercise).

Goal : to study the mapping property of H in L^p, $1 \le p \le \infty$

In fact, the kernel of H is a tempered distribution, i.e.

$$p.v.\frac{1}{x}(\psi) = \lim_{\epsilon\to 0}\int_{|x|>\epsilon}\frac{\psi(x)}{x}dx \quad \psi \in \mathcal{S}(\mathbb{R})$$

3.6 Connect to complex analysis

Let $u(x,t) = (P_t * f)(x)$, where $P_t(x)$, $x \in \mathbb{R}^n, t > 0$ is the Poisson kernel of the half plane

$$\widehat{P}_t(\xi) = e^{-2\pi t|\xi|} \Leftrightarrow P_t(x) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}\frac{t}{(t^2+|x^2|)^{\frac{n+1}{2}}}$$

and

$$u(x,t) = \int e^{-2\pi t|\xi|}\hat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi$$

Also,$\lim_{t\to 0}(P_t * f)(x) = f(x)$ in L^p, $1 \le p < \infty$ and a.e.

Now we take $n = 1$,

$$P_t(x) = \frac{1}{\pi}\frac{t}{(t^2+x^2)}$$

Let $z = x + it$, then

$$u(x,t) = u(z) = \int_0^\infty \hat{f}(\xi)e^{i2\pi z\xi}d\xi + \int_{-\infty}^0 \hat{f}(\xi)e^{i2\pi\bar{z}\xi}d\xi$$

Now if we let

$$iv(x,t) = iv(z) = \int_0^\infty \hat{f}(\xi)e^{i2\pi z\xi}d\xi - \int_{-\infty}^0 \hat{f}(\xi)e^{i2\pi\bar{z}\xi}d\xi$$

then $u + iv = 2\int_0^\infty \hat{f}(\xi)e^{i2\pi z\xi}d\xi$ is analytic in $\text{Im}z > 0$

Note that u and v are harmonic. Also, u and v are real if f is real. Prove it (exercise). So v is a harmonic conjugate of u. Observe that

$$\begin{aligned} iv(x,t) &= \int_0^\infty \hat{f}(\xi)e^{i2\pi(x+it)\xi}d\xi - \int_{-\infty}^0 \hat{f}(\xi)e^{i2\pi(x-it)\xi}d\xi \\ &= \int_0^\infty e^{-2\pi t\xi}\hat{f}(\xi)e^{2\pi ix\xi}d\xi + \int_{-\infty}^0 (-1)e^{-2\pi t(-\xi)}\hat{f}(\xi)e^{i2\pi x\xi}d\xi \\ &= \int_{-\infty}^\infty \text{sign}(\xi)e^{-2\pi t|\xi|}\hat{f}(\xi)e^{i2\pi x\xi}d\xi \end{aligned}$$

Then

$$v(x,t) = \int_{-\infty}^\infty -i\text{sign}(\xi)e^{-2\pi t|\xi|}\hat{f}(\xi)e^{i2\pi x\xi}d\xi = (Q_t * f)(x)$$

where $\widehat{Q}_t = -i\text{sign}(\xi)e^{-2\pi t|\xi|}$. We can compute

$$Q_t = \frac{1}{\pi}\frac{x}{(t^2+x^2)} \quad \text{(Conjugate Poisson Kernel)}$$

If we write

$$P_t + iQ_t = \frac{1}{\pi}\frac{t+ix}{(t^2+x^2)} = \frac{1}{\pi}\frac{i\bar{z}}{z\bar{z}} = \frac{1}{\pi}\frac{i}{z}$$

the second equivalent let $z = x + it$.

Lemma 3.12.

$$\lim_{t\searrow 0} Q_t = \frac{1}{\pi}p.v\frac{1}{x}$$

as a tempered distribution

Proof. Need to show that $\forall\ \psi \in \mathcal{S}(\mathbb{R})$,

$$\lim_{t\searrow 0}(Q_t - \frac{1}{\pi p.v.\frac{1}{x}})(\psi) = 0$$

Meaning

$$\begin{aligned} & \lim_{t\to 0}(\int_{-\infty}^{\infty}\frac{x\psi(x)}{t^2+x^2}dx - \int_{|x|>t}\frac{\psi(x)}{x}dx) = 0 \\ = \quad & \lim_{t\to 0}(\int_{|x|leqt}\frac{x\psi(x)}{t^2+x^2}dx + \int_{|x|>t}\frac{x\psi(x)}{t^2+x^2}dx - \int_{|x|>t}\frac{\psi(x)}{x}dx) \\ = \quad & \lim_{t\to 0}(\int_{|x|leq1}\frac{x\psi(tx)}{1+x^2}dx + \int_{|x|>1}(\frac{x\psi(tx)}{1+x^2}dx - \frac{\psi(tx)}{x})dx). \end{aligned}$$

□