Introduction to the classical DeGiorgi method and its application in the regularity theory of incompressible Navier-Stokes equations

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Abstract

The note is mainly for personal record, if you want to read it, please be careful. This note is given by Prof. *Chi Hin Chan*'s lecture during the summer course in NCTS, 2015. We will introduce the De-Giorgi's method in the classical setting of elliptic PDE's in the divergence form with rough diffusive coefficients. It was in this setting that De-Giorgi used his method to establish the regularity of solutions to general linear elliptic PDE's in the framework of Holder's continuity. After this preparation in the classical elliptic setting, we go ahead to discuss a piece of recent work of Alexis Vasseur in which he applied the De-Giorgi's method to yield an alternative proof of the key proposition in the famous partial regularity theory of Caffarelli, Kohn, Nirenberg. Further, we will introduce notions such as the Serrin's regularity criteria, and the notion of suitable weak solutions in the theory of weak solutions to the incompressible Navier-Stokes equations.

1 Introduction to DeGiorgi's ideas

Let $B(1) \subset \mathbb{R}^N$ $(N \geq 2)$. We consider a function $u \in H^1(B(1))$ which is a weak solution to

$$\partial_\alpha \{a^{\alpha\beta}\partial_\beta u\} = 0 \text{ in } B(1). \tag{1}$$

Here $(a^{\alpha\beta}) : B(1) \to \mathbb{R}^{N^2}$ satisfies the following constraint:

$$\frac{1}{\Lambda}|\xi|^2 \leq a^{\alpha\beta}(x)\xi_\alpha\xi_\beta \leq \Lambda|\xi|^2, \ \forall x \in B(1), \ \xi \subset \mathbb{R}^N.$$

Or regard $\xi \in T_x^*(B(1))$. Relation with Riemannian geometry,

$$g = g_{\alpha\beta}dx^\alpha \otimes dx^\beta$$

and

$$\Delta f = \frac{1}{\sqrt{G}}\partial_\alpha\{g^{\alpha\beta}\sqrt{G}\partial_\beta f\},$$

where $g^{\alpha\beta} = d(dx^\alpha, dx^\beta)$.

In this note, we focus on the DeGiorgi method, 1957-1960. Compare to the Schauder's estimate, we need the coefficients lying in $C^\alpha(\Omega)$, then the solution will lie in $C^{2,\alpha}(\Omega)$. The idea is that if we rescale $u_\lambda(x) = u(\lambda x)$ and $a_\lambda^{\alpha\beta}(x) = a^{\alpha\beta}(\lambda x)$, if we make λ smaller and smaller, then $a_\lambda^{\alpha\beta}$ will behave like a piecewise constant function. This scaling just "flatten" the graph of solutions, it will not change the height of functions. Therefore, the regularity will similar as the Laplacian case.

1.1 First DeGiorgi lemma

The first step is from the L^2 to L^∞. If the L^2 energy of the solution u in $B(1)$ is small enough, then

$$\|u\|_{L^\infty(B(\frac{1}{2}))} \leq 1.$$

Lemma 1.1. *There exists an absolute constant $\epsilon_0 = \epsilon_0(N, \Lambda) > 0$ such that the following statement holds for any weak solution $u \in H^1(B(1))$ to (1). If $\|u_+\|^2_{L^2(B(1))} < \epsilon_0$, then*

$$u_+ \leq 1 \text{ in } B(\frac{1}{2}).$$

Remark 1.2. For the whole information of u, just multiply the negative sign and consider the previous lemma again. What if $\|u\|_{L^2(B(1))}$ is not small ? Rescale $V = \dfrac{\epsilon_0}{2\|u\|_{L^2(B(1))}} u$, then $\|V\|_{L^2(B(1))} < \epsilon_0$. This will imply

$$\|u\|_{L^\infty(B(\frac{1}{2}))} \leq \frac{2}{\epsilon_0} \|u\|_{L^2(B(1))}.$$

Set up the Game: Let $R_k = \dfrac{1}{2}(1 + \dfrac{1}{2^k})$, $k \geq 0$. Let $\varphi_k \in C_0^\infty(B(R_k))$ with

$$\varphi_k = \begin{cases} 1 & \text{in } B(R_{k+1}), \\ 0 & \text{outside } B(R_k), \end{cases}$$

and $|\nabla \varphi_k| \leq \dfrac{2}{R_k - R_{k+1}} \chi_{B(R_k)} = 2^{k+3} \chi_{B(R_k)}$.

Cutting level. Let $V_0 = u_+$, $V_k = [u - (1 - \dfrac{1}{2^k})]_+$ with $V_k \leq V_{k-1}$ and $\nabla V_k = \nabla u \cdot \chi_{\{v_k > 0\}}$. $U_k = \int_{B(R_k)} |V_k|^2$.

Rough idea: Build up the following

$$U_k \leq C_0^k U_{k-1}^\beta, \ \beta > 1. \tag{2}$$

If $U_0 = \|u_+\|^2_{L^2(B(1))} < \epsilon_0$, then $U_k \to 0$ as $k \to \infty$. $\int_{B(\frac{1}{2})} [u - 1]_+ = \lim_{k \to \infty} U_k = 0$ which implies $u \leq 1$ in $B(\dfrac{1}{2})$.

Test against (1) by $\varphi_k^2 V_k$, $V_k \notin H_0^1(B(1))$ but $\varphi_k^2 V_k \in H_0^1(B(1))$, which will be the test function. Then

$$\begin{aligned} 0 &= \int_{B(1)} a^{\alpha\beta} \partial_\beta u \partial_\alpha \{\varphi_k^2 V_k\} \\ &= \int_{B(1)} a^{\alpha\beta} \partial_\beta u \cdot 2\varphi_k \cdot \partial_\alpha \varphi_k \cdot V_k + \int_{B(1)} a^{\alpha\beta} \partial_\beta u \cdot \varphi_k^2 \partial_\alpha V_k \\ &= \int_{B(1)} a^{\alpha\beta} \partial_\beta V_k \cdot 2\varphi_k \cdot \partial_\alpha \varphi_k \cdot V_k + \int_{B(1)} a^{\alpha\beta} \partial_\beta V_k \varphi_k^2 \partial_\alpha V_k, \end{aligned}$$

and let $g_x(A,B) = a^{\alpha\beta}(x)\cdot A_\alpha B_\beta$,

$$\begin{aligned}\int_{B(1)} a^{\alpha\beta}\partial_\alpha V_k \partial_\beta V_k \varphi_k^2 &= -2\int_{B(1)} a^{\alpha\beta}\partial_\beta V_k\cdot\varphi_k\cdot\partial_\alpha\varphi_k\cdot V_k \\ &= -2\int_{B(1)} g_x(\varphi_k\nabla V_k, V_k\nabla\varphi_k)|_x dx \\ &\le \epsilon\int_{B(1)} g_x(\varphi_k\nabla V_k,\varphi_k\nabla V_k)|_x dx \\ &\quad + \frac{1}{\epsilon}\int_{B(1)} g_x(V_k\nabla\varphi_k, V_k\nabla\varphi_k)|_x dx.\end{aligned}$$

This implies

$$\begin{aligned}\int_{B(1)} a^{\alpha\beta}\partial_\alpha V_k\partial_\beta V_k\varphi_k^2 &\le 4\int_{B(1)} a^{\alpha\beta}\partial_\alpha\varphi_k\partial_\beta\varphi_k\cdot V_k^2 \\ &\le 4\Lambda\int_{B(1)}|\nabla\varphi_k|^2 V_k^2 \\ &\le 4\Lambda 2^{2k+6}\int_{B(R_k)} V_k^2 \\ &= 128\Lambda 4^k U_k\end{aligned}$$

and use ellipticity condition, we have

$$\begin{aligned}\int_{B(1)}|\nabla(\varphi_k V_k)|^2 &\le 2\int_{B(1)}|\nabla\varphi_k|^2V_k^2 + 2\int_{B(1)}\varphi_k^2|\nabla V_k|^2 \\ &\le 256\cdot 4^k\int_{B(R_k)}|V_k|^2 + 256\Lambda 4^k U_k \\ &= 256(1+\Lambda)4^k U_k.\end{aligned}$$

For $N\ge 3$, use the Sobolev embedding theorem,

$$\|\varphi_k V_k\|_{L^{\frac{2N}{N-2}}(B(1))} \le C_0\left(\int_{B(1)}|\nabla(\varphi_k V_k)|^2\right)^{1/2}.$$

For $N=2$, use the Ladyzhenskaya inequality,

$$\|\varphi_k V_k\|_{L^4(B(1))} \le \|\varphi_k V_k\|_{L^2(B(1))}^{1/2}\cdot\|\nabla(\varphi_k V_k)\|_{L^2(B(1))}^{1/2}.$$

Note that, $\{x\in B(1): V_k>0\}\subset\{x\in B(1): V_{k-1}(x)>\frac{1}{2^k}\}$, $\chi_{\{V_k>0\}}\le (2^kV_{k-1})^{\frac{4}{N-2}}$, and $\chi_{B(R_k)}\le\varphi_{k-1}^{\frac{2N}{N-2}}$,

$$\begin{aligned}U_k &= \int_{B(R_k)} V_k^2\chi_{\{V_k>0\}} \le \int_{B(R_k)} V_k^2(2^{\frac{4}{N-2}})^k V_{k-1}^{\frac{4}{N-2}} \\ &\le (2^{\frac{4}{N-2}})^k\int_{B(R_k)} V_{k-1}^{\frac{2N}{N-2}} \le (2^{\frac{4}{N-2}})^k\|\varphi_{k-1}V_{k-1}\|_{L^{\frac{2N}{N-2}}(B(1))}^{\frac{2N}{N-2}},\end{aligned}$$

which raised up the index. Finally,

$$\begin{aligned} U_k &\le (2^{\frac{4}{N-2}})^k C_0 (\int_{B(1)} |\nabla(\varphi_{k-1}V_{k-1})|^2)^{\frac{N}{N-2}} \\ &\le (2^{\frac{4}{N-2}})^k C_0 (256(1+\Lambda)4^k U_k)^{\frac{N}{N-2}} \\ &\le [C_0(N,\Lambda)]^k U_{k-1}^{\frac{N}{N-2}}, \ k \ge 1. \end{aligned}$$

Thinking: $\{U_k\}_{k=1}^{\infty} \subset [0,\infty)$, $U_{k+1} \le (C_0)^{k+1} U_k^{\beta}$, $k \ge 0$, $C_0 > 1$, $\beta > 1$. $U_{l+1} \sim (C_0^{k+1} U_k^{\beta-1}) U_k$ and $U_{k+1} \sim \frac{1}{A} U_k$, where $A > 1$. It means $C_0^{k+1} U_k^{\beta} \sim \frac{1}{A}$, then $U_k \sim \frac{1}{(AC_0^{k+1})^{\frac{1}{\beta-1}}}$ and $U_{k+1} \sim \frac{1}{A} U_k \sim \frac{1}{A} \frac{1}{(AC_0^{k+1})^{\frac{1}{\beta-1}}}$,......., $A \sim C_0^{\frac{1}{\beta-1}}$. Want $C_0^{k+1} U^{\beta-1} \le \frac{1}{C_0^{\frac{1}{\beta-1}}}$ for all $k \ge 0$. $\|u_+\|_{L^2(B(1))}^2 \le \frac{1}{(C_0 C_0^{\frac{1}{\beta-1}})^{\frac{1}{\beta-1}}}$.

1.2 A detour: DeGiorgi's isoperimetric inequality

Take $u \in H^1(B(1)) \cap C^\infty(\overline{B(1)})$, $\mathcal{A} = \{x \in B(1) : u(x) \le 0\}$, $\mathcal{B} = \{x \in B(1) : u(x) \ge 1\}$, $\mathcal{C} = \{x \in B(1) : 0 < u(x) < 1\}$.

Proposition 1.3. *The following estimate holds:*

$$|\mathcal{A}| \cdot |\mathcal{B}| \le C_N \|\nabla u_+\|_{L^2(B(1))} |\mathcal{C}|^{\frac{1}{2}}.$$

Take $\widetilde{u} = \max\{0, \inf\{1, u\}\}$, $x \in \mathcal{A}$, $y \in \mathcal{B}$, (Trick: Think of $|\nabla \widetilde{u}| = |\nabla \widetilde{u}| \chi_{\mathcal{C}}$ as a function on $\mathbb{R}^N$.)

$$\begin{aligned} \widetilde{u}(y) - \widetilde{u}(x) = 1 &= \int_0^{|y-x|} \frac{d}{d\tau} \{\widetilde{u}(x + \tau \frac{y-x}{|y-x|})\} d\tau \\ &= \int_0^{|y-x|} \nabla \widetilde{u}|_{(x+\tau\frac{y-x}{|y-x|})} \cdot \frac{y-x}{|y-x|} d\tau. \end{aligned}$$

Then

$$1 \le \int_0^{|y-x|} |\nabla \widetilde{u}(x + \tau \frac{y-x}{|y-x|})| d\tau \le \int_0^{\infty} |\nabla \widetilde{u}(x + \tau \frac{y-x}{|y-x|})| d\tau,$$

and since $|x| \le 1$, if $y \in \mathcal{B}$, $|y - x| \le 2$,

$$\begin{aligned} |\mathcal{B}| &\le \int_{y\in\mathcal{B}} \int_0^{\infty} |\nabla \widetilde{u}(x + \tau \frac{y-x}{|y-x|})| d\tau dy \\ &\le \int_{|y-x|\le 2} \int_0^{\infty} |\nabla \widetilde{u}(x + \tau \frac{y-x}{|y-x|})| d\tau \\ &\le \int_0^2 r^{N-1} \int_{S^{N-1}} \int_0^{\infty} |\nabla \widetilde{u}(x + \tau\omega)| d\tau dS(\omega) dr \\ &= \frac{2^N}{N} \int_0^{\infty} \int_{S^{N-1}} \tau^{N-1} |\nabla \widetilde{u}(x + \tau\omega)| dS(\omega) d\tau \\ &= \frac{2^N}{N} \int_{\mathbb{R}^N} \frac{|\nabla \widetilde{u}(x+y)}{|y|^{N-1}} dy = \frac{2^N}{N} \int_{\mathcal{C}} \frac{|\nabla \widetilde{u}(y)|}{|x-y|^{N-1}} dy, \end{aligned}$$

then

$$|\mathcal{A}|\cdot|\mathcal{B}| \le \frac{2^N}{N}\int_{y\in\mathcal{C}}\int_{x\in\mathcal{A}}\frac{|\nabla\widetilde{u}(y)|}{|x-y|^{N-1}}dxdy$$
$$= \frac{2^N}{N}\int_{y\in\mathcal{C}}|\nabla\widetilde{u}(y)|\cdot\int_{x\in\mathcal{A}}\frac{1}{|x-y|^{N-1}}dxdy,$$

$x\in\mathcal{A}$ and $|x-y|\le 2$. $\mathcal{A}\subset\{x\in|x-y|\le 2\}$ provided $|y|\le 1$,

$$\int_{\mathcal{A}}\frac{1}{|x-y|^{N-1}}dy \le \int_{|x-y|\le 2}\frac{1}{|x-y|^{N-1}}dy = C_N.$$

Therefore,

$$|\mathcal{A}|\cdot|\mathcal{B}| \le C_N\int_{y\in\mathcal{C}}|\nabla\widetilde{u}(y)|dy.$$

2 Oscillating lemma

Suppose we have a solution $u: B(2)\to\mathbb{R}$ to (1) which satisfies

1. $u\le 2$ in $B(2)$,
2. $|\{x\in B(1): u(x)\le 0\}| \ge \frac{|B(1)|}{2}$.

The second DeGiorgi lemma shows that

Lemma 2.1. *$\exists$ another absolute constant $\lambda=\lambda(\Lambda,N)\in(0,1)$ if the solution $u: B(2)\to\mathbb{R}$ satisfies 1,2, then*

$$u\le 2-\lambda \text{ in } B(\frac{1}{2}).$$

Is it possible that

$$\int_{B(1)}(2(u-1)_+)^2 < \epsilon_0,$$

we care since $u_1 = 2(u-1)$ is another solution to (1). If this happens, which means $\int_{B(1)}(u_1)^2<\epsilon_0$, by the first DeGiorgi lemma, we have

$$u_1\le 1 \text{ in } B(\frac{1}{2}), \text{ equivalently, } u\le\frac{3}{2}.$$

This may be false (do successive cuttings...) and treat u_1 just in the same manner as u and let $u_2=2(u_1-1)$, and iterate these steps until we find some $u_k\le\frac{3}{2}$. Note that $u_2=2(u_1-1)=2^2u-2^{2-2}$, then

$$u_k = 2^k u-(2^k+2^{k-1}+\cdots+2)$$
$$= 2^k\{u-2(1-\frac{1}{2^k})\}.$$

Look at $u=u_0$ and so on, set $u_k\in H^1(B(1))$, $\mathcal{A}_k=\{x\in B(1): u_k(x)\le 0\}$, $\mathcal{B}_k=\{x\in B(1): u_k(x)\ge 1\}$, $\mathcal{C}_k=\{x\in B(1): 0<u_k(x)<1\}$. $|\mathcal{A}_1| = |\mathcal{A}_0|+|\mathcal{C}_0|$, $|\mathcal{A}_k|=|\mathcal{A}_{k-1}|+|\mathcal{C}_{k-1}|$ and property 2 gives that $|\mathcal{A}_k|\ge\frac{|B(1)|}{2}$.

Remark 2.2. The structure of (1) allows us to obtain: For each solution $u_k$to (1) in $B(2)$,

$$\int_{B(1)} |\nabla(u_k)_+|^2 \leq C_{\Lambda,N} \int_{B(2)} |(u_k)_+|^2 \leq 4C_{\Lambda,N}|B(2)|,$$

i.e.,

$$\|\nabla(u_k)_+\|_{L^2(B(1))} \leq C_{\Lambda,N}.$$

$|\mathcal{A}_k| \cdot |\mathcal{B}_k| \leq C_N \|\nabla(u_k)_+\|_{L^2(B(1))} |\mathcal{C}_k|^{\frac{1}{2}} \leq C_{\Lambda,N}|\mathcal{C}_k|^{\frac{1}{2}}$.

Remember: We want $|\mathcal{B}_k|$ to be small eventually. We want $4|\mathcal{B}_k| < \epsilon_0$, i.e.,

$$\int_{B(1)} (u_{k+1})_+^2 = 4\int_{B(1)} (u_k - 1)_+^2 \chi_{\mathcal{B}_k} \leq 4|\mathcal{B}_k| < \epsilon_0.$$

If in the worst case

$$|\mathcal{B}_k| \leq \frac{\epsilon_0}{4} \text{ holds for all } k,$$

then

$$|\mathcal{C}_k| \geq (\frac{|B(1)|}{2}\frac{\epsilon_0}{4}\frac{1}{C_{\Lambda,N}})^2 := \alpha_{N,\Lambda}, \ \forall k \geq 0.$$

But $|B(1)| \geq |\mathcal{A}_k| \geq k\alpha_{N,\Lambda} + \dfrac{|B(1)|}{2}$ or $k \leq \dfrac{|B(1)|}{2\alpha_{N,\Lambda}}$, choose $K_0 := \left\lfloor \dfrac{|B(1)|}{2\alpha_{N,\Lambda}} \right\rfloor + 1$. In other words, the propagation on the property $|\mathcal{B}_k| \geq \dfrac{\epsilon_0}{4}$ would stop at some $0 \leq k \leq K_0$, i.e., $|\mathcal{B}_{K_0}| < \dfrac{\epsilon_0}{4}$ will imply $\int_{B(1)} (u_{K_0+1})_+^2 < \epsilon_0$ with u_{K_0+1} solves (1). Apply the first DeGiorgi lemma to u_{K_0+1} and get $u_{K_0+1} \leq 1$ in $B(\dfrac{1}{2})$, i.e.,

$$\{u - 2(1 - \frac{1}{2^{K_0+1}})\} \leq \frac{1}{2^{K_0+1}} \text{ in } B(\frac{1}{2}),$$

then $u \leq 2 - \dfrac{1}{2^{K_0+1}}$.

Remark 2.3. You can always choose d_1 with $|d_1| \leq \dfrac{\lambda}{2}$ such that $|u - d_1| \leq 2 - \dfrac{\lambda}{2}$. Let

$$W^{(1)} = \frac{2}{2 - \frac{\lambda}{2}}[u(\frac{1}{2}x) - d_1]$$

be a rescaled solution. Finally, we will see

$$\text{osc}_{B(\frac{1}{2^k})} u \leq (\frac{2 - \frac{\lambda}{2}}{2})^k \text{osc}_{B(1)} u$$

or ($|u| \leq 2$)

$$\text{osc}_{B(r^k)} u \leq 4\delta^k.$$

Thinking: $|x| \sim r^k$ which means $k \sim \dfrac{\ln|x|}{\ln r}$, then $\delta^k \sim \delta^{\frac{\ln|x|}{\ln r}} = (\delta^{\log_\delta |x|})^{\frac{1}{\ln r}\frac{1}{\log_\delta e}} = |x|^{\frac{1}{\ln r}\frac{1}{\log_\delta e}} \sim |x|^\gamma$. Then $|u(x) - u(0)| \sim |x|^\gamma$.

3 Introduction to the elements of incompressible Navier-Stokes (NS) equations

3.1 Historical view

Starting point: The linear theory. Let Ω be a bounded domain with smooth boundary in $\mathbb{R}^N$, $N = 2, 3$. Consider

$$\begin{cases} \partial_t u - \Delta u + \nabla P = 0 \\ \nabla \cdot u = 0 \end{cases} \quad \text{in } [0, T] \times \Omega. \tag{3}$$

To write down the weak formulation, we need the right functional space for the "test-vector fields" on Ω. Let

$\Lambda^1_{c,\sigma}(\Omega) =$ the space of al smooth, compactly supported divergence free v.f on Ω,

$$V(\Omega) = \overline{\Lambda^1_{c,\sigma}}^{\|\cdot\|_{H^1(\Omega)}}, \text{ and } H_\Omega = \overline{\Lambda^1_{c,\sigma}}^{L^2(\Omega)}.$$

If Ω is bounded, $\|\varphi\|_{H^1_0(\Omega)} = \|\varphi\|_V = \|\nabla\varphi\|_{L^2(\Omega)}$ since $\|\varphi\|_{L^2(\Omega)} \le C(\Omega)\|\nabla\varphi\|_{L^2(\Omega)}$. If $\Omega = \mathbb{R}^N$, $\|\varphi\|_V = \|\varphi\|_{L^2(\mathbb{R}^N)} + \|\nabla\varphi\|_{L^2(\mathbb{R}^N)}$.

Let $\widetilde{V}(\Omega) = \{u \in H^1_0(\Omega) | \nabla \cdot u = 0\}$, then $V(\Omega) \subset \widetilde{V}(\Omega)$. Are they the same ? If Ω is either $\mathbb{R}^N$, $\mathbb{R}^N_+$ or any bounded domain with smooth boundary in $\mathbb{R}^N$ for $N = 2, 3$ (usual domains), then $V(\Omega) = \widetilde{V}(\Omega)$. The example when $V(\Omega) \subsetneq \widetilde{V}(\Omega)$ is to take

$$\Omega = \mathbb{R}^3 - \{\text{this } (x_1, x_3)\text{-plane with a disc removed}\}.$$

Heywood (1970) proved $\widetilde{V}(\Omega) =$ finite dimesional space $\oplus V_\Omega$, which has something to do non-uniqueness, in general situation.

Theorem 3.1. *Let Ω be a "usual domain" in $\mathbb{R}^N (N = 2, 3)$. There exists a unique element $u \in C^0([0, T]; H_\Omega) \cap L^2(0, T; V_\Omega)$ which satisfies*

A. $\partial_t v \in L^2(0, T; V'_\Omega)$

B. For a.e. $t \in [0, T]$, and $\varphi \in V_\Omega$, we have

$$\langle \partial_t v|_t, \varphi\rangle_{V'_\Omega \otimes V_\Omega} + \int_\Omega \nabla v|_t : \nabla\varphi = (f(t), \varphi)_{V'_\Omega \otimes V_\Omega} \ \textit{(weak formulation of (3))},$$

C. $u(0) = a$.

The following properties are fundamental fact to further build the theory of weak solutions to NS equations. Why do we care about f ? Back to the original NS equations

$$\begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla P = 0 \\ \nabla \cdot u = 0 \end{cases} \quad \text{in } [0, T] \times \Omega \tag{4}$$

Weak formulation for (4). For a.e. $t \in [0, T]$, $\forall \varphi \in V_\Omega$, we have

$$\begin{aligned} \langle \partial_t u, \varphi\rangle_{V'_\Omega \otimes V_\Omega} + \int_\Omega \nabla u : \nabla\varphi &= -\int_\Omega (u \cdot \nabla u) \cdot \varphi \\ &= \int_\Omega (u \otimes u) : \nabla\varphi = \int_\Omega u_\alpha u_\beta \partial_\alpha \varphi_\beta. \end{aligned}$$

Reference: Seregin, Lecture notes on Regularity theory to NS equations, Oxford (Ch5) and Leray-Schauder fixed point method Twice. Big Glory (Leray-Hopf, 1930-1940).

Theorem 3.2. *Take $a \in H_{\mathbb{R}^N}$. There exists $u \in L^\infty(0,T;H_{\mathbb{R}^N}) \cap L^2(0,T;V_\Omega)$ which satisfies*

D. For any $\varphi \in L^2(\mathbb{R}^N)$, the function $t \to \int_{\mathbb{R}^N} u(t)\varphi$ is continuous,

E. $\partial_t u \in L^1(0,T;V'_{\mathbb{R}^N})$

F. For a.e. $t \in [0,T]$, and any $\varphi \in V_{\mathbb{R}^N}$, we have

$$\langle \partial_t u, \varphi\rangle + \int_{\mathbb{R}^N} \nabla u : \nabla\varphi = \int_{\mathbb{R}^N} (u \otimes u) : \nabla\varphi,$$

G.$\|u(t) - a\|_{L^2(\mathbb{R}^N)} \to 0$ as $t \to 0$.

H.For a.e. $t \in [0,T]$, we have

$$\frac{1}{2}\|u(t)\|^2_{L^2(\mathbb{R}^N)} + \int_0^T \int_{\mathbb{R}^N} |\nabla u(t,x)|^2 dxdt \le \frac{1}{2}\|a\|^2_{L^2(\mathbb{R}^N)}.$$

The result is not easy by any standard method. Teman (abstract): either bounded or $\mathbb{R}^N$. Seregin (concrete): bounded domain with smooth boundary. Moreover, in 2-dimesional space, the condition H will be an equality, and note that we only discuss the *partial regularity* properties for higher dimensional ($N \ge 3$) spaces.

Recall in the Evans' book, we have $\partial_t f \in L^2(0,T;H^{-1}(\Omega))$, $f \in L^2(0,T;H^1_0(\Omega))$, $\frac{d}{dt}|f|^2 = \frac{1}{2}\langle \partial_t f(t), f(t)\rangle_{H^{-1}(\Omega)\otimes H^1_0(\Omega)}$. If we have $\partial_t u \in L^2(0,T;V^1_{\mathbb{R}^N})$, $u \in L^2(0,T;V_{\mathbb{R}^N})$ ($u \in C^0([0,T];H_{\mathbb{R}^N})$), $V_{\mathbb{R}^N} \subset H_{\mathbb{R}^N} \subset V'_{\mathbb{R}^N}$ and

$$\|u(t)\|^2_{L^2(\mathbb{R}^N)} - \|u(0)\|^2_{L^2(\mathbb{R}^N)} = \int_0^T \langle \partial_t u|_T, u|_T\rangle_{V'_{\mathbb{R}^N}\otimes V_{\mathbb{R}^N}} d\tau.$$

Look at (4) and define a projection $\mathbb{P} : L^2(\mathbb{R}^N) \to H_{\mathbb{R}^N}$, $L^2(\mathbb{R}^N) = H_{\mathbb{R}^N} \oplus \{dF \in L^2(\mathbb{R}^N) : F \in W^{1,2}_{loc}(\mathbb{R}^N)\}$.

$$|\langle -\Delta u, \varphi\rangle| = |\int_{\mathbb{R}^N} \nabla u : \nabla\varphi| \le \|\nabla u\|_{L^2(\mathbb{R}^N)}\|\varphi\|_{V_{\mathbb{R}^N}},\ \forall\varphi \in V_{\mathbb{R}^N},$$

which implies

$$\|(-\Delta)u|_t\|_{V'_{\mathbb{R}^N}} \le \|\nabla u|_t\|_{L^2(\mathbb{R}^N)}$$

and

$$\|(-\Delta)u\|_{L^2(0,T;V'_{\mathbb{R}^N})} \le \|u\|_{L^2(0,T;V_{\mathbb{R}^N})} < \infty.$$

Moreover, in 2D, we have

$$\begin{aligned} |\langle u \cdot \nabla u|_t, \varphi\rangle &= |\int_{\mathbb{R}^2} u_\alpha u_\beta \partial_\alpha \varphi_\beta| \\ &\le \int_{\mathbb{R}^2} |u|^2|\nabla\varphi| \le (\int_{\mathbb{R}^2} |u|^4)^{\frac{1}{2}}\|\varphi\|_{V_{\mathbb{R}^N}}. \end{aligned}$$

Note that we have used the Ladyzhenskaya's inequality, which states $\forall f \in H_0^1(\mathbb{R}^2)$,

$$\|f\|_{L^4(\mathbb{R}^2)} \leq C_0 \|f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \|\nabla f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}},$$

so we have

$$\|u\cdot\nabla u\|_{V'_{\mathbb{R}^N}} \leq \|u(t)\|_{L^4(\mathbb{R}^2)}^2 \leq C_0\|u(t)\|_{L^2(\mathbb{R}^2)}\|\nabla u(t)\|_{L^2(\mathbb{R}^2)},$$

and

$$\int_0^T \|u\cdot\nabla u\|_{V'_{\mathbb{R}^N}} dt \leq C_0 \int_0^T \|u(t)\|_{L^2(\mathbb{R}^2)}\|\nabla u(t)\|_{L^2(\mathbb{R}^2)}.$$

3.2 More weak solutions for the NS equations

In 1980's, suitable weak solutions which enable us to "localize" the NS equation.

Definition 3.3. (Version 1) A suitable weak solution to (NS) which arises from $a \in H^1_{\mathbb{R}^3}$ is an element $u \in L^\infty(0,T;H_{\mathbb{R}^3}) \cap L^2(0,T;V_{\mathbb{R}^3})$ which satisfies condition D to H, and for a.e. $t\in[0,T]$, $\varphi \in C_c^\infty((0,T]\times\mathbb{R}^3)$, we have

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{R}^3}\varphi(t,x)|u(t,x)|^2 + \int_0^t\int_{\mathbb{R}^3}|\nabla u(t,x)|^2\varphi(t,x)dxdy\\ \leq&\frac{1}{2}\int_0^t\int_{\mathbb{R}^3}(\partial_t\varphi+\Delta\varphi)|u|^2dxdt + \int_0^t\int_{\mathbb{R}^3}\nabla\varphi\cdot u(\frac{1}{2}|u|^2+P)dxdt.\end{aligned}$$

We do not know if each Leray-Hopf weak solutions will satisfy this definition.

Remark 3.4. It is a fact that for each initial data $a \in H_{\mathbb{R}^3}$, there is at least one suitable weak solution to (4) which arises from a.

Key lemma of the Partial Regularity theory.

Lemma 3.5. *(Vasseur) For each $p > 1$, there exists $\epsilon_0 = \epsilon_0(p) \in (0,1)$ such that for any suitable weak solution (u,P) in $[-1,1]\times B(1)$, we have the following implication: If*

$$\sup_{t\in[-1,1]}\|u(t)\|_{L^2(B(1))}^2 + \int_{-1}^1\int_{B(1)}|\nabla u|^2dxdt + (\int_{-1}^1[\int_{B(1)}|P|dx]^p dt)^{\frac{2}{p}} < \epsilon_0,$$

then

$$|u| \leq 1 \ in \ [-\frac{1}{2},1]\times B(\frac{1}{2}).$$

Remark 3.6. For CKN-version: If

$$\|u\|_{L^3([-1,1]\times B(1))} + \|P\|_{L^{\frac{3}{2}}([-1,1]\times(B(1))} < \epsilon_0,$$

then

$$|u| \leq 1 \text{ in } [-\frac{1}{2},1]\times B(1).$$

From the (4), take divergence on both sides, we have

$$-\Delta P = R_\alpha R_\beta\{u_\alpha u_\beta\},$$

where R_ℓ stands for the Riesz transformation. When $u \in L^3$, i.e., $|u|^2 \in L^{\frac{3}{2}}([-1,0]\times\mathbb{R}^3)$, then $P \in L^{\frac{3}{2}}([-1,0]\times\mathbb{R}^3)$.

Now, (u,P) satisfies the following: $\forall \varphi \in C_c^\infty([-1,1]\times B(1))$, a.e. $t\in(-1,1]$, then

$$\begin{aligned}&\frac{1}{2}\int_{B(1)}\varphi(t,\cdot)|u(t,\cdot)|^2+\int_{-1}^{t}\int_{B(1)}|\nabla u|^2\varphi\\&\le\frac{1}{2}\int_{-1}^{t}(\partial_t\varphi+\Delta\varphi)|u|^2+\int_{-1}^{t}\int_{B(1)}u\cdot\nabla\varphi\{\frac{1}{2}|u|^2+P\}.\end{aligned}\tag{5}$$

3.3 Do truncation to $|u|$

$v_k=[|u|-(1-\frac{1}{2^k})]_+$, $B_k=B(1+\frac{1}{2^{3k}})$, $B_{k-\frac{1}{3}}=B(1+\frac{2}{2^{3k}})$, $B_{k-\frac{2}{3}}=B(1+\frac{2^2}{2^{3k}})$. Take a bump function $\eta_k\in C_c^\infty(B_{k-\frac{1}{3}})$ such that

$$\begin{aligned}&\chi_{B_k}\le\eta_k\le\chi_{B_{k-\frac{1}{3}}},\\&|\nabla\eta_k|\le 2\cdot 2^{3k}\chi_{B_{k-\frac{1}{3}}},\\&|\nabla^2\eta_k|\le 4\cdot 2^{6k}\chi_{B_{k-\frac{1}{3}}}.\end{aligned}$$

Find the inequality satisfied by v_k, and (5) can be abstractly phrased as follows:

$$\partial_t\frac{|u|^2}{2}+|\nabla u|^2-\Delta(\frac{|u|^2}{2})+\frac{1}{2}\nabla\cdot\{u|u|^2\}+u\cdot\nabla P\le 0.$$

Note that $\partial_t v_k^2=\frac{v_k}{|u|}u\cdot\partial_t u$. You know somehow v_k should satisfy

$$\frac{1}{2}\partial_t v_k^2+D_k^2-\Delta(\frac{v_k^2}{2})+\frac{1}{2}\nabla\cdot\{u\cdot v_k^2\}+(\frac{v_k}{|u|}-1)u\cdot\nabla P+u\cdot\nabla P\le 0,$$

but this is not the right way to obtain the inequality. The problem shows up on the big part of v_k.

3.4 Look at $(\frac{v_k}{|u|}-1)u$

1. If $v_k>0$, then $(\frac{v_k}{|u|}-1)=-\frac{(1-\frac{1}{2^k})}{|u|}|u|$.
2. If $v_k(x)=0$, $|u|\le 1-\frac{1}{2^k}(\frac{v_k}{|u|}-1)u\in H^1(B(1))\cap L^\infty(B(1))$ and $|\frac{v_k}{|u|}-1|\cdot|u|\le 1-\frac{1}{2^k}<1$.

Note that the idea is $\frac{v_k}{|u|}$ and u themselves maybe worse, but combine them together and after canceling out, the term $(\frac{v_k}{|u|}-1)u\cdot\eta_k$ will be nice. We use for each $t\in[-1,1]$, $(\frac{v_k}{|u|}-1)u\cdot\eta_k$ to test against (4)

$$0=\left\langle(\partial_t u-\Delta u+u\cdot\nabla u+\nabla P)|_t,(\frac{v_k}{|u|}-1)u\cdot\eta_k\right\rangle_{H^{-1}(B(1))\otimes H_0^1(B(1))},$$

the weak formulation of (4) matters. The idea is let $\frac{1}{2}\nabla\cdot\{u|u|^2\}$ handle the large part of $|v_k|^2$ for us.

$$\left\langle u\cdot\nabla u, (\frac{v_k}{|u|}-1)u\eta_k\right\rangle = \int_{B(1)} u_\alpha\partial_\alpha u_\beta(\frac{v_k}{|u|}-1)u_\beta\cdot\eta_k$$
$$= \int_{B(1)} u_\alpha\cdot\partial\partial\{\frac{v_k^2-|u|^2}{2}\}\cdot\eta_k$$
$$= \int_{B(1)} \nabla\cdot\{\frac{u}{2}\cdot(v_k^2-|u|^2)\}\eta_k,$$

since

$$\partial_\alpha\{\frac{v_k^2-|u|^2}{2}\} = v_k\partial_\alpha v_k - u\cdot\nabla u = v_k\partial_\alpha|u| - u\cdot\nabla u$$
$$= v_k\frac{u}{|u|}\nabla u - u\cdot\nabla u = (\frac{v_k}{|u|}-1)u\cdot\nabla u,$$

$$|\nabla(\frac{{v_k}^2-|u|^2}{2}) \leq |\nabla u| \text{ and } \frac{v_k^2-|u|^2}{2}\in W^{1,2}(B(1)).$$

Now, look at Δu:

$$\left\langle -\Delta u, (\frac{v_k}{|u|}-1)u\cdot\eta_k\right\rangle_{H^1(B(1))\otimes H_0^1(B(1))} = \int_{B(1)}\nabla u:\nabla\{(\frac{v_k}{|u|}-1)u\cdot\eta_k\}$$
$$= \int_{B(1)}\nabla u:\nabla((\frac{v_k}{|u|}-1)u)\cdot\eta_k + \int_{B(1)}\nabla u:(\frac{v_k}{|u|}-1)u\nabla\eta_k := I_1+I_2.$$

For I_2,

$$I_2 = \int_{B(1)}\partial_\alpha u_\beta(\frac{v_k}{|u|}-1)u_\beta\partial_\alpha\eta_k = \int_{B(1)}\frac{1}{2}\partial_\alpha(v_k^2-|u|^2)\partial_\alpha\eta_k$$
$$= -\left\langle\Delta(\frac{v_k^2-|u|^2}{2}),\eta_k\right\rangle.$$

For I_1, note that $\nabla|u|^2 = 2u\cdot\nabla u = 2|u|\nabla|u|$

$$I_1 = \int_{B(1)}\nabla u:\nabla((\frac{v_k}{|u|}-1)u)\cdot\eta_k = \int_{B(1)}\nabla u:\nabla(\frac{v_k}{|u|}u) - |\nabla u|^2$$
$$= \int_{B(1)}\nabla u:\nabla(\frac{v_k}{|u|})\cdot u + \nabla u:\frac{v_k}{|u|}\nabla u - |\nabla u|^2$$
$$= \nabla u\nabla(\frac{v_k}{|u|})\cdot u + (\frac{v_k}{|u|}-1)|\nabla u|^2$$
$$= |u|\nabla|u|^2\cdot\frac{1-\frac{1}{2^k}}{|u|^2}\nabla|u|\chi_{\{v_k>0\}} + (\frac{v_k}{|u|}-1)|\nabla u|^2$$
$$= [\frac{1-\frac{1}{2^k}}{|u|}\nabla|u|^2\chi_{\{v_k>0\}} + (\frac{v_k}{|u|}-1)|\nabla u|^2].$$

Note that $v_k = [|u|-(1-\frac{1}{2^k})]_+$ and $D_k^2 = \frac{v_k}{|u|}|\nabla|u||^2\chi_{\{v_k>0\}} + \frac{v_k}{|u|}|\nabla u|^2$. $u = (1-\frac{v_k}{|u|})u + \frac{v_k}{|u|}u$ and $|u|\frac{v_k^2}{2} \leq (1-\frac{1}{2^k})\frac{v_k^2}{2} + \frac{v_k^3}{2}$.

3.5 Short summary

$$\partial_t(\frac{v_k^2-|u|^2}{2})+\nabla\cdot(\frac{u}{2}(v_k^2-|u|^2))+D_k^2-|\nabla u|^2+(\frac{v_k}{|u|}-1)u\cdot\nabla P\equiv 0 \quad (6)$$

hold in $\mathcal{D}'$, whichever

$$\partial_t(\frac{|u|^2}{2})+|\nabla u|^2-\frac{1}{2}|\Delta(|u|^2)+\nabla\cdot(\frac{1}{2}u|u|^2)+u\cdot\nabla P\le 0 \quad (7)$$

which is (5) to test against nonnegative test functions. From (6) and (7), we obtain

$$\partial_t(\frac{v_k^2}{2})+D_k^2-\Delta(\frac{v_k^2}{2})+\nabla\cdot(\frac{u}{2}-v_k^2)+(\frac{v_k}{|u|}-1)u\cdot\nabla P+u\cdot\nabla P\le 0. \quad (8)$$

Define

$$U_k:=\sup_{T_k\le t\le 1}\|v_k(t)\|^2_{L^2(B_k)}+\int_{T_k}^1\int_{B_k}D_k^2,$$

by choosing $T_k=-\frac{1}{2}(1+\frac{1}{2^k})$ and derive

$$U_k\le C_0^k[1+\|P\|_{L_t^pL_x^1}]U_{k-1}^\beta$$

to imply the first DeGiorgi lemma holding.

For $\varphi\ge 0\in C_c^\infty((-1,1)\times B(1))$ works for (7) but not for (8). For $t>0$,

$$\begin{aligned}&\frac{1}{2}\int_{B(1)}v_k^2(t,\cdot)\varphi(t,\cdot)+\int_{-1}^1\int_{B(1)}d_k^2\varphi\\ \le&\int_{-1}^1\int_{B(1)}(\partial_t\varphi+\Delta\varphi)v_k^2+\int_{-1}^1\int_{B(1)}\frac{u}{2}v_k^2\cdot\nabla\varphi\\ &+\int_{-1}^1\int_{B(1)}[(\frac{v_k}{|u|}-1)u\cdot\nabla P+u\nabla P]\cdot\varphi dt.\end{aligned}$$

In the final step, let $\psi_\epsilon(t)$ be a bump function with support in $(\sigma-\epsilon,s+\epsilon)$ $(\sigma<0,s>0)$ and $\psi_\epsilon(t)=1$ in $(\sigma+\epsilon,s-\epsilon)$, and connect 1 and 0 by straight lines. Plug the test function $\varphi=\psi_\epsilon(t)\cdot\eta_k(x)$ into previous calculations, then we have

$$\begin{aligned}&\int_{\sigma-\epsilon}^{\sigma+\epsilon}\int_{B(1)}\psi_\epsilon(t)\eta_k(x)D_k^2(t,x)dxd\\ \le&\int_{-1}^1\psi'_\epsilon(t)\cdot\int_{B(1)}\eta_k(x)v_k^2(t,x)dxdy+\int_{-1}^1\int_{B(1)}\psi_\epsilon(t)\Delta\eta_k(x)v_k^2(t,x)dxdt\\ &+\int_{\sigma-\epsilon}^{\sigma+\epsilon}\psi_\epsilon(t)\int_{B_{k-\frac{1}{3}}}\frac{u}{2}v_k^2\nabla\eta_k+\int_{\sigma-\epsilon}^{s+\epsilon}\int_{B_{k-\frac{1}{3}}}[(\frac{v_k}{|u|}-1)u\cdot\nabla P+u\cdot\nabla P]\psi_\epsilon(t)\eta_k(x),\end{aligned}$$

and

$$\begin{aligned}&\int_{-1}^1\psi'_\epsilon(t)\cdot\int_{B(1)}\eta_k(x)v_k^2(t,x)dxdt\\ =&-\frac{1}{2\epsilon}\int_{s-\epsilon}^{s+\epsilon}\int_{B(1)}\eta_k(x)u_k^2(t,x)dxdt+\frac{1}{2\epsilon}\int_{\sigma-\epsilon}^{\sigma+\epsilon}\int_{B(1)}\eta_k(x)v_k^2dxdt.\end{aligned}$$

Until now, we have

$$\partial_t \frac{v_k^2}{2} + D_k^2 - \Delta(\frac{v_k^2}{2}) + \nabla\cdot(u\frac{v_k^2}{2}) + (\frac{v_k}{|u|} - 1)u\cdot\nabla P + \nabla\cdot(uP) \le 0 \text{ in } (-1,1)\times B(1).$$

Select any $-1 < \sigma < t < 1$,

$$\begin{aligned}
&\int_{B(1)} \eta_k |v_k(t,\cdot)|^2 + \int_\sigma^t \int_{B(1)} D_k^2 \cdot \eta_k dx dt \\
\le &\int_{B(1)} \eta_k |v_k(\sigma,\cdot)|^2 + \int_\sigma^t \frac{v_k^2}{2}\Delta\eta_k + \int_\sigma^t \int_{B(1)} u\cdot\frac{v_k^2}{2}\nabla\eta_k \\
&+ \int_\sigma^t |\int_{B(1)} \eta_k\{(\frac{v_k}{|u|} - 1)u\cdot\nabla P + \nabla\cdot(uP)\}dt.
\end{aligned}$$

Fix $t \in [T_k, 1]$, take average of the above inequality over $\sigma \in [T_{k-1}, T_k]$,

$$\begin{aligned}
&\int_{B(1)} \eta_k |v_k(t)|^2 + \int_{T_k}^t \int_{B(1)} D_k^2 \eta_k \le \text{Aerage of LHS} \le \text{Average of LHS} \\
\le &\frac{1}{T_k - T_{k-1}} \int_{T_{k-1}}^{T_k} \eta_k |v_k(\sigma,\cdot)|^2 d\sigma + \int_{T_{k-1}}^t \int_{B(1)} \frac{v_k^2}{2}|\Delta\eta_k| + |u|\frac{v_k^2}{2}|\nabla\eta_k| \\
&+ \int_{T_{k-1}}^t |\int_{B(1)} \eta_k\{(\frac{v_k}{|u|} - 1)u\cdot\nabla P + \nabla\cdot(uP)|d\tau.
\end{aligned}$$

Use $|\nabla\eta_k| \le 2\cdot 2^{3k}$ and $|\nabla^2 u_k| \le 42^{6k}$, then

$$\begin{aligned}
\frac{1}{2}U_k \le &\sup_{T_k\le t\le 1} \int_{B(1)} \eta_k \frac{|v_k(t)|^2}{2} + \int_{T_k}^1 \int_{B(1)} D_k^2 \eta_k \\
\le &2^k \int_{T_k}^1 \int_{B_{K-\frac{1}{3}}} |v_k|^2 + \int_{Q_{k-1}} (4\cdot 2^{\sigma k})[|v_k|^2 + |u|\frac{v_k^2}{2}] \\
&+ \int_{T_{k-1}}^t |\int_{B(1)} \eta_k\{(\frac{v_k}{|u|} - 1)u\cdot\nabla P + \nabla\cdot(uP)|d\tau.
\end{aligned}$$

$u \in L^\infty(-1,1;L^2(B(1)) \cap L^2(-1,1;H^1(B(1))$, $v_k \in L^\infty(T_k,1;L^2(B_k))$. Note that $L^2(T_k,1;H^1(B(1)) \subset L^2(T_k,1;L^6(B(1))$.

$$\|v_k\|_{L^6(B_k)} \le C_0\{\|v_k\|_{L^2(B_k)} + \|\nabla v_k\|_{L^2(B_k)}\},\ B(\frac{1}{2}) \subset B_k \subset B(1).$$

For $\theta \in (0,1)$, $\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{\infty} = \frac{\theta}{6} + \frac{1-\theta}{2}$, implying $p = \frac{10}{3}$. Therefore,

$$\|v_k\|_{L^{\frac{10}{3}}(Q_k)} \le \|v_k\|^{1-\theta}_{L^\infty(T_k,1;L^2(B_k)} \cdot \|v_k\|^\theta_{L^2(T_k,1;H^1(B_k))} \le U_k^{\frac{1}{2}}.$$

4 Deal with the pressure

Motivation: For Leray Hopf weak solution in $\mathbb{R}^3$, $-\Delta P = \partial_\alpha\partial_\beta\{u_\alpha u_\beta\}$, $u \in L^\infty(-1,1;L^2(\mathbb{R}^3))$, $|u|(t,\cdot) \in L^1(\mathbb{R}^3)$ and $|u|(t,\cdot) \in L^6(\mathbb{R}^3)$.

Lemma 4.1. *Suppose we have $P \in L^p(T_{k-1}, 1; L^2(B_{k-1}))$ and some symmetric matrix $(G_{\alpha\beta}) \in L^\infty(T_{k-1}, 1; L^1(B_{k-1}))$ such that $(-\Delta)P = \partial_\alpha\partial_\beta\{G_{\alpha\beta}\}$ (Not the whole space setting). Then*

$$P|_{B_{k-\frac{1}{3}}} = P_{k1} + P_{k2},$$

where P_{k1} is the nonlocal part and $P_{k2} = (-\Delta)^{-1}\partial_\alpha\partial_\beta[G_{\alpha\beta} \cdot \varphi_k]$ is the local part. Moreover,

$$\|P_{k1}\|_{L^p(T_{k-1},1;L^\infty(B_{k-\frac{1}{3}}))} + \|\nabla P_{k1}\|_{L^p(T_{k-1},1;L^\infty(B_{k-\frac{1}{3}}))}.$$

Observe that

$$\begin{aligned} -\Delta(\varphi_k P) &= -\Delta\varphi_k \cdot P - 2\nabla\varphi_k \cdot \nabla P - \Delta P \cdot \varphi_k \\ &= -P\Delta\varphi_k - 2\nabla \cdot \{P\nabla\varphi_k\} + 2\varphi_k\Delta P - \varphi_k\Delta P \\ &= -P\Delta\varphi_k - 2\nabla \cdot \{P\nabla\varphi_k\} + \varphi_k\Delta P \end{aligned}$$

and

$$\begin{aligned} -\varphi_k\Delta P &= \varphi_k\partial_\alpha\partial_\beta\{G_{\alpha\beta}\} \\ &= \partial_\alpha\{\varphi_k\partial_\beta G_{\alpha\beta}\} - \partial_\alpha\varphi_k \cdot \partial_\beta G_{\alpha\beta} \\ &= \partial_\alpha\partial_\beta[\varphi_k G_{\alpha\beta}] - \partial_\alpha\{\partial_\beta\varphi_k \cdot G_{\alpha\beta}\} - \partial_\alpha\varphi_k\partial_\beta G_{\alpha\beta} \\ &= (-\Delta)P_{k2} - \partial_{\alpha\beta}\varphi_k \cdot G_{\alpha\beta} - 2\partial_\beta\varphi_k \cdot \partial_\alpha G_{\alpha\beta}, \end{aligned}$$

or

$$\begin{aligned} (-\Delta)[\varphi_k P] = (-\Delta)P_{k2} - \partial_{\alpha\beta}\varphi_k \cdot G_{\alpha\beta} - 2\partial_\beta\varphi_k \cdot \partial_\alpha G_{\alpha\beta} \\ - \Delta\varphi_k \cdot P - 2\nabla \cdot \{P \cdot \nabla\varphi_k\}. \end{aligned}$$

Further,

$$\begin{aligned} (-\Delta)P_{k1} = \frac{1}{4\pi}\frac{1}{|x|} * \{-\partial_{\alpha\beta}\varphi_k \cdot G_{\alpha\beta} - 2\partial_\beta\varphi_k\partial_\alpha G_{\alpha\beta} \\ - \Delta\varphi_k \cdot P - 2\nabla \cdot \{P\nabla\varphi_k\}. \end{aligned}$$

Since $\text{supp}\nabla\varphi_k$, $\text{supp}\nabla^2\varphi_k \subset B_{k-1}\backslash B_{k-\frac{2}{3}}$, $(-\Delta)P_{k1} = 0$ in $B_{k-\frac{2}{3}}$, we estimate $\|P_{k1}\|_{L^\infty(B_{k-\frac{1}{3}})}$ and $\|\nabla P_{k1}\|_{L^\infty(B_{k-\frac{1}{3}})}$.

For $x \in B_{k-\frac{1}{3}}$,

$$\begin{aligned} \int_{\mathbb{R}^3} \frac{1}{|x-y|}\nabla \cdot \{P\nabla\varphi_k\}|_y dy &= \int_{y \in B_k - B_{k-\frac{2}{3}}} \frac{1}{|x-y|}\nabla_y \cdot \{P\nabla\varphi_k\}dy \\ &= -\int_{B_{k-1}\backslash B_{k-\frac{2}{3}}} \frac{1}{|y-x|^2}(\frac{y-x}{|y-x|})P(y)\nabla\varphi_k|_y dy \\ &\leq 2 \cdot 2^{6k} \cdot 2^{3k}\int_{B_{k-1}} |P|. \end{aligned}$$

The conclusion is that

$$\begin{aligned} &\|\nabla P_{k1}(t,\cdot)\|_{L^\infty(B_{k-\frac{1}{3}})} + \|P_{k1}(t,\cdot)\|_{L^\infty(B_{k-\frac{1}{3}})} \\ \leq &C_0 2^{12k}\int_{B(1)} \{|\{P(t,\cdot)| + |G|\}. \end{aligned}$$

Lemma 4.2. *Let $p > 1$, consider $P \in L^p(T_{k-1}, 1; L^1(B_{k-1}))$ and $(G_{\alpha\beta}) \in L^\infty(T_{k-1}, 1; L^1(B_{k-1}))$ such that $(-\Delta P) = \partial_{\alpha\beta} G_{\alpha\beta}$. Then*

$$P|_{B_{k-\frac{1}{3}}} = P_{k1} + P_{k2},$$

where $P_{k2} = (-\Delta)\partial_{\alpha\beta}(G_{\alpha\beta}\varphi_k)$ and P_{k1} satisfies the following

$$(-\Delta)P_{k1} = 0 \text{ in } [T_{k-1}, 1] \times B_{k-\frac{2}{3}}$$

and

$$\begin{aligned} &\|P_{k1}\|_{L^p(T_{k-1},1;L^\infty(B_{k-\frac{1}{3}}))} + \|\nabla P_{k1}\|_{L^p(T_{k-1},1;L^\infty(B_{k-\frac{1}{3}}))} \\ \le & C_0 2^{24k} \{\|P\|_{L^p(T_{k-1},1;L^1(B_{k-1}))} + \|G\|_{L^p(T_{k-1},1;L^1(B_{k-1}))}\}. \end{aligned}$$

The details of proof, we refer to Seregin's big paper and Sverak's work.