# The development of the enclosure method in an anisotropic background and the strong unique continuation for the elasticity with residual stress

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# Outline

1. The Enclosure Method (introduced by Ikehata)
   - Applied to Anisotropic Maxwell system

2. The Strong Unique Continuation Property
   - For the Residual Stress System with Gevrey Coefficients

# Part 1: The enclosure method for the anisotropic Maxwell system

**What is the enclosure method ?**

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Applications: Nondestructive/Noninvasive testing, Sonar, Radar, EIT, etc.

# Inverse obstacle problems

**Discontinuity** in a medium affects propagation of various physical quantity (signal) in a medium.

Cavity, **Inclusion**, Crack, Obstacle...etc
(the interface of jump discontinuity of the medium)
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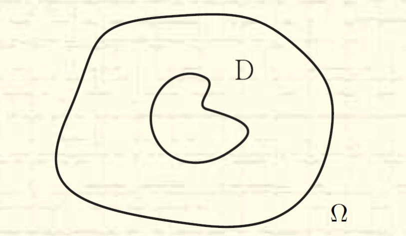
# Problem description

We consider unknown obstacles in electromagnetic fields with anisotropic medium lies in $\Omega$.
Let $\Omega$ be a bounded domain in $\mathbb{R}^3$. $D$ is an inclusion in $\Omega$.




**Problem**: How to find $D$?

# Various methods

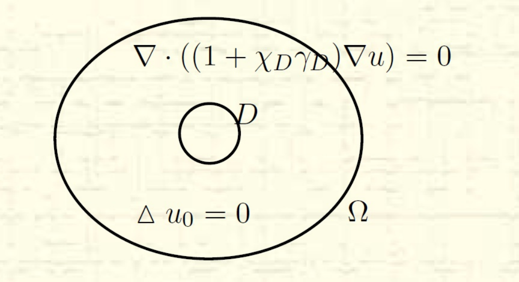
There are several methods to retrieve the information of obstacles $D$ inside $\Omega$.

1. Probe Method (Ikehata)
2. Enclosure Method (Ikehata)
3. Linear Sampling Method (Colton-Kirsch)
4. Factorization Method (Kirsch)
5. Singular Source Method (Potthast)

We focus on the **enclosure method** to find the unknown inclusions.

# The enclosure method

In order to understand ideas, here we consider the simplest case: $\Delta u_0 = 0$ in $\Omega$.




**Goal**: Reconstruct unknown $D$ by the boundary measurements.

Consider the function $u$ satisfying

$$\nabla \cdot ((1 + \gamma_D \chi_D)\nabla u) = 0 \text{ in } \Omega.$$

The corresponding **Dirichlet-to-Neumann** (DN) map is given by

$$\Lambda_D(u|_{\partial\Omega}) = \frac{\partial u}{\partial \nu}|_{\partial\Omega},$$

where $\nu$ is the unit outer normal on $\partial\Omega$.

# Ideas

Key ideas in the enclosure type method:

(A) Let $\Lambda_\emptyset$ be the DN map when $D = \emptyset$ and $\Lambda_\emptyset(u_0|_{\partial\Omega}) = \frac{\partial u_0}{\partial \nu}|_{\partial\Omega}$.

By energy method, one can show the following *energy integral*

$$\int_{\partial\Omega} (\Lambda_D - \Lambda_\emptyset) f \cdot \bar{f}\, dS \approx \int_{D} |\nabla u_0|^2 dx \tag{1.1}$$

where $u_0$ is the solution of the unperturbed equations (without $D$, $\Delta u_0 = 0$ in $\Omega$).

(1.1) is true due to the positivity of the equation.

# Special solution

(B) One can find solutions of the Laplace equation in the following form

$$u_{0,d,h} = e^{\frac{1}{h}[\omega\cdot x - d + i\omega^{\perp}\cdot x]}.$$

This is the **complex geometrical optics** (CGO) solution of the Laplace equation.

Behavior of $u_{0,h}$:

$$\begin{cases} u_{0,d,h} \downarrow 0 \text{ as } h \to 0^{+} & \text{for } \omega\cdot x < d, \\ u_{o,d,h} \uparrow \infty \text{ as } h \to 0^{+} & \text{for } \omega\cdot x > d. \end{cases}$$

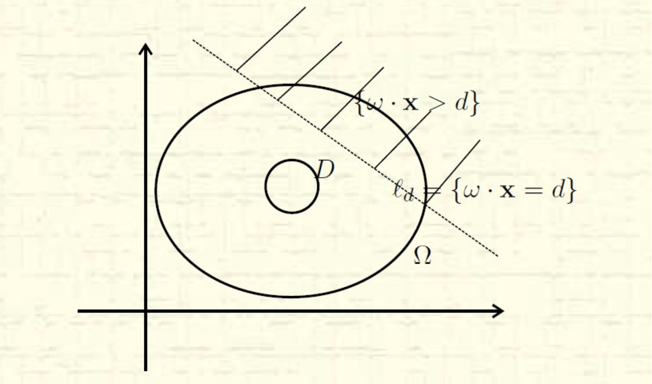
# Indicator function

From the *energy integral*, we can define the **indicator function** $I$: for any $f \in H^{1/2}(\partial\Omega)$,

$$I(f) = \int_{\partial\Omega} (\Lambda_D - \Lambda_\emptyset) f \cdot \bar{f} \, dS,$$

$I(f)$ is completely determined by the boundary measurements. We can take $f := f_{d,h} = u_{0,d,h}|_{\partial\Omega}$ into the indicator functional $I = I(d,h)$.

If $D \cap \{\omega \cdot x > d\} = \emptyset$, then $I(d,h) \to 0$ as $h \to 0^+$.



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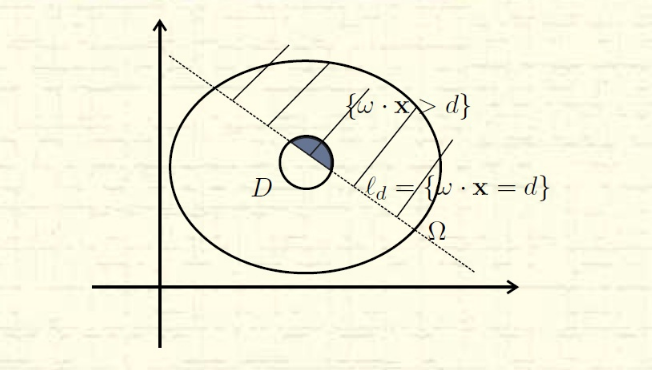


Change the direction $\omega$ and move the level set $\{x \cdot \omega = d\}$, we can **enclose** the unknown obstacle $D$.

The enclosure reconstruction method consists of: **CGO solutions** and **Energy integral**.

# Anisotropic Maxwell system

This is a joint work with Rulin Kuan and Mourad Sini.

Let $D$ be an unknown obstacle and let $k > 0$ be the wave number. We consider the anisotropic Maxwell system

$$\begin{cases} \nabla \times E - ik\mu H = 0 & \text{in } \Omega, \\ \nabla \times H + ik\varepsilon E = 0 & \text{in } \Omega, \\ \nu \times E = f & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

# Description of the problem

For the inclusion case, the coefficients $\varepsilon(x) = \varepsilon_0(x) - \varepsilon_D(x)\chi_D$, where $\varepsilon_0(x)$ to be a $C^\infty$ positive definite matrix-valued function, $\varepsilon_D(x)$ is a matrix-valued function which is regarded as a perturbation in the unknown obstacle $D$. $\mu(x) > 0$ is a $C^\infty$ scalar function, and $\nu$ is the unit outer normal on $\partial\Omega$
As before, $D$ is an inclusion.

Assume $k$ is not an eigenvalue for the spectral problem to (2.1), then (2.1) is well posed.

**Impedance Map**: We define the impedance map $\Lambda_D : TH^{-\frac{1}{2}}(\partial\Omega) \to TH^{-\frac{1}{2}}(\partial\Omega)$ by

$$\Lambda_D(\nu \times H|_{\partial\Omega}) = (\nu \times E|_{\partial\Omega}),$$

where $TH^{-\frac{1}{2}}(\partial\Omega) := \{f \in H^{-\frac{1}{2}}(\partial\Omega) | \nu \cdot f = 0\}$ and $\times$ is the standard cross product in $\mathbb{R}^3$.

We denote by $\Lambda_\emptyset$ the impedance map for the domain without an obstacle.

**Inverse Problem**: Find $D$ from $\Lambda_D$.

# Difficulties

For the anisotropic Maxwell's system, until now, there are **NO** CGO solutions. Similar to anisotropic elliptic case, we will construct new special solutions: The **oscillating-decaying** (OD) solutions.

The OD solutions were first constructed by G. Nakamura, G. Uhlmann, J. N. Wang, 2005.

# Construction OD solutions

Consider the anisotropic Maxwell system

$$\begin{cases} \nabla \times E - ik\mu H = 0 & \text{in } \Omega, \\ \nabla \times H + ik\varepsilon E = 0 & \text{in } \Omega, \end{cases} \tag{2.2}$$

where $\varepsilon$ is a matrix and $\mu$ is a scalar and we want to construct OD solutions.

The first step of constructing OD solutions is to reduce the **anisotropic Maxwell system** to a **strongly elliptic system**.

# From Maxwell to Elliptic

If $E$ and $H$ are of the following forms

$$\begin{cases} E = -\frac{i}{k}\varepsilon^{-1}\nabla\times(\mu^{-1}(\nabla\times B)) - \varepsilon^{-1}(\nabla\times A) \\ H = \frac{i}{k}\mu^{-1}\nabla\times(\varepsilon^{-1}(\nabla\times A)) - \mu^{-1}(\nabla\times B) \end{cases}$$

with $A, B$ satisfying the **strongly elliptic systems**

$$\begin{cases} L_A(A) = \mu\nabla tr(M^A\nabla A) - \nabla\times(\varepsilon^{-1}(\nabla\times A)) + k^2\mu A = 0 \\ L_B(B) = \varepsilon\nabla tr(M^B\nabla B) - \nabla\times(\mu^{-1}(\nabla\times B)) + k^2\varepsilon B = 0 \end{cases},$$

where $M^A = m\mu^{-1}I_3$ and $M^B = m\mu^{-1}\varepsilon$ for arbitrary positive constant $m$. Then $(E, H)$ satisfies (2.2).

# Oscillating decaying solutions for Elliptic

Then we represent $A$ and $B$ to be two oscillating-decaying solution in the following form: Let $\omega$ be a unit vector, then

$$\begin{cases} A = w^A_{\chi_t,b,t,N,\omega} + r^A_{\chi_t,b,t,N,\omega} \text{ in } \Omega_t(\omega) := \Omega \cap \{x\cdot\omega > t\}, \\ w^A_{\chi_t,b,t,N,\omega} = \chi_t(x')Q_t e^{i\tau x\cdot\xi} e^{-\tau(x\cdot\omega - t)A^A_t(x')} b + \gamma^A_{\chi_t,b,t,N,\omega}(x,\tau), \\ B = w^B_{\chi_t,b,t,N,\omega} + r^B_{\chi_t,b,t,N,\omega} \text{ in } \Omega_t(\omega), \\ w^B_{\chi_t,b,t,N,\omega} = \chi_t(x')Q_t e^{i\tau x\cdot\xi} e^{-\tau(x\cdot\omega - t)A^B_t(x')} b + \gamma^B_{\chi_t,b,t,N,\omega}(x,\tau) \end{cases}$$

with suitable decay in $\tau$ for $\gamma^A, \gamma^B, r^A$ and $r^B$.

# Oscillating decaying solutions for Maxwell

Via the relations between $E, H, A, B$, for the inclusion obstacle case, we have

$$\begin{cases} E_t = & F_A^1(x) e^{i\tau x\cdot\xi} e^{-\tau(x\cdot\omega - t)A_t^A(x')} b \\ & +\Gamma^{A,1}_{\chi_t,b,t,N,\omega} + r^{A,1}_{\chi_t,b,t,N,\omega} \\ H_t = & F_A^2(x) e^{i\tau x\cdot\xi} e^{-\tau(x\cdot\omega - t)A_t^A(x')} b \\ & +\Gamma^{A,2}_{\chi_t,b,t,N,\omega} + r^{A,2}_{\chi_t,b,t,N,\omega} \end{cases} \text{ in } \Omega_t(\omega),$$

where $F_A^1(x) = O(\tau)$, $F_A^2(x) = O(\tau^2)$ are some smooth functions and for $|\alpha| = j$, $j = 1, 2$, we have

$$\begin{cases} \|\Gamma^{A,j}_{\chi_t,b,t,N,\omega}(x,\tau)\|_{L^2(\Omega_t(\omega))} \le c\tau^{|\alpha|-3/2} e^{-\tau(s-t)a_A}, \\ \|r^{A,j}_{\chi_t,b,t,N,\omega}(x,\tau)\|_{L^2(\Omega_t(\omega))} \le c\tau^{j-N+1/2}, \end{cases}$$

for some positive constants $a_A$ and $c$.

# Energy integral

The impedance maps $\Lambda_D : \nu \times H|_{\partial\Omega} \to \nu \times E|_{\partial\Omega}$ and $\Lambda_0 : \nu \times H_0|_{\partial\Omega} \to \nu \times E_0|_{\partial\Omega}$. By energy method, one can show that

$$\int_{\partial\Omega} (\nu \times H_0) \cdot (\overline{(\Lambda_D - \Lambda_\emptyset)(\nu \times H_0)} \times \nu) dS \approx \int_D |\nabla \times H_0|^2 dx.$$

However, the OD solutions are not well-define on the whole domain $\Omega$, we cannot obtain the full boundary information from the OD solutions.

# Runge approximation property

In fact, we can find a sequence of solutions solving the anisotropic Maxwell's system and they will approximate to the OD solution. This property is called the **Runge approximation property**. If we set $u = \begin{pmatrix} H \\ E \end{pmatrix}$ and

$$L := i \begin{pmatrix} \varepsilon^{-1} & 0 \\ 0 & \mu^{-1} I_3 \end{pmatrix} \begin{pmatrix} 0 & \nabla\times \\ -\nabla\times & 0 \end{pmatrix} + k I_6,$$

then we have

$$Lu = 0,$$

where $I_j$ means $j \times j$ identity matrix for $j = 3, 6$.

## Theorem (Runge approximation property)

*Let $D$ and $\Omega$ be two open bounded domains with $C^\infty$ boundary in $\mathbb{R}^3$ with $D \Subset \Omega$. If $u \in (H(curl, D))^2$ satisfies*

$$Lu = 0 \text{ in } D.$$

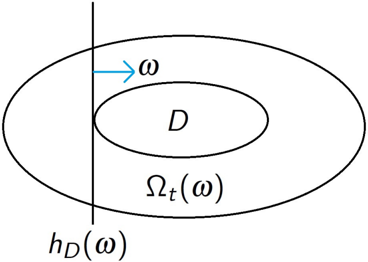
*Given any compact subset $K \subset D$ and any $\varepsilon > 0$, there exists $U \in (H(curl, \Omega))^2$ such that*

$$LU = 0 \text{ in } \Omega,$$

*and $\|U - u\|_{H(curl,K)} < \varepsilon$, where $\|f\|_{H(curl,K)} = \left(\|f\|_{L^2(K)} + \|curl f\|_{L^2(K)}\right)$.*

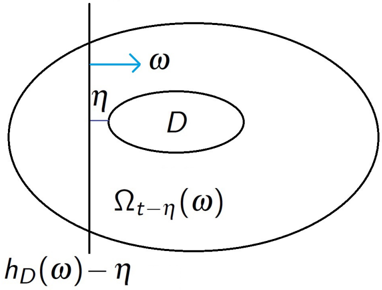
# Technical part

**Support function**: For $\omega \in \mathbb{S}^2$, we define the support function of $D$ by $h_D(\omega) = \inf_{x \in D} x \cdot \omega$.




When $t = h_D(\rho)$, $\Omega_t(\omega) \cap \partial D \neq \emptyset$. We cannot use the Runge approximation property directly.

Let $\eta$ be *any* positive real number.




This will imply $D \Subset \Omega_{t-\eta}(\omega)$.
We denote $(E_{t-\eta}, H_{t-\eta})$ to be the OD solution defined on $\Omega_{t-\eta}(\omega)$.

By the Runge approximation property, we can find a sequence of functions $\{(E_{\eta,\ell}, H_{\eta,\ell})\}_{\ell=1}^{\infty}$ satisfying the Maxwell system in $\Omega$ such that $(E_{\eta,\ell}, H_{\eta,\ell})$ approximates to $(E_{t-\eta}, H_{t-\eta})$ as $\ell \to \infty$ in $L^2(\Omega_{t-\eta}(\omega))$ and in $H(curl, D)$ by interior estimates since $D \Subset \Omega_{t-\eta}(\omega)$.

In addition, we can show that $(E_{t-\eta}, H_{t-\eta})$ converges to $(E_t, H_t)$ in $H(curl, D)$ as $\eta \to 0$.

From the energy integral, we can define the indicator function as follows.

# Indicator function

**Indicator function**: For $\omega \in S^2$, $\tau > 0$ and $t > 0$ we define the indicator function

$$I_\omega(\tau,t) := \lim_{\eta\to 0}\lim_{\ell\to\infty} I_\omega^{\eta,\ell}(\tau,t),$$

where

$$I_\omega^{\eta,\ell}(\tau,t) := ik\tau\int_{\partial\Omega}(\nu\times H_{\eta,\ell})\cdot(\overline{(\Lambda_D-\Lambda_\emptyset)(\nu\times H_{\eta,\ell})}\times\nu)dS.$$

**Goal**: We want to characterize the convex hull of the obstacle $D$ from the impedance map $\Lambda_D$.

Integral Inequalities

$$|\tau^{-1} I_\omega^{\eta,\ell}| \leq c \int_D |\nabla \times H_{\eta,\ell}|^2 dx + k^2 \int_\Omega \mu |\widetilde{H_{\eta,\ell}}|^2 dx,$$

$$|\tau^{-1} I_\omega^{\eta,\ell}| \geq c \int_D |\nabla \times H_{\eta,\ell}|^2 dx - k^2 \int_\Omega \mu |\widetilde{H_{\eta,\ell}}|^2 dx,$$

where $\widetilde{H_{\eta,\ell}} = H - H_{\eta,\ell}$ be the reflected solution, then $\widetilde{H_{\eta,\ell}}$ satisfies

$$\begin{cases} \nabla \times (\varepsilon^{-1} \nabla \times \widetilde{H_{\eta,\ell}}) - k^2 \mu \widetilde{H_{\eta,\ell}} = -\nabla \times ((\varepsilon^{-1}(x) - \varepsilon_0^{-1}(x)) \nabla \times H_{\eta,\ell}) \text{ in } \Omega \\ \nu \times \widetilde{H_{\eta,\ell}} = 0 \text{ on } \partial\Omega. \end{cases}$$

Then we have some estimate for $\widetilde{H_{\eta,\ell}}$.

# Meyers $L^p$ estimate

**Proposition (Estimate for $\widetilde{H_{\eta,\ell}}$)**

*Assume $\Omega$ is a smooth domain and $D \Subset \Omega$. Then there exist a positive constant $C$ and $\delta > 0$ such that*

$$\|\widetilde{H_{\eta,\ell}}\|_{L^2(\Omega)} \leq C\|\nabla \times H_{\eta,\ell}\|_{L^p(D)}$$

*for every $p \in (\frac{4}{3}, 2]$.*

The proof is by using a global $L^p$ estimate for the curl of the solutions of the anisotropic Maxwell system.

**Theorem (Main Theorem)**

*Let $\omega \in \mathbb{S}^2$, we have the following characterization of $h_D(\omega)$.*

$$\begin{cases} \lim_{\tau\to\infty} |I_\omega(\tau,t)| = 0 \text{ when } t < h_D(\omega), \\ \liminf_{\tau\to\infty} |I_\omega(\tau,h_D(\omega))| > 0. \end{cases}$$

# Conclusion for part 1

•We develop an enclosure type method for identifying inclusion obstacles in anisotropic Maxwell system. Our main tool is the OD solutions for the anisotropic Maxwell system.

•Our theory shows that we are able to determine the **convex hull** of inclusions by the impedance map.

# Part 2: SUCP for residual stress system with Gevrey coefficients

Unique Continuation Property

We call $u$ has the unique continuation property If $u \in H^1_{loc}(\Omega)$ satisfies $Lu = 0$ in $\Omega$ and $u$ vanishes on an open subset of $\Omega$, then $u$ must vanish identically in $\Omega$.

# Strong Unique Continuation Property (SUCP)

We call $u$ has the SUCP if $u \in H^1_{loc}(\Omega)$ satisfies $Lu = 0$ in $\Omega$ and vanishes to infinity order at a point $x_0 \in \Omega$,i.e., for all $N > 0$

$$\int_{R \leq |x-x_0| \leq 2R} |u|^2 dx = O(R^N), \quad R \to 0, \tag{3.1}$$

then $u$ must vanish identically in $\Omega$. If $u$ is smooth, the condition (3.1) is equivalent to all partial derivatives of $u$ vanishing at $x_0$, which means $\partial^\beta f(x_0) = 0$ for all multiindices $\beta$.

# Residual Stress System

Let $\Omega$ be a open connected domain in $\mathbb{R}^3$ and consider the time-harmonic elasticity system

$$\nabla \cdot \sigma + \kappa^2 \rho u = 0 \text{ in } \Omega, \tag{3.2}$$

where $\sigma = (\sigma_{ij})_{i,j=1}^3$ is the stress tensor field, $\kappa \in \mathbb{R}$ is the frequency and $\rho = \rho(x) > 0$ denotes the density of the medium.

The vector field $u(x) = (u_i(x))_{i=1}^3$ is the displacement vector. Suppose that the stress tensor is given by

$$\sigma(x) = T(x) + (\nabla u)T(x) + \lambda(x)(\mathrm{tr}E)I + 2\mu(x)E,$$

where $E(x) = \dfrac{\nabla u + \nabla u^t}{2}$ is the infinitesimal strain and $\lambda(x), \mu(x)$ are the Lamé parameters.

The second-rank tensor $T(x) = (t_{ij}(x))_{i,j=1}^{3}$ is the residual stress and satisfies

$$t_{ij}(x) = t_{ji}(x), \ \forall i,j = 1,2,3 \text{ and } x \in \Omega$$

and

$$\nabla \cdot T = \sum_j \partial_j t_{ij} = 0 \text{ in } \Omega, \ \forall i = 1,2,3.$$

If we define the elastic tensor $C = (C_{ijkl})_{i,j,k,l=1}^{3}$ with

$$C_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu(\delta_{jk}\delta_{jl} + \delta_{jk}\delta_{il}) + t_{jl}\delta_{ik},$$

then (3.2) is equivalent to

$$\nabla \cdot (C\nabla u) + \kappa^2 \rho u = 0 \text{ in } \Omega.$$

# Brief History

Results on (strong) unique continuation property for the residual stress system has been proved by:

• G. Nakamura and J.N. Wang (2003) proved the unique continuation property for (3.2) under the condition $\max_{i,j} \|t_{ij}\|_\infty$ is small and $T(x), \lambda(x)$, $\mu(x) \in W^{2,\infty}$ and $\rho(x) \in W^{1,\infty}$.

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• C.L. Lin (2004) proved the SUCP for (3.2) under the assumptions that $T(0) = 0$, $\max_{i,j} \|t_{ij}\|_\infty$ is small, $\lambda(x), \mu(x)$ and $\rho(x)$ are in $C^2$.

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- G. Uhlmann and J.N. Wang (2009) proved unique continuation principle for (3.2) under the conditions $T(x), \lambda(x),\ \mu(x) \in W^{2,\infty}$, $\rho(x) \in W^{1,\infty}$ and general residual stress.

# Reduction - First step

We want to prove the SUCP for

$$\nabla \cdot (C\nabla u) + \kappa^2 \rho u = 0 \text{ in } \Omega.$$

We define

$$Ru = \nabla \cdot (\nabla u T) \tag{3.3}$$

with $Ru = ((Ru)_1, (Ru)_2, (Ru)_3)$, where $(Ru)_i = \sum_{jk} t_{jk} \partial^2_{jk} u_i$, $i = 1, 2, 3$.

Set $U = (u, v, w)^t$, where $v = \nabla \cdot u$, $w = \nabla \times u$ and $u$ satisfies

$$\nabla \cdot (C\nabla u) + \kappa^2 \rho u = 0 \text{ in } \Omega. \tag{3.4}$$

Take divergence on (3.4) and take curl on (3.4), we can find two equations for $v, w$. We write differential equations for $u, v, w$ in the following form.

$(u, v, w)$ satisfies

$$\widetilde{P_1}(x, D)u := \frac{Ru}{\mu} + \Delta u = \sum_{m=0}^{1} A_{1,m}(u, v)$$

$$\widetilde{P_2}(x, D)v := \frac{Rv}{(\lambda + 2\mu)} + \Delta v = -\frac{1}{\lambda + 2\mu} \sum_{jk} \nabla(t_{jk}) \cdot \partial_{jk}^2 u + \sum_{m=0}^{1} A_{2,m}(u, v, v$$

$$\widetilde{P_1}(x, D)w := \frac{Rw}{\mu} + \Delta w = -\frac{1}{\mu} \sum_{jk} \nabla(t_{jk}) \times \partial_{jk}^2 u + \sum_{m=0}^{1} A_{3,m}(u, v, w),$$

where $A_{\ell,m}$ are $m$-th order differential operator, $\ell = 1, 2, 3,\ m = 0, 1.$

# Difficulties

We want to prove the SUCP for general residual stress $T$. The main difficulty is:

1. The system of $(u, v, w)$ is not decoupled.

2. We cannot find a change of coordinates such that the differential operators $\widetilde{P_1}(x, D)$ and $\widetilde{P_2}(x, D)$ are Laplacian at $x = 0$ simultaneously (If $T(0) = 0$, then $\widetilde{P_1}(0, D) = \widetilde{P_2}(0, D) = \Delta$).

# Main Trick

Our main tool is to reduce (3.2) into a special **fourth order elliptic system**.
We need to derive suitable **Carleman estimates** in order to get the SUCP for this elliptic system.

In general, the SUCP doe not hold even the coefficients are smooth, Alinhac (1980) gave a counterexample. Thus, we consider the coefficients of (3.2) lie in the Gevrey class.

# Gevrey Class

We say that $f \in C^\infty(\Omega)$ belongs to the Gevrey class of order $s$, denote it as $G^s(\Omega)$ (or $G^s$), if there exist constants $c$, $A$ and multiindices $\beta$ such that

$$|\partial^\beta f| \leq cA^{|\beta|}(|\beta|!)^s \text{ in } \Omega.$$

We give several useful properties for Gevrey class $G^s$.

The Gevery class collects functions between smooth and analytic.

# Properties for Gevrey functions

1. $f \in G^s$ and vanishes to infinity order at 0. If $s-1<\rho$, then

$$|f(x)| \leq e^{-|x|^{-1/\rho}} \text{ near } x=0.$$

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2. $e^{-|x|^{-1/\rho}} \in G^s$ provided $1 + \rho = s$.
3. (**Gevrey regularity**) Let $P(x, D)u = f$ in $\Omega$ be an elliptic differential system with coefficients and $f$ are in the Gevrey class $G^s$. Then $u \in G^s(O)$ for all bounded $O \Subset \Omega$.

We assume $P_1$ and $P_2$ are two strongly elliptic operators, where

$$P_1(x,D) := \sum_{jk} a^1_{jk}(x)\partial^2_{x_jx_k} := \sum_{jk}(\mu\delta_{jk}+t_{jk})\partial^2_{x_jx_k}, \quad (4.1)$$

$$P_2(x,D) := \sum_{jk} a^2_{jk}(x)\partial^2_{x_jx_k} := \sum_{jk}((\lambda+2\mu)\delta_{jk}+t_{jk})\partial^2_{x_jx_k} \quad (4.2)$$

with $a^1_{jk}(x)=\mu(x)\delta_{jk}+t_{jk}(x)$ and
$a^2_{jk}(x)=(\lambda(x)+2\mu(x))\delta_{jk}+t_{jk}(x)$.

The elliptic condition means that there exists $c_0 > 0$ such that for any $\xi = (\xi_i)_{i=1}^3 \in \mathbb{R}^3$

$$\sum_{jk} a^1_{jk}(x)\xi_j\xi_k = \sum_{jk} t_{jk}\xi_j\xi_k + \mu|\xi|^2 \geq c_0|\xi|^2 \tag{4.3}$$

$$\sum_{jk} a^2_{jk}(x)\xi_j\xi_k = \sum_{jk} t_{jk}\xi_j\xi_k + (\lambda + 2\mu)|\xi|^2 \geq c_0|\xi|^2 \tag{4.4}$$

for all $x \in \Omega$.

Colombini, Grammatico and Tataru (2006) derive the following condition for the appropriate index $s$ for $G^s$. $\exists \alpha > 0$ such that the eigenvalues $\lambda_1^\ell \leq \lambda_2^\ell \leq \lambda_3^\ell$ of $(a_{jk}^\ell(0))$ satisfying

$$\alpha > \frac{\lambda_3^\ell - \lambda_1^\ell}{\lambda_1^\ell}$$

and

$$s < 1 + \frac{1}{\alpha}$$

uniformly in $x$ and for $\ell = 1, 2$.

# Main result

**Theorem (Main Theorem)**

*Let the residual stress $(t_{ij}(x))_{i,j=1}^{3}$, the Lamé parameters $\lambda(x)$, $\mu(x)$ and the density of the medium $\rho(x)$ be in the Gevrey class $G^s(\Omega)$ with $s$ satisfying $s < 1 + \frac{1}{\alpha}$. Then for all $u \in H^2_{loc}(\Omega; \mathbb{R}^3)$ solving (3.2) and for all $N > 0$*

$$\int_{R \le |x| \le 2R} |u|^2 dx = O(R^N) \text{ as } R \to 0,$$

*then $u$ is identically zero in $\Omega$.*

However, it is not easy to prove the main theorem directly for the general residual stress, we will introduce a reduction method to transform (3.2) into a new fourth order elliptic system with principally diagonal leading terms. Moreover, we need to derive suitable Carleman estimates in order to obtain the SUCP.

# Reduction - Second step

Recall that in the first step, we have reduce the residual stress system into

$$P_1(x,D)u = Ru + \mu\Delta u = \sum_{m=0}^{1} B_{1,m}(u,v)$$

$$P_2(x,D)v = Ru + (\lambda+2\mu)\Delta v = -\sum_{jk} \nabla(t_{jk}) \cdot \partial_{jk}^2 u + \sum_{m=0}^{1} B_{2,m}(u,v,w),$$

$$P_1(x,D)w = Rw + \mu\Delta w = -\sum_{jk} \nabla(t_{jk}) \times \partial_{jk}^2 u + \sum_{m=0}^{1} B_{3,m}(u,v,w),$$

where $B_{\ell,m}$ are $m$-th order differential operators, $m = 0,1$ and $\ell = 1,2,3$.

# Key observation 1

If we act $P_2$ on $P_1(x,D)u$-equation, $P_1(x,D)w$-equation and act $P_1$ on $P_2(x,D)v$ equation, we will obtain fourth order elliptic equations $P_2P_1$ for $u,w$ and $P_1P_2$ for $v$.

$$P_2\left(P_1(x,D)u\right) = P_2\left(\sum_{m=0}^{1} B_{1,m}(u,v)\right),$$

$$P_1\left(P_2(x,D)v\right) = P_1\left(-\sum_{jk}\nabla(t_{jk})\cdot\partial_{jk}^2 u + \sum_{m=0}^{1} B_{2,m}(u,v,w)\right),$$

$$P_2\left(P_1(x,D)w\right) = P_2\left(-\sum_{jk}\nabla(t_{jk})\times\partial_{jk}^2 u + \sum_{m=0}^{1} B_{3,m}(u,v,w)\right),$$

Use $P_1P_2 = P_2P_1 + [P_1,P_2]$, where $[P_1,P_2]$ is a commutator, a third order differential operator.

# Key observation 2

We also observe that

$$P_1(x,D)[\sum_{jk}\nabla(t_{jk})\cdot\partial_{jk}^2 u]=\sum_{m\leq 3}D_m^1(u,v,w)$$

and

$$P_2(x,D)[\sum_{jk}\nabla(t_{jk})\times\partial_{jk}^2 u]=\sum_{m\leq 3}D_m^2(u,v,w),$$

where $D_m^1, D_m^2$ are $m$-th order differential operators.

Let $U = (u, v, w)^t$, we transform (3.2) into a fourth order principally diagonal elliptic system

$$P_2 P_1 U = \sum_{m=0}^{3} B_m(U), \tag{4.5}$$

where $B_m(U)$ is an $m$-th order differential operator. Note that the leading coefficients of $U$ is principally diagonal and $U \in G^s$ since the coefficients of (4.5) are in $G^s$.

# Main result for fourth order elliptic system

**Theorem (Main Theorem)**

*Let $P = P_2 P_1$ be a fourth order elliptic operator with coefficients in $G^s$. Then the SUCP holds for the elliptic system*

$$PU = \sum_{|\beta| \leq 3} B_m(U)$$

*provided that all the coefficients are in $G^s$.*

If $U$ has the SUCP, then $u$ has the SUCP.

The main theorem was first proved by Colombini and Koch in 2010, they use iterative method to derive the Carleman estimate for higher order elliptic equations. They proved the Carleman estimates in the following type:
Let $P_1, P_2, \cdots, P_M$ be second order elliptic operators with Gevrey coefficients. Let $P = P_M P_{M-1} \cdots P_1$ be an $2M$ order elliptic operator, then the SUCP holds for

$$Pu = \sum_{|\beta| \leq [\frac{3M}{2}]} a_\beta \partial^\beta u$$

provided $a_\beta \in G^s$, $\forall |\beta| \leq [\frac{3M}{2}]$.

The proof of the theorem is based on the iteration from the Carleman estimates for the 2nd order elliptic operator. In our case, when $M=2$, $2M=4$ and $[\frac{3M}{2}]=3$, then we can apply the Colombini-Koch's result directly.

# Carleman Estimates

We are going to derive the Carleman estimates for the weight $e^{\tau|x|^{-\alpha}}$ for the fourth order elliptic operator $P = P_2 P_1$. The main point is that $U \in G^s$ and vanishes to infinity order, then we have $|U| \leq e^{-|x|^{-\alpha}}$ near 0 provided that $s < 1 + \dfrac{1}{\alpha}$.

We have the following Carleman estimates for $P$:

$$
\begin{aligned}
&\sum_{j=0}^{4} \tau^{6-2j} \int |x|^{-8-6\alpha} |x|^{2j(1+\alpha)} e^{2\tau|x|^{-\alpha}} |D^j V|^2 dx \\
&\lesssim \sum_{j=9}^{2} \tau^{3-2j} \int |x|^{-4-3\alpha} |x|^{j(1+\alpha)} e^{2\tau|x|^{-\alpha}} |D^j (P_1 V)|^2 dx \\
&\lesssim \int e^{2\tau|x|^{-\alpha}} |(P_2 P_1 V)|^2 dx = \int e^{2\tau|x|^{-\alpha}} |PV|^2 dx.
\end{aligned}
$$

Use the vanishing to infinity order of $U$ at 0 and the Carleman estimate we can obtain $U \equiv 0$ in a small neighborhood of 0, then $u \equiv 0$ in a small neighborhood of 0.

Furthermore, by using the unique continuation principal was proved by Uhlmann-Wang's result in 2009, therefore, we can obtain $u \equiv 0$ in $\Omega$, then we are done.

# Conclusion 2

Under the assumptions for the Gevrey coefficients with appropriate indices $s$, we can prove if $u$ vanishes to infinity order at $x_0 \in \Omega$, then $u$ vanishes identically in $\Omega$ provided that $\Omega$ is a simply connected domain.

Thank you for your attention !

# Proof of the Main Theorem

The operator $P = P_2 P_1$ is strongly elliptic in the Gevrey class $G^s$, then $U$ is also in the Gevrey class $G^s$. Therefore, we have the vanishing of infinite order implies that

$$|u| \lesssim e^{-|x|^{-\gamma}}$$

for some $\gamma > \alpha$.

Let $\chi \in C_0^\infty(\mathbb{R}^3)$ be such that $\chi \equiv 1$ for $|x| \leq R$ and $\chi \equiv 0$ for $|x| \geq 2R$ ($R > 0$ is small enough). Then we can apply the Carleman estimates to the function $\chi U$, which means

$$\begin{aligned} & C \sum_{|\beta|=0}^{4} \tau^{6-2|\beta|} \int_{|x|<R} |x|^{(2|\beta|-6)(1+\alpha)-2} e^{2\tau|x|^{-\alpha}} |D^\beta U|^2 dx \qquad (5.1) \\ & \leq \int e^{2\tau|x|^{-\alpha}} |PU|^2 dx \\ & \leq \int_{|x|<R} e^{2\tau|x|^{-\alpha}} |PU|^2 dx + \int_{|x|>R} e^{2\tau|x|^{-\alpha}} |P(\chi U)|^2 \\ & \leq \int_{|x|<R} e^{2\tau|x|^{-\alpha}} |\sum_{m=0}^{3} \widehat{E_m}(U)|^2 dx + \int_{|x|>R} e^{2\tau|x|^{-\alpha}} |P(\chi U)|^2, \end{aligned}$$

If $\tau$ is large and $R$ is sufficiently small, then we have

$$C\sum_{|\beta|=0}^{4}\tau^{6-2|\beta|}\int_{|x|<R}|x|^{(2|\beta|-6)(1+\alpha)-2}e^{2\tau|x|^{-\alpha}}|D^\beta U|^2dx \tag{5.2}$$
$$\leq \int_{|x|>R}e^{2\tau|x|^{-\alpha}}|P(\chi U)|^2,$$

for some constant $C>0$.

Notice that $e^{\tau|x|^{-\alpha}} \geq e^{\tau R^{-\alpha}}$ for $|x| < R$ and $e^{\tau|x|^{-\alpha}} \leq e^{\tau R^{-\alpha}}$ for $|x| > R$. Therefore, we can use (5.2) to obtain

$$C \sum_{|\beta|=0}^{4} \tau^{6-2|\beta|} \int_{|x|<R} |x|^{(2|\beta|-6)(1+\alpha)-2} |D^{\beta} U|^2 dx$$
$$\leq \int_{|x|>R} |P(\chi U)|^2.$$

Let $\tau \to \infty$, we get $U = 0$ in $\{|x| < R\}$ for $R$ small, which implies $u = 0$ in $\{|x| < R\}$.