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包圍重構法在非等向性介質下的發展與殘留應力的彈性系統之強唯一連續性

The development of the Enclosure Method in an Anisotropic Background and the Strong Unique Continuation for the Elasticity with Residual Stress

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本論文係林奕亘君（D01221001）在國立臺灣大學數學學系、所完成之博士學位論文，於民國 104 年 12 月 23 日承下列考試委員審查通過及口試及格，特此證明

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誌 謝

本文能順利呈現，要感謝我的指導老師王振男教授，在這當中他給我許多的幫忙，也不辭辛勞的與我討論這相關的內容。王振男老師在我從碩士念到博士的這六年的歲月中，不但給我學業上的激勵指點，生涯規劃的提醒以及面對工作的態度，都讓我獲益良多。在 2014 年的 1 月底時，王老師推薦我代表理學院去參加在新加坡舉辦的 GYSS-全球青年科學家論壇，這個論壇雖然不是針對數學家舉辦，但是同為科研的人，互相討論互相激勵是一件很重要的事情。在那個論壇中我認識了很多來自全世界各地的年輕科學家，也認識了幾位費爾茲獎得主，這個論壇完全打開了我對科學研究的眼界。而在該年二月，王老師給了我另一個機會去奧地利的 Linz，RICAM 學術機構訪問 Mourad Sini 教授，也開啟了我對反問題中另一個問題分支的興趣，也在當時奠定了完成 Maxwell 這個問題的基礎。在 2014 年的 12 月時，王老師給我一個機會在台灣大學舉辦的逆問題研討會中做第一次報告，報告的內容是我的第一篇文章。這次的經驗非常重要，因為這是我第一次在國際性的研討會做一個報告，讓我知道如何在真正的國際會議中給出比較有水準的演講。

在 2015 年的時候，王老師給了我更多的機會，包括一月份在東京明治大學的台灣日本年輕學者應用數學研討會，給了一個小演講；在同年五月，到芬蘭赫爾辛基參加逆問題領域中規模最大的會議：應用逆問題大會。超過五百位研究反問題的專家學者學生，齊聚赫爾辛基大學，這兩周的收穫完全無法用言語形容。在六月時，去上海的華東師範大學參加第六屆華東偏微分方程年會，也學了很多有關方程研究的看法。在同年七月也去了京都大學訪問 Iso 教授，在京都大學也給了兩個演講。九月底去香港科技大學參加 Prof. Uhlmann 舉辦的逆問題會議，也是認識很多同行的新朋友還有對於數學的見識更上一層樓。以上總總的出國增廣見聞機會，皆是王振男老師不遺餘力的對我無私的付出。

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中文摘要

這篇論文的目的是在三維中非等向性介質下重構可穿透與不可穿透障礙物。我們將會示範如何利用包圍法重構對於以下兩種數學模型：非等向性的橢圓方程以及非等向性的馬克士威方程。到目前為止，對於非等向性的數學模型，沒有可以利用的複幾何光學解用來重構未知障礙物。因此我們將會使用另一種特別解：震盪遞減解使用在我們的逆問題之中。

特別的，在這篇文章中，我們會介紹一種新的轉換法，把非等向性的馬克士威方程轉變成一個二階線性強橢圓系統。這個方法是用來建構非等向性的馬克士威方程的震盪遞減解。而在此篇文章的最後，我們將會討論強唯一連續性質對於 Gevrey 係數的殘留應力系統。

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Abstract

The goal of this dissertation is to develop reconstruction schemes to determine penetrable and impenetrable obstacles in a region in 3-dimensional in an anisotropic background. We demonstrate the enclosure-type method for two different mathematical models: The anisotropic elliptic equation and the anisotropic Maxwell system. So far, in the anisotropic case, there are no complex geometrical optics solutions which we can use to reconstruct the unknown obstacles in a given medium. Therefore, we use another special type solution: the oscillating decaying solutions, which are useful in our inverse problems.

In particular, for the anisotropic Maxwell system model, we also introduce a new reduction method to transform the Maxwell system into a second order strongly elliptic system. This reduction method is the main tool to construct the oscillating decaying solutions for the anisotropic Maxwell system. In addition, we prove the strong unique continuation for a residual stress system with Gevrey coefficients.

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Chapter 1

Preliminaries

Inverse boundary value problem is a field of discussing the inverse problems of partial differential equations. The inverse boundary value problems have become a popular field since A.P. Calderón published his pioneering work "On an inverse boundary value problem" [1] in 1980s. The problem proposed by Calderón is: "Is possible to determine the electrical conductivity of a medium by making voltage and current measurements on its boundary ?" More specifically, for each voltage density on the boundary, there would be the corresponding current which can be measured theoretically on the same periphery. In addition, the Calderón problem is also called the inverse conductivity problem.

Under the assumptions of no sources or sinks of current in Ω, a voltage potential f at the boundary $\partial\Omega$ induces a voltage potential u in Ω, which solves the Dirichlet problem for the conductivity equation,

$$\begin{cases} \nabla \cdot (\gamma \nabla u) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases}$$

Since γ is positive, there exists a unique weak solution $u \in H^1(\Omega)$ for any boundary value $f \in H^{1/2}(\partial\Omega)$. One can define the *Dirichlet-to-Neumann map* (termed as DN map hereafter) formally as

$$\Lambda_\gamma : f \to \gamma \frac{\partial u}{\partial \nu}|_{\partial\Omega}.$$

The question is whether this DN map uniquely determines the conductivity γ in Ω. This problem led to the development of the Electrical Impedance Tomography (EIT), an imaging method with potential applications in medical imaging and nondestructive testing. The ideas of solving Calderón problem is based mainly on gaining information from boundary data, which can be extended to tackle many physical issues in the reality. The questions evolve from theoretical determinations to practical reconstructions. For example, boundary measurements determines the information of

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unknown obstacles in a given medium.

By extending Calderón's ideas, we not only can determine the conductivity γ from the boundary information but also we can reconstruct unknown obstacles in a given subject. There are several reconstruction methods to know the information of unknown obstacle inside a given domain: The enclosure method(Ikehata), the linear sampling method(Colton-Kirsch), the factorization method(Kirsch) and the singular source method(Potthast). The enclosure method can be applied in the following: the subject contains unknown obstacles and the conductivity is unknown in the unknown obstacle which is different from the background. The enclosure method is not only a theoretical detection method but also provides an algorithm to draw the unknown obstacle. In this article, we will illustrate how to reconstruct unknown obstacles in a known background from boundary information. We will employ a nondestructive method: "the *enclosure method*", which was first introduced by Ikehata [19].

The simplest inverse obstacle problem has the following formulation. Let u be a solution of the conductivity equation

$$\begin{cases} \nabla \cdot (\gamma \nabla u) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases}$$

where $\gamma(x) := 1 + \gamma_D \chi_D$, where $D \Subset \Omega$ is an unknown obstacle in Ω. By defining the DN map $\Lambda_D : f \to \gamma \dfrac{\partial u}{\partial \nu}|_{\partial\Omega}$, we are able to explore the shape of D through the above reconstruction method and the DN map Λ_D. This geometrical inverse problem is quite well studied in the literature see [24] and several methods have been proposed to solve it. In this chapter, we focus on the enclosure method, which is initiated by Ikehata, see for examples [17, 19], and developed by many researchers [27, 30, 44, 53, 55, 61], [26, 55] for the acoustic model, [25, 30] for the Lamé model and [27, 66] for the Maxwell model. The testing functions used in [27, 66] are complex geometric optics (CGO) solutions of the isotropic Maxwell's equation. The construction of CGO solutions for isotropic inhomogeneous Maxwell's equations is first proposed in [51]. After that, the authors in [28] also constructed CGO solutions for some special anisotropic Maxwell's equations. However, there are not yet of CGO solutions for general anisotropic Maxwell system. Besides, CGO solutions, another kind of special solutions for anisotropic elliptic system was proposed for substitution in [48] and [49]. They are called oscillating-decaying (OD) solutions.

This thesis is organized as follows. In Chapter 2, we first review the idea of the enclosure method for the isotropic scalar elliptic equations and generalize such a concept to the anisotropic scalar elliptic equations and fully examine the enclosure method including the *complex geometric optics* (CGO) solutions and the *indicator functional* (or indicator function). Consequently, the theoretical linkage between the enclosure method and the Calderón's problem will be presented. In the absence of CGO solutions for the anisotropic elliptic equation in $\mathbb{R}^3$, we introduce another

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special solutions called the *oscillating-decaying* solutions and use the *Runge approximation property* to obtain our reconstruction algorithm in the anisotropic case, which primarily steams from the enclosure method (intuitively, viewed as the enclosure-type method). In addition, the traditional indicator function requires some modifications.

In Chapter 3, we confine the framework of the enclosure method to the isotropic Maxwell system, which has been addressed in [66], in the similar approach of the CGO solutions and suitable choice of the indicator function. Instead, we can define the *impedance map*, the counterpart for the elliptic case (that is, the DN map). The indicator function and the reconstruction algorithm are adjusted due to the slight differences between the impedance map and the DN map. Stretching the result of isotropic case to the anisotropic case poses plenty of challenges. We thereby propose a new reduction method which transforms the anisotropic Maxwell system into a second order strongly elliptic system in $\mathbb{R}^3$. Further, given the relationship between the oscillating-decaying solutions and the strongly elliptic system, we utilize the newly-proposed (reduction) method to convey such the relation to the anisotropic Maxwell systems and, in turn, derive the representation of the oscillating-decaying solutions of the anisotropic Maxwell system.

In the following chapter, we prove the strong unique continuation property (SUCP) for a residual stress system with Gevrey coefficients on the basis of the SUCP for the scalar elliptic equations with coefficients in the Gevrey class. Finally, we provide some guidelines for the future works.

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Chapter 2

The enclosure method for second order elliptic equations

The enclosure method is to reconstruct an unknown obstacle in a known background, which was first introduced by Ikehata, see [17]. The main tool of this reconstruction algorithm is by the indicator functional and the complex geometric optics (CGO) solutions. The idea of these tools can be traced back to the Calderón's pioneering work. In the following, we will show the relations between Calderón's work and these tools. Moreover, if the mathematical models are complicated, we need to introduce appropriate elliptic regularity estimates (C^α estimates or Meyers' L^p estimates), we will discuss these details in the following sections.

2.1 Calderón's problem

In 1980s', Calderón published his pioneer work "On an inverse boundary value problem" [1]. His work affected the development of the inverse boundary value problem and the inverse conductivity problem. We will give a brief introduction about the Calderón problem how to relate to the enclosure method.

Let us begin by giving the mathematical model. Let $\Omega \subset \mathbb{R}^n$ be a bounded open subset in $\mathbb{R}^n$ for $n = 2, 3$ with C^∞ boundary. Assume that $\gamma > 0$ is a C^2 function defined on $\overline{\Omega}$. Let $u \in H^1(\Omega)$ satisfy

$$\begin{cases} \nabla \cdot (\gamma \nabla u) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases} \tag{2.1.1}$$

where $f \in H^{1/2}(\partial\Omega)$. It is well-known that (2.1.1) has a unique weak solution $u \in H^1(\Omega)$. We can

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define the Dirichlet-to-Neumann (DN) map formally as

$$\Lambda_\gamma f = \gamma \frac{\partial u}{\partial \nu}|_{\partial\Omega}.$$

More precisely, the DN map is defined weakly as

$$(\Lambda_\gamma f, g)_{\partial\Omega} = \int_\Omega \gamma \nabla u \cdot \nabla v dx,\ \ f, g \in H^{1/2}(\partial\Omega),$$

where u is the solution of (2.1.1) and v is any function in $H^1(\Omega)$ with $v|_{\partial\Omega} = g$. The pairing on the boundary is integration with respect to the surface measure

$$(f, g)_{\partial\Omega} = \int_{\partial\Omega} fg dS.$$

With the definition, we know that Λ_γ is a bounded linear map from $H^{1/2}(\partial\Omega)$ into $H^{-1/2}(\partial\Omega)$.

The Calderón problem (also called the inverse conductivity problem) is to determine the conductivity function γ from the knowledge of the map Λ_γ. That is, if the measured current $\Lambda_\gamma f$ is known for all boundary voltages $f \in H^{1/2}(\partial\Omega)$, one would like to determine the conductivity γ. There are several aspects of this inverse problem which are interesting to both the mathematical theory and the practical applications. When $\Omega \subset \mathbb{R}^n$ for $n \geq 3$, we have the following results.

1. **Uniqueness**. If $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, we have $\gamma_1 = \gamma_2$. The result was proved by Sylvester-Uhlmann [57] in 1987.

2. **Reconstruction**. Given the boundary measurements Λ_γ, find a procedure to reconstruct the conductivity γ. There is a convergent algorithm which was found by Nachman [42].

3. **Stability**. If Λ_{γ_1} is close to Λ_{γ_2} in a suitable sense, then γ_1 and γ_2 are close. In 1988, Alessandrini [2] proved that if $\gamma_j \in H^s(\Omega)$ for $s > \frac{n}{2} + 2$, $\|\gamma_j\|_{H^s(\Omega)} \leq M$ and $\frac{1}{M} \leq \gamma_j \leq M$ $(j = 1, 2)$. Then

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \leq \omega(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)}),$$

   where $\omega(t) = C|\log t|^{-\sigma}$ for small $t > 0$ and $C = C(\Omega, M, n, s) > 0$, $\sigma = \sigma(n, s) \in (0, 1)$.

4. **Partial data**. If Γ is a subset of $\partial\Omega$ and if $\Lambda_{\gamma_1} f|_\Gamma = \Lambda_{\gamma_2} f|_\Gamma$ for all boundary voltages f, show that $\gamma_1 = \gamma_2$. When Ω is convex and Γ is any open subset of $\partial\Omega$, Kenig-Sjöstrand-Uhlmann then proved this result in [29].

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In order to deal with the Calderón problem, Calderón considered the following nonlinear map

$$Q_\gamma(f) := \int_\Omega \gamma |\nabla u|^2 dx = \int_{\partial\Omega} \Lambda_\gamma(f) \cdot \overline{f} ds, \tag{2.1.2}$$

where ds is the standard surface measure and u solves (2.1.1) with $u|_{\partial\Omega} = f$. Calderón proved Q_γ is analytic and the Frechet derivative of Q_γ at γ_0 is injective whenever γ_0 is a constant, which means the map from γ to Q_γ is injective for constant conductivity γ.

Calderón's work has made the huge influences in the inverse problems. The nonlinear map Q_γ can be widely applied to other areas of the inverse problems. For example, Q_γ is also called the indicator functional, which is useful in the *enclosure method* for the reconstruction of unknown obstacles. Moreover, Calderón took a special harmonic function $u = e^{x\cdot(\rho + i\rho^\perp)}$ to show the injectivity of the linearized map, where $\rho \in \mathbb{C}^n$ with $\rho \cdot \rho^\perp = 0$. In addition, we call $e^{x\cdot(\rho+i\rho^\perp)}$ to be the *complex geometric optics* (CGO) solutions and the CGO solution plays an important role in the inverse problem, for more details, we refer readers to [60].

2.2 Ideas of the enclosure method

We give two different examples to demonstrate ideas of the enclosure method. Here is the mathematical setting: Let $\Omega \subset \mathbb{R}^n$, for $n = 2, 3$ and $D \Subset \Omega$ be an unknown obstacle. We consider the simplest case in the following: Let $\gamma_0 \equiv 1$ be a given conductivity on the background medium and $\widetilde{\gamma}(x) = \gamma_0 + \gamma_D(x)\chi_D = 1 + \gamma_D\chi_D$ be a total conductivity defined on Ω, where $\chi_D = \begin{cases} 1, & \text{if } x \in D \\ 0, & \text{otherwise} \end{cases}$ is the characteristic function of domain D and $\gamma_D(x) > 0$ is a bounded function on Ω. Then we have two different conductivity equations with boundary value $f \in H^{1/2}(\partial\Omega)$,

$$\begin{cases} \Delta u_0 = 0 & \text{in } \Omega, \\ u_0 = f & \text{on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} \nabla \cdot (\widetilde{\gamma}(x)\nabla u) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases}$$

where u_0 is the voltage when $D = \emptyset$ (no unknown obstacles in Ω) and u is the voltage when $D \neq \emptyset$(Ω contains unknown obstacles). Then we can define the Dirchlet-to-Neumann maps: For $f \in H^{1/2}(\partial\Omega)$,

$$\Lambda_{\gamma_0}(f) = \gamma_0 \frac{\partial u_0}{\partial \nu}|_{\partial\Omega},$$

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and

$$\Lambda_{\widetilde{\gamma}}(f)=\widetilde{\gamma}\frac{\partial u}{\partial \nu}|_{\partial\Omega},$$

where ν is a unit outer normal on $\partial\Omega$. The enclosure method consists of two tools: The *indicator functional* and the *special solutions* (*CGO* solutions).

First, we introduce the indicator functional and the ideas came from the nonlinear map Q_γ we have defined in the previous section (see (2.1.2)). We consider the indicator functional

$$E(f):=\int_{\partial\Omega}(\Lambda_{\widetilde{\gamma}}(f)-\Lambda_{\gamma_0}(f))\cdot\overline{f}dS.$$

If given voltage f on $\partial\Omega$, we can regard $E(f)$ as a difference of currents or energies corresponding to the situations with and without D. Furthermore, due to the positivity property of the equations, we can easily derive

$$E(f)\approx C\int_D|\nabla u_0|^2dx,$$

where $C>0$ is independent of f and u_0 and recall that u_0 solves $\Delta u_0=0$ with $u_0|_{\partial\Omega}=f$.

Second, it is not hard to see for any $h>0$,

$$u_0=e^{\frac{1}{h}(\rho\cdot x+i\rho^\perp\cdot x)}$$

is a solution of the Laplace equation, where $\rho,\rho^\perp\in\mathbb{S}^{n-1}$ (for $n=2,3$) and $\rho\cdot\rho^\perp=0$. Note that the special function u_0 was also appeared in [1], which was proposed by Calderón. Moreover, let $d\in\mathbb{R}$ be arbitrary,

$$u_{0,d,h}=e^{-\frac{d}{h}}e^{\frac{1}{h}(\rho\cdot x+i\rho^\perp\cdot x)}$$

satisfies the Laplace equation, i.e. $\Delta u_{0,d,h}=0$.

We set $f_{0,d,h}=u_{0,d,h}|_{\partial\Omega}$ and take $f_{0,d,h}$ into the indicator functional $E(f)=E(f_{0,d,h})$, then we have

$$\begin{aligned}E(f_{0,d,h}) &\approx C\int_D|\nabla u_{0,d,h}|^2dx\\ &\approx C'\frac{1}{h^2}\int_D e^{\frac{2}{h}(\rho\cdot x-d)}dx\end{aligned}$$

for some positive constants C,C' independent of h. Now, we define the support function $h_D(\rho)$

$$h_D(\rho):=\sup_{x\in D}x\cdot\rho$$

and let $d:=h_D(\rho)$, then we have the following two situations:

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1. If $x \in \{\rho \cdot x > d\}$, then we can see that

$$u_{0,d,h} \to \infty \text{ as } h \to 0+ .$$

In addition, we also have

$$E(f_{0,d,h}) \to \infty \text{ as } h \to 0+ .$$

2. If $x \in \{\rho \cdot x < d\}$, then we can see that

$$u_{0,d,h} \to 0 \text{ as } h \to 0+ .$$

In addition, we also have

$$E(f_{0,d,h}) \to 0 \text{ as } h \to 0+ .$$

Then from the limiting behaviors of $E(f_{0,d,h})$ as h tends to 0, we can conclude that if we choose $f_{0,d,h}$ to be our testing boundary measurements, then the limit behavior of $E(f_{0,d,h})$ will tell us whether the level set $\{x \cdot \rho = d\}$ touches ∂D or not. By varying the direction ρ and the real value d, we can reconstruct a *convex hull* for the unknown obstacle D theoretically.

Remark 2.1. We call $E(f)$ to be the indicator functional. In fact, in [20], Ikehata called $E(f_{0,d,h})$ the indicator function.

Let us summarize the ideas of previous reconstruction procedures. First, we define the indicator function $E(f)$ from the DN map on the boundary. Second, we construct a sequence of special solutions $u_{0,d,h}$ (CGO solutions) for the Laplace equation, and let $f_{0,d,h} = u_{0,d,h}|_{\partial\Omega}$ be the boundary testing functions, then the limit behavior of $E(f_{0,d,h})$ will tell us whether the level set $\{x \cdot \rho = d\}$ touches ∂D or not when h tends to 0. It looks like to use the hyperplanes to *enclose* the unknown obstacle D in Ω, and named the *enclosure method*.

2.3 Complex geometric optics solutions and related topics

Since Ikehata proposed the idea of the enclosure method, there are many applications of this method to other physical problems. We will show how to extend the ideas to different physical settings and related results.

Recall that the enclosure method contains two different tools: The indicator function and the special solutions. In different mathematical problems, we can define similar indicator functions via the Dirichlet-to-Neumann map (for the Maxwell system, we define the impedance map, it will be seen in Chapter 3). The main problem lies on how to find a suitable sequence of testing functions, which satisfy the specific partial differential equation. For example, we know that

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$u_{0,d,h} = e^{\frac{1}{h}(x\cdot\rho - d + ix\cdot\rho^{\perp})}$ solves the Laplace equation. Notice that $u_{0,d,h}$ are harmonic functions with complex phases. By using the following form of solutions,

$$e^{i\frac{1}{h}\rho(x)}(a(x) + R_h(x)),$$

one can construct approprate testing data with a complex phase function $\rho(x)$ and $R_h(x) \ll a(x)$ as $h \to 0+$. The solutions with this form are so-called the *complex geometric optics* (CGO) solutions, which play an essential role in the enclosure-type method.

The results to the existence of CGO solutions for various mathematical problems and CGO solutions are useful for the inverse boundary value problem, for example, see [56, 57, 51, 52, 18, 16, 61]. In particular, CGO solutions play an important role of the probing method in the enclosure type method, we refer readers to [16, 17, 19, 22, 23, 43, 55, 54, 58, 61, 66].

From linear phase to general phase

From Ikehata's previous work, he used the Calderón's harmonic function $e^{x\cdot(\rho+\rho^{\perp})}$ to construct the boundary testing data. The phase function $x \cdot (\rho + \rho^{\perp})$ is linear and we use it to enclose the unknown obstacle. By using the linear phase type harmonic function, we can only reconstruct the *convex hull* of the unknown obstacle. One can refer to a survey paper [21] for detailed explanation and early development of this theory. In [54, 45, 16], the writers used the complex spherical wave solutions to detect concave parts of the unknown obstacles. Moreover, in [61], the researchers proposed a framework to construct the CGO solutions with general phases for some elliptic systems in 2 dimension. This work provides more choices for the phase function of the CGO solutions in 2D. They also gave a concrete example: the CGO solutions with complex polynomial phases and apply these CGO solutions for the conductivity equations to determine unknown obstacles with more general shapes. This type of CGO solutions were also applied to elastic system [64] and Helmholtz equation [43].

More results for the Helmholtz type equation

Recall that we know that $e^{\frac{1}{h}(x\cdot\rho - d + ix\cdot\rho^{\perp})}$ are CGO solutions for various $h, d \in \mathbb{R}$ and $\rho \in \mathbb{S}^{n-1}$ for $n \in \mathbb{N}$ (we only consider $n = 2, 3$). For more general mathematical models, we can consider the following problem

$$\begin{cases} \nabla \cdot (\widetilde{\gamma}(x)\nabla u + k^2 u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases} \tag{2.3.1}$$

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where $\widetilde{\gamma}(x) = 1 + \gamma_D \chi_D$, for some $\gamma_D > 0$, $\gamma_D \in L^\infty(D)$ and χ_D is the characteristic function defined on D. For the unperturbed case, i.e. when $D = \emptyset$, we have the Helmholtz equation

$$\begin{cases} \Delta u_0 + k^2 u_0 = 0 & \text{in } \Omega, \\ u_0 = f & \text{on } \partial\Omega. \end{cases} \tag{2.3.2}$$

Now, we want to know the information of the unknown obstacle $D \Subset \Omega$.

In the beginning, we need to define the DN map

$$\Lambda_D f := \frac{\partial u}{\partial \nu}|_{\partial\Omega} \text{ and } \Lambda_\emptyset f := \frac{\partial u_0}{\partial \nu}|_{\partial\Omega},$$

where u and u_0 are solutions of (2.3.1) and (2.3.2), respectively and ν is a unit outer normal on $\partial\Omega$. Similarly, we can define the indicator function

$$E(f) := \int_{\partial\Omega} (\Lambda_D - \Lambda_\emptyset) f \cdot \overline{f} dS,$$

and use integration by parts many times, we will obtain the upper bound estimates and the lower bound estimates for $E(f)$:

$$E(f) \leq C \int_D |\nabla u_0|^2 dx + k^2 \int_\Omega |w|^2 dx$$

and

$$E(f) \geq c \int_D |\nabla u_0|^2 dx - k^2 \int_\Omega |w|^2 dx,$$

where c, C are independent of u_0, w and $w = u - u_0$ is called the *reflected solution* satisfying

$$\begin{cases} \nabla \cdot (\widetilde{\gamma}(x) \nabla w) + k^2 w = -\nabla \cdot (\widetilde{\gamma}(x) - 1) \nabla u_0 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.3.3}$$

For more calculation details, we refer readers to [43]. Note that the upper and lower bounds only involve u_0 and w. Our remaining task is to find appropriate estimates for $\int_\Omega |w|^2$.

In fact, there are two different approaches for $\int_\Omega |w|^2 dx$: One is the C^α-*estimates* method which was first introduced by [43] and the other is *Meyers' L^p estimates* method which was first introduced by [55]. We give a brief comparison with C^α-estimates method and Meyers' L^p estimate method. Note that in the following estimates, the constants C may change line to line, and they are independent of u_0 and w.

1. C^α-*estimates method*: This method was introduced in [43]. Recall that we have an upper

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bound for the indicator function

$$E(f) \le C \int_D |\nabla u_0|^2 dx + k^2 \int_\Omega |w|^2 dx.$$

By (2.3.3) and the standard elliptic regularity estimates, we have

$$\int_\Omega |w|^2 dx \le C \int_D |\nabla u_0|^2 dx,$$

then we obtain

$$E(f) \le C \int_D |\nabla u_0|^2 dx.$$

The main problem appears on the lower bound for $E(f)$. In [43], the authors defined a new function

$$I_{x_0,\alpha} := \int_{\partial D} \left| \frac{\partial u_0}{\partial \nu} \right| |x - x_0|^\alpha dS,$$

for any $x_0 \in \Omega$, then they derived

$$\int_\Omega |w|^2 dx \le C_{q,\alpha} \{ I_{x_0,\alpha}^2 + I_{x_0,\alpha} \|\nabla u_0\|_{L^q(D)} + \|u_0\|_{L^2(D)}^2 \}, \tag{2.3.4}$$

for any $\alpha \in (0,1)$ and $q \in (2,4]$. The estimate (2.3.4) relies on the C^α-estimates for the elliptic equation, which were proved in the paper [35]. In order to apply this type C^α-estimate, we need to add regularity assumptions on the unknown obstacle D, which is $\partial D \in C^2$. In addition, we know that

$$u_0 := e^{\frac{1}{h}(x\cdot\rho - d) + i\sqrt{\tau^2+k^2} x\cdot\rho^\perp} \tag{2.3.5}$$

are CGO solutions for the Helmholtz equation. Combine the lower bound of $E(f)$, (2.3.4) and put the CGO solutions (2.3.5) into the indicator function $E(d,h) := E(f_{0,d,h}) = E(u_{0,d,h}|_{\partial\Omega})$, then we can obtain

$$\begin{cases} E(d,h) \to 0 \text{ as } h \to 0+ & \text{if } \omega \cdot x < h_D(\rho), \\ E(d,h) \to \infty \text{ as } h \to 0+ & \text{if } \omega \cdot x > h_D(\rho), \end{cases} \tag{2.3.6}$$

where $h_D(\rho) = \inf_{x\in D} x \cdot \rho$ is the support function we have mentioned before.

2. *Meyers' L^p-estimates method*: This method was introduced in [55]. Similarly, since the upper bound of $E(f)$ can be obtained by the standard elliptic regularity, we only need to take care of the lower bound of $E(f)$. Recall that w is the reflected solution of (2.3.3), and in [39], the author derived the following estimates (Meyers' L^p estimates): Assume $D \Subset \Omega$ and ∂D is Lipschitz. For every $p_0 > 2$, there exists a positive constant C_{p_0} independent of w and u_0

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such that

$$\|w\|_{L^2(\Omega)} \leq C_{p_0}\|u_0\|_{W^{1,p}(D)}, \tag{2.3.7}$$

for $p \in (\frac{6}{5}, p_0]$. In addition, by (2.3.7), we have

$$E(f) \geq c\int_{\Omega} |\nabla u_0|^2 dx - c\|u_0\|^2_{W^{1,p}(D)}.$$

In [55], the authors used a decomposition technique to obtain the lower bound of $E(f)$, and we will give details in the next chapter (section 3.4.2). Note that the key point is that we only need ∂D is Lipschitz. In summary, we can use the Meyers' L^p estimates to obtain the same result (2.3.6). For more enclosure methods for the Helmholtz-type equations, we refer readers to the survey paper [65].

From Laplacian leading term to general elliptic operator

Until now, we only considered the case when the mathematical models with the Laplacian as the leading order term. For the leading term - Laplacian, we call this mathematical model to be *isotropic*. In order to consider more general situation, we need to consider the equations or systems with non-Laplacian leading terms and we call the case to be *anisotropic*. However, the anisotropy of the non-Laplacian prevents us from constructing CGO solutions by using the standard methods. As a result, in [48], the authors constructed another special type of solutions which is called the *oscillating-decaying* (OD) solutions. The OD solutions are also useful in the inverse problems, especially for the reconstruction problems. In two-dimensional case, we can use the isothermal coordinates to transform a general second order elliptic equation into Laplacian type equations. However, for three-dimensional case, we do not know how to construct CGO solutions yet, we will use OD solutions to reconstruct the unknown obstacles. We will give all the details in the next section.

2.4 The enclosure-type method: Second order anisotropic elliptic equations

In this section, we develop an enclosure-type reconstruction scheme to identify penetrable obstacles in acoustic waves with anisotropic medium in $\mathbb{R}^3$. The main difficulty of treating this problem lies in the fact that there are no complex geometrical optics solutions available for the acoustic equation with anisotropic medium in $\mathbb{R}^3$. Instead, we will use another type of special solutions called oscillating-decaying solutions. Even though that oscillating-decaying solutions are defined only on the half space, we are able to give necessary boundary inputs by the Runge approximation

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property. Moreover, since we are considering a Helmholtz-type equation, we turn to Meyers' L^p estimate to compare the integrals coming from oscillating-decaying solutions and those from reflected solutions.

2.4.1 Problem for the anisotropic elliptic equation

In the study of inverse problems, we are interested in the special type of solutions for elliptic equations or systems which play an essential role since the pioneer work of Caldéron. Sylvester and Uhlmann [57] introduced complex geometric optics (CGO) solutions to solve the inverse boundary value problems of the conductivity equation. Based on CGO solutions, Ikehata proposed the so called enclosure method to reconstruct the impenetrable obstacle, for more details, see [17, 20, 21]. There are many results concerning this reconstruction algorithm, such as [43, 62]. The researchers constructed CGO-solutions with polynomial-type phase function of the Helmholtz equation $\Delta u + k^2 u = 0$ or the elliptic system with the Laplacian as the principal part.

When the medium is anisotropic, we need to consider more general elliptic equations, such as anisotropic scalar elliptic equation in a bounded domain $\Omega \subset \mathbb{R}^3$,

$$\nabla \cdot (A^0(x)\nabla u) + k^2 u = 0, \tag{2.4.1}$$

where $A^0(x) = (a^0_{ij}(x))$, $a^0_{ij}(x) = a^0_{ji}(x)$, and we assume the uniform ellipticity condition, that is, for all $\xi = (\xi_1, \xi_2, \cdots \xi_n) \in \mathbb{R}^n$, $\lambda^0|\xi|^2 \leq \sum_{i,j} a^0_{ij}(x)\xi_i\xi_j \leq \Lambda^0|\xi|^2$ and $x \in \Omega$. In two dimensional case, we can transform (2.4.1) to an isotropic equation by using isothermal coordinates, then we can apply the CGO-solutions for this case, which can be found in [58]. When $\Omega \subset \mathbb{R}^3$, we cannot directly transform (2.4.1) to an isotropic equation as we do in $\mathbb{R}^2$, thus we need to use the oscillating-decaying solutions in our reconstruction algorithm. In [46], the author introduced oscillating-decaying solutions for the conductivity equation $\nabla \cdot (\gamma(x)\nabla u) = 0$ with the isotropic conductivity.

We make the following assumptions.

1. Let $\Omega \subset \mathbb{R}^3$ be a bounded C^∞-smooth domain and assume that D is an unknown obstacle with Lipschitz boundary such that $D \Subset \Omega \subset \mathbb{R}^3$ with an inhomogeneous index of refraction subset of a larger domain Ω.

2. Let $A(x) = (a_{ij}(x))$ and $A^0(x) = (a^0_{ij}(x))$ be symmetric matrices with $a_{ij}(x) = a^0_{ij}(x) + \widetilde{a_{ij}}(x)\chi_D$, where each $a^0_{ij}(x)$ is bounded C^∞-smooth, $\widetilde{A}(x) = (\widetilde{a_{ij}}(x)) \in L^\infty(D)$ is regarded as a perturbation in the unknown obstacle D and $\widetilde{A}(x)\xi \cdot \xi \geq \widetilde{\lambda}|\xi|^2$ for any $\xi \in \mathbb{R}^3$ and $x \in D$ with some $\widetilde{\lambda} > 0$. Further $A(x)$ satisfies $\lambda|\xi|^2 \leq A(x)\xi \cdot \xi \leq \Lambda|\xi|^2$ for some constants

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$0 < \lambda \leq \Lambda.$

Now, let $k > 0$ and consider the steady state anisotropic acoustic wave equation with Dirichlet boundary condition

$$\begin{cases} \nabla \cdot (A(x)\nabla u) + k^2 u = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega. \end{cases} \tag{2.4.2}$$

For the unperturbed case, we have

$$\begin{cases} \nabla \cdot (A^0(x)\nabla u_0) + k^2 u_0 = 0 & \text{in } \Omega \\ u_0 = f & \text{on } \partial\Omega. \end{cases} \tag{2.4.3}$$

In this paper, we assume that k^2 is not a Dirichlet eigenvalue of the operator $-\nabla \cdot (A\nabla\bullet)$ and $-\nabla \cdot (A^0\nabla\bullet)$ in Ω. It is known that for any $f \in H^{1/2}(\partial\Omega)$, there exists a unique solution u to (2.4.2). We define the Dirichlet-to-Neumann map $\Lambda_D : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ in the anisotropic case as the following.

Definition 2.2. $\Lambda_D f := A\nabla u \cdot \nu = \sum_{i.j=1}^{3} a_{ij}\partial_j u \cdot \nu_i$ and $\Lambda_\emptyset f := A^0\nabla u_0 \cdot \nu = \sum_{i.j=1}^{3} a_{ij}\partial_j u_0 \cdot \nu_i$, where $\nu = (\nu_1, \nu_2, \nu_3)$ is a unit outer normal on $\partial\Omega$.

Inverse problem: Identify the location and the convex hull of D from the DN-map Λ_D.

The domain D can also be considered as an inclusion embedded in Ω. The aim of this work is to give a reconstruction algorithm for this problem. Note that the information on the medium parameter $\widetilde{A}(x) = (\widetilde{a_{ij}}(x))$ inside D is not known a priori.

The main tool in our reconstruction method is the oscillating-decaying solutions for the second order anisotropic elliptic differential equations. We use the results from the paper [47] to construct the oscillating-decaying solution. In the next section, we will construct the oscillating-decaying solutions for anisotropic elliptic equations. Note that even if $k = 0$, which means the equation is $\nabla \cdot (A(x)\nabla u) = 0$, we do not know of any CGO-type solutions. Roughly speaking, given a hyperplane, an oscillating-decaying solution is oscillating very rapidly along this plane and decaying exponentially in the direction transverse to the same plane. Oscillating-decaying solutions are special solutions with the imaginary part of the phase function non-negative. Note that the domain of the oscillating-decaying solutions is not over the whole Ω, so we need to extend such solutions to the whole domain. Fortunately, the Runge approximation property provides us a good approach to extend this special solution.

In Ikehata's work, the CGO-solutions are used to define the indicator function (see [21] for the definition). In order to use the oscillating-decaying solutions to the inverse problem of identifying an inclusion, we employ the Runge approximation property to redefine the indicator function. It

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was Lax [31] that first recognized the Runge approximation property is a consequence of the weak unique continuation property. In our case, it is clear that the anisotropic elliptic equation has the weak unique continuation property if the leading part is Lipschitz continuous. Finally, the main theorem and reconstruction algorithm will be presented in the end of this chapter.

2.4.2 Construction of oscillating-decaying solutions

In this section, we follow the paper [47] to construct the oscillating-decaying solution in the anisotropic elliptic equations. In our case, since we only consider a scalar elliptic equation, its construction is simpler than that in [47]. Consider the anisotropic Helmholtz type equation

$$\nabla \cdot (A(x)\nabla u) + k^2 u = 0 \text{ in } \Omega. \tag{2.4.4}$$

Note that the oscillating-decaying solutions of

$$\nabla \cdot (A(x)\nabla u) = 0 \text{ in } \Omega$$

will have the same form as the equation (2.4.4), which means the lower order term k^2u will not affect the representation of the oscillating-decaying solutions, the following are the construction details. Now, we assume that the domain Ω is an open, bounded smooth domain in $\mathbb{R}^3$ and the coefficients $A(x) = (a_{ij}(x))$ is a symmetric 3×3 matrix satisfying uniformly elliptic condition, which means $\sum_{i.j=1}^{3} a_{ij}(x)\xi_i\xi_j \geq c|\xi|^2$, $\forall \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ for some $c > 0$.

Assume that

$$A(x) = (a_{ij}(x)) \in B^\infty(\mathbb{R}^3) = \{f \in C^\infty(\mathbb{R}^3) : \partial^\alpha f \in L^\infty(\mathbb{R}^3),\ \forall \alpha \in \mathbb{Z}_+^3\}$$

is the anisotropic coefficients. Note that $A(x) \in B^\infty$ already implies that A is Lipschitz continuous and the Lipschitz continuity property of $A(x)$ will apply the weak unique continuation property of (2.4.4) (see [15] for example).

We give several notations as follows. Assume that $\Omega \subset \mathbb{R}^3$ is an open set with smooth boundary and $\omega \in S^2$ is given. Let $\eta \in S^2$ and $\zeta \in S^2$ be chosen so that $\{\eta, \zeta, \omega\}$ forms an orthonormal system of $\mathbb{R}^3$. We then denote $x' = (x \cdot \eta, x \cdot \zeta)$. Let $t \in \mathbb{R}$, $\Omega_t(\omega) = \Omega \cap \{x \cdot \omega > t\}$ and $\Sigma_t(\omega) = \Omega \cap \{x \cdot \omega = t\}$ be a non-empty open set. We consider a scalar function $u_{\chi_t,t,b,N,\omega}(x,\tau) := u(x,\tau) \in C^\infty(\overline{\Omega_t(\omega)} \backslash \overline{\Sigma_t(\omega)}) \cap C^0(\overline{\Omega_t(\omega)})$ with $\tau \gg 1$ satisfying:

$$\begin{cases} L_A u = \nabla \cdot (A(x)\nabla u) + k^2 u = 0 & \text{in } \Omega_t(\omega) \\ u = e^{i\tau x\cdot\xi}\{\chi_t(x')Q_t(x')b + \beta_{\chi_t,t,b,N,\omega}\} & \text{on } \Sigma_t(\omega), \end{cases} \tag{2.4.5}$$

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where $\xi \in S^2$ lying in the span of η and ζ and fixed $\chi_t(x') \in C_0^\infty(\mathbb{R}^2)$ with $\mathrm{supp}(\chi_t) \subset \Sigma_t(\omega)$, $Q_t(x')$ is a nonzero smooth function and $0 \neq b \in \mathbb{C}^3$. Moreover, $\beta_{\chi_t,b,t,N,\omega}(x',\tau)$ is a smooth function supported in $\mathrm{supp}(\chi_t)$ satisfying:

$$\|\beta_{\chi_t,b,t,N,\omega}(\cdot,\tau)\|_{L^2(\mathbb{R}^2)} \leq c\tau^{-1}$$

for some constant $c > 0$. From now on, we use c, c' and their capitals to denote general positive constants whose values may vary from line to line. As in the paper [47], $u_{\chi_t,b,t,N,\omega}$ can be written as

$$u_{\chi_t,b,t,N,\omega} = w_{\chi_t,b,t,N,\omega} + r_{\chi_t,b,t,N,\omega}$$

with

$$w_{\chi_t,b,t,N,\omega} = \chi_t(x')Q_t e^{i\tau x\cdot\xi} e^{-\tau(x\cdot\omega - t)A_t(x')} b + \gamma_{\chi_t,b,t,N,\omega}(x,\tau) \tag{2.4.6}$$

and $r_{\chi_t b,t,N,\omega}$ satisfying

$$\|r_{\chi_t,b,t,N,\omega}\|_{H^1(\Omega_t(\omega))} \leq c\tau^{-N-1/2}, \tag{2.4.7}$$

where $A_t(\cdot) \in B^\infty(\mathbb{R}^2)$ is a complex function with its real part $\mathrm{Re}A_t(x') > 0$, and $\gamma_{\chi_t,b,t,N,\omega}$ is a smooth function supported in $\mathrm{supp}(\chi_t)$ satisfying

$$\|\partial_x^\alpha \gamma_{\chi_t,b,t,N,\omega}\|_{L^2(\Omega_s(\omega))} \leq c\tau^{|\alpha|-3/2} e^{-\tau(s-t)a} \tag{2.4.8}$$

for $|\alpha| \leq 1$ and $s \geq t$, where $a > 0$ is some constant depending on $A_t(x')$.

Without loss of generality, we consider the special case where $t = 0$, $\omega = e_3 = (0,0,1)$ and choose $\eta = (1,0,0)$, $\zeta = (0,1,0)$. The general case can be obtained from this special case by change of coordinates. Define $L = L_A$ and $\widetilde{M}\cdot = e^{-i\tau x'\cdot\xi'} L(e^{i\tau x'\cdot\xi'}\cdot)$, where $x' = (x_1, x_2)$ and $\xi' = (\xi_1, \xi_2)$ with $|\xi'| = 1$, then $\widetilde{M}$ is a differential operator. To be precise, by using $a_{jl} = a_{lj}$, we calculate $\widetilde{M}$ to be given by

$$\begin{aligned}
\widetilde{M} &= -\tau^2 \sum_{jl} a_{jl}\xi_j\xi_l + 2\tau \sum_{jl} a_{jl}(i\xi_l)\partial_j + \sum_{jl} a_{jl}\partial_j\partial_l \\
&\quad + \sum_{jl} (\partial_j a_{jl})(i\tau\xi_l) + \sum_{jl} (\partial_j a_{jl})\partial_l + k^2 \\
&= -\tau^2 \sum_{jl} a_{jl}\xi_j\xi_l + 2\tau \sum_{l} a_{3l}(i\xi_l)\partial_3 + a_{33}\partial_3\partial_3 \\
&\quad + 2\tau \sum_{j\neq 3,l} a_{jl}(i\xi_l)\partial_j + \sum_{j\neq 3,l\neq 3} a_{jl}\partial_j\partial_l \\
&\quad + \sum_{jl} (\partial_j a_{jl})(i\tau\xi_l) + \sum_{jl} (\partial_j a_{jl})\partial_l + k^2
\end{aligned}$$

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with $\xi_3 = 0$. Now, we want to solve

$$\widetilde{M}v = 0,$$

which is equivalent to $Mv = 0$, where $M = a_{33}^{-1}\widetilde{M}$. Now, we use the same idea in [47], define $\langle e, f\rangle = \sum_{ij} a_{ij} e_i f_j$, where $e = (e_1, e_2, e_3)$, $f = (f_1, f_2, f_3)$ and denote $\langle e, f\rangle_0 = \langle e, f\rangle|_{x_3=0}$. Let P be a differential operator, and we define the order of P, denoted by $ord(P)$, in the following sense:

$$\|P(e^{-\tau x_3 A(x')}\varphi(x')\|_{L^2(\mathbb{R}^3_+)} \leq c\tau^{ord(P)-1/2},$$

where $\mathbb{R}^3_+ = \{x_3 > 0\}$, $A(x')$ is a smooth complex function with its real part greater than 0 and $\varphi(x') \in C_0^\infty(\mathbb{R}^2)$. In this sense, similar to [47], we can see that τ, ∂_3 are of order 1, ∂_1, ∂_2 are of order 0 and x_3 is of order -1.

Now according to this order, the principal part M_2 (order 2) of M is:

$$M_2 = -\{D_3^2 + 2\tau \langle e_3, e_3\rangle_0^{-1} \langle e_3, \rho\rangle_0 D_3 + \tau^2 \langle e_3, e_3\rangle_0^{-1} \langle \rho, \rho\rangle_0\}$$

with $D_3 = -i\partial_3$ and $\rho = (\xi_1, \xi_2, 0)$. Note that the principal part M_2 does not involve the lower order term $k^2\cdot$, so we can follow all the constructions in the same procedures as in [47] and we omit details.

2.4.3 Runge approximation property

Definition 2.3. [31] Let L be a second order elliptic operator, solutions of an equation $Lu = 0$ are said to have the Runge approximation property if, whenever K and Ω are two simply connected domains with $K \subset \Omega$, any solution in K can be approximated uniformly in compact subsets of K by a sequence of solutions which can be extended as solution to Ω.

There are many applications for the Runge approximation property in inverse problems. Similar results for some elliptic operators can be found in [31], [37]. The following theorem is a classical result for Runge approximation property for second order elliptic equations.

Theorem 2.4. *(Runge approximation property) Let $L_0\cdot = \nabla(A^0(x)\nabla\cdot) + k^2\cdot$ be a second order elliptic differential operator with $A^0(x)$ to be Lipschitz. Assume that k^2 is not a Dirichlet eigenvalue of $-\nabla(A^0(x)\nabla\cdot)$ in Ω. Let O and Ω be two open bounded domains with smooth boundary in $\mathbb{R}^3$ such that $O \Subset \Omega$.*

Let $u_0 \in H^1(O)$ satisfy

$$L_0 u_0 = 0 \text{ in } O.$$

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Then for any compact subset $K \subset O$ and any $\epsilon > 0$, there exists $U \in H^1(\Omega)$ satisfying

$$L_0 U = 0 \text{ in } \Omega,$$

such that

$$\|u_0 - U\|_{H^1(K)} \leq \epsilon.$$

Proof. The proof is standard and it is based on the weak unique continuation property for the anisotropic second order elliptic operator L_0 and the Hahn-Banach theorem. For more details, how to derive the Runge approximation property from the weak unique continuation, we refer readers to [31] □

It remains to use the same ideas which comes from the reflected solutions. Here we use the useful elliptic estimates, which is called the *Meyers' L^p estimates.*

2.4.4 Meyers' L^p estimates and some identities

We need some estimates for solutions to some Dirichlet problems which will be used in next section. Recall that, for $f \in H^{1/2}(\partial\Omega)$, let u and u_0 be solutions to the Dirichlet problems (2.4.2) and (2.4.3), respectively. Note that $a_{ij}(x) = a^0_{ij}(x) + \widetilde{a_{ij}}(x)\chi_D$ and we set $w = u - u_0$, then w satisfies the Dirichlet problem

$$\begin{cases} \nabla \cdot (A(x)\nabla w) + k^2 w = -\nabla \cdot ((\widetilde{A}\chi_D)\nabla u_0) & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases} \tag{2.4.9}$$

where $A(x) = (a_{ij}(x))$, $A^0(x) = (a^0_{ij}(x))$ and $\widetilde{A}(x) = (\widetilde{a_{ij}}(x))$. Then we have some estimates for w.

Lemma 2.5. *There exists a positive constant C independent of w such that we have*

$$\|w\|_{L^2(\Omega)} \leq C\|\nabla w\|_{L^p(\Omega)}$$

for $\frac{6}{5} \leq p \leq 2$ if $n = 3$.

Proof. The proof follow from [55] by Freidrich's inequality, see [38] p.258 and use a standard elliptic regularity. □

Lemma 2.6. *There exists $\epsilon \in (0, 1)$, depending only on Ω, $A^0(x) = (a^0_{ij}(x))$ and $\widetilde{A}(x) = (\widetilde{a_{ij}}(x))$ such that*

$$\|\nabla w\|_{L^p(\Omega)} \leq C\|u_0\|_{W^{1,p}(D)}$$

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for $\max\{2-\epsilon, \frac{6}{5}\} < p \leq 2$ *if* $n = 3$.

Proof. The proof is also followed from [55]. Set $f := -(\widetilde{A}\chi_D)\nabla u_0$. Let w_0 be a solution of

$$\begin{cases} \nabla \cdot (A(x)\nabla w_0) = \nabla \cdot f & \text{in } \Omega, \\ w_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.4.10}$$

The following L^p-estimate of w_0, known as Meyers estimate, followed from [39], then we can get

$$\|\nabla w_0\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)} \tag{2.4.11}$$

for $p \in (\max\{2-\epsilon, \frac{6}{5}\}, 2]$, where $\epsilon \in (0,1)$ depends on Ω, $A^0(x) = (a^0_{ij}(x))$ and $\widetilde{A}(x) = (\widetilde{a_{ij}}(x))$. We set $W := w - w_0$, then since $w = w_0 + W$, we have

$$\|\nabla w\|_{L^p(\Omega)} \leq C(\|\nabla w_0\|_{L^p(\Omega)} + \|\nabla W\|_{L^p(\Omega)}). \tag{2.4.12}$$

Moreover, W satisfies

$$\begin{cases} \nabla \cdot (A(x)\nabla W) + k^2 W = -k^2 w_0 & \text{in } \Omega, \\ W = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.4.13}$$

By the standard elliptic regularity, we have

$$\|W\|_{H^1(\Omega)} \leq C\|w_0\|_{L^2(\Omega)}.$$

Thus, we get for $p \leq 2$,

$$\|\nabla W\|_{L^p(\Omega)} \leq C\|\nabla W\|_{L^2(\Omega)} \leq C\|W\|_{H^1(\Omega)} \leq C\|w_0\|_{L^2(\Omega)}. \tag{2.4.14}$$

By Sobolev embedding theorem, we get

$$\|w_0\|_{L^2(\Omega)} \leq C\|w_0\|_{W^{1,p}(\Omega)} \tag{2.4.15}$$

for $p \geq \frac{6}{5}$ if $n = 3$. Use Poincare's inequality in L^p spaces ($w_0|_{\partial\Omega} = 0$), we have

$$\|w_0\|_{L^2(\Omega)} \leq C\|\nabla w_0\|_{L^p(\Omega)} \tag{2.4.16}$$

for $p \geq \frac{6}{5}$ if $n = 3$. Combining (2.4.11) with (2.4.12), (2.4.14) and (2.4.16), we can obtain

$$\|\nabla w\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)} \leq C\|u_0\|_{W^{1,p}(D)}$$

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for $\max\{2-\epsilon, \frac{6}{5}\} < p \leq 2$ if $n = 3$. $\square$

Recall the Dirichlet-to-Neumann map which we have defined in the section 1: $\Lambda_D f := A\nabla u \cdot \nu$ and $\Lambda_\emptyset f := A^0 \nabla u_0 \cdot \nu$, where $\nu = (\nu_1, \nu_2, \nu_3)$ is a unit outer normal on $\partial\Omega$. We next prove some useful identities.

Lemma 2.7. *$\int_{\partial\Omega} (\Lambda_D - \Lambda_\emptyset) f\bar{f} d\sigma = Re \int_D \widetilde{A} \nabla u_0 \cdot \overline{\nabla u} dx$.*

Proof. It is clear that

$$\begin{aligned}
\int_{\partial\Omega} (A\nabla u) \cdot \nu \bar{\varphi} d\sigma &= \int_\Omega \nabla \cdot (A\nabla u \bar{\varphi}) dx \\
&= \int_\Omega \nabla \cdot (A\nabla u)\bar{\varphi} + A\nabla u \cdot \overline{\nabla \varphi} dx \\
&= -k^2 \int_\Omega u\bar{\varphi} dx + \int_\Omega A\nabla u \cdot \overline{\nabla \varphi} dx
\end{aligned}$$

$\forall \varphi \in H^1(\Omega)$. Since $u = u_0 = f$ on $\partial\Omega$, the left hand side of the identity has the same value whether we take $\varphi = u$ or $\varphi = u_0$, and it is equal to $\int_{\partial\Omega} \Lambda_D f\bar{f} d\sigma$.

$$\begin{aligned}
\int_{\partial\Omega} \Lambda_D f\bar{f} d\sigma &= -k^2 \int_\Omega u\overline{u_0} dx + \int_\Omega A\nabla u \cdot \overline{\nabla u_0} dx \\
&= -k^2 \int_\Omega |u|^2 dx + \int_\Omega A\nabla u \cdot \overline{\nabla u} dx.
\end{aligned}$$

The right hand side of the identity above is real. Hence, by taking the real part, we have

$$\int_{\partial\Omega} \Lambda_D f\bar{f} d\sigma = -k^2 \mathrm{Re} \int_\Omega u\overline{u_0} dx + \mathrm{Re} \int_\Omega A\nabla u \cdot \overline{\nabla u_0} dx$$

and

$$\int_{\partial\Omega} \Lambda_\emptyset f\bar{f} d\sigma = -k^2 \mathrm{Re} \int_\Omega u\overline{u_0} dx + \mathrm{Re} \int_\Omega A^0\nabla u \cdot \overline{\nabla u_0} dx.$$

Therefore, we have

$$\begin{aligned}
\int_{\partial\Omega} (\Lambda_D - \Lambda_\emptyset) f\bar{f} d\sigma &= \mathrm{Re} \int_\Omega (A - A^0)\nabla u \cdot \overline{\nabla u_0} dx \qquad (2.4.17) \\
&= \mathrm{Re} \int_D \widetilde{A} \nabla u \cdot \overline{\nabla u_0} dx.
\end{aligned}$$

$\square$

The estimates in the following lemma play an important role in our reconstruction algorithm.

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Lemma 2.8. *We have the following identities:*

$$\int_{\partial\Omega}(\Lambda_D-\Lambda_\emptyset)f\bar{f}d\sigma = -\int_\Omega A\nabla w\cdot\overline{\nabla w}dx+k^2\int_\Omega|w|^2dx + \int_D\widetilde{A}\nabla u_0\cdot\overline{\nabla u_0}dx, \tag{2.4.18}$$

$$\int_{\partial\Omega}(\Lambda_D-\Lambda_\emptyset)f\bar{f}d\sigma = \int_\Omega A^0\nabla w\cdot\overline{\nabla w}dx-k^2\int_\Omega|w|^2dx + \int_D\widetilde{A}\nabla u\cdot\overline{\nabla u}dx. \tag{2.4.19}$$

In particular, we have

$$\int_{\partial\Omega}(\Lambda_D-\Lambda_\emptyset)f\bar{f}d\sigma\le k^2\int_\Omega|w|^2dx+C\int_D|\nabla u_0|^2dx, \tag{2.4.20}$$

$$\int_{\partial\Omega}(\Lambda_D-\Lambda_\emptyset)f\bar{f}d\sigma\ge c\int_\Omega|\nabla u_0|^2dx-k^2\int_\Omega|w|^2dx, \tag{2.4.21}$$

where $C>0$ is a constant depending on $\widetilde{A}(x)$ and c is a constant depending on A, A^0 and $\widetilde{A}$.

Proof. Multiplying the identity

$$\nabla\cdot(A(x)\nabla w)+k^2w+\nabla\cdot(\widetilde{A}\chi_D\nabla u_0)=0$$

by $\bar{w}$ and integrating over Ω, we get

$$\begin{aligned}
0 &= \int_\Omega\nabla\cdot(A\nabla w)\bar{w}dx-\int_\Omega\nabla\cdot(\widetilde{A}\chi_D\nabla u_0)\bar{w}dx+k^2\int_\Omega|w|^2dx\\
&= -\int_\Omega A\nabla w\cdot\overline{\nabla w}dx+\int_{\partial\Omega}(A\nabla w\cdot\nu)\bar{w}d\sigma-\int_\Omega\widetilde{A}\chi_D\nabla u_0\cdot\overline{\nabla w}dx\\
&\quad+\int_{\partial\Omega}(\widetilde{A}\chi_D\nabla u_0\cdot\nu)\bar{w}d\sigma+k^2\int_\Omega|w|^2dx\\
&= -\int_\Omega A\nabla w\cdot\overline{\nabla w}dx-\int_D\widetilde{A}\nabla u_0\cdot\overline{\nabla w}dx+k^2\int_\Omega|w|^2dx\\
&= -\int_\Omega A\nabla w\cdot\overline{\nabla w}dx-\int_D\widetilde{A}\nabla u_0\cdot\overline{\nabla u}dx+k^2\int_\Omega|w|^2dx\\
&\quad+\int_D\widetilde{A}\nabla u_0\cdot\overline{\nabla u_0}dx,
\end{aligned}$$

and use (2.4.17) we can obtain

$$\int_{\partial\Omega}(\Lambda_D-\Lambda_\emptyset)f\bar{f}d\sigma=-\int_\Omega A\nabla w\cdot\overline{\nabla w}dx+\int_D\widetilde{A}\nabla u_0\cdot\overline{\nabla u_0}dx+k^2\int_\Omega|w|^2dx.$$

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Similarly, multiplying the identity

$$\nabla \cdot (\widetilde{A}\chi_D \nabla u) + \nabla \cdot (A^0 \nabla w) + k^2 w = 0$$

by $\bar{w}$ and integrating over Ω, we get

$$\begin{aligned} 0 &= \int_\Omega \nabla \cdot (\widetilde{A}\chi_D \nabla u)\bar{w}dx + \int_\Omega \nabla \cdot (A^0\nabla w)\bar{w}dx + k^2\int_\Omega |w|^2 dx \\ &= -\int_D \widetilde{A}\nabla u \cdot \overline{\nabla w}dx - \int_\Omega A^0\nabla w \cdot \overline{\nabla w}dx + k^2 \int_\Omega |w|^2 dx \\ &= -\int_D \widetilde{A}\nabla u \cdot \overline{\nabla u}dx + \int_D \widetilde{A}\nabla u \cdot \overline{\nabla u_0}dx + k^2\int_\Omega |w|^2 dx \\ &\quad -\int_\Omega A^0 \nabla w \cdot \overline{\nabla w}dx, \end{aligned}$$

and use (2.4.17) again, we can obtain

$$\int_{\partial\Omega} (\Lambda_D - \Lambda_\emptyset) f\bar{f}d\sigma = \int_\Omega A^0\nabla w \cdot \overline{\nabla w}dx - k^2 \int_\Omega |w|^2 dx + \int_D \widetilde{A}\nabla u \cdot \overline{\nabla u}dx.$$

For the remaining part, (2.4.20) is an easy consequence of (2.4.18)

$$\begin{aligned} \int_{\partial\Omega} (\Lambda_D - \Lambda_\emptyset) f\bar{f}d\sigma &\le k^2 \int_\Omega |w|^2 dx + \int_D \widetilde{A}\nabla u_0 \cdot \overline{\nabla u_0}dx \\ &= k^2\int_\Omega |w|^2 dx + C\int_D |\nabla u_0|^2 dx, \end{aligned}$$

since $\widetilde{A} \in L^\infty(D)$.

Finally, for the lower bound, we use

$$\begin{aligned} A^0\nabla w \cdot \overline{\nabla w} + \widetilde{A}\chi_D \nabla u \cdot \overline{\nabla u} &= A\nabla u \cdot \overline{\nabla u} - 2\mathrm{Re}A^0\nabla u \cdot \overline{\nabla u_0} + A^0 \nabla u_0 \cdot \overline{\nabla u_0} \\ &= A(\nabla u - (A)^{-1}A^0\nabla u_0)\cdot (\overline{\nabla u - (A)^{-1}A^0\nabla u_0}) \\ &\quad +(A^0 - (A^0)(A)^{-1}(A^0))\nabla u_0 \cdot \overline{\nabla u_0} \\ &\ge (A^0 - (A^0)(A)^{-1}(A^0))\nabla u_0 \cdot \overline{\nabla u_0} \\ &\ge c|\nabla u_0|^2, \end{aligned}$$

since $A(\nabla u - (A)^{-1}A^0\nabla u_0) \cdot (\overline{\nabla u - (A)^{-1}A^0\nabla u_0}) \ge 0$ and note that $A^0 - (A^0)(A)^{-1}(A^0) = A^0(A)^{-1}(A - A^0) = A^0(A)^{-1}\widetilde{A}\chi_D$ is a positive definite matrix by our previous assumptions in section 1. □

Before stating our main theorem, we need to estimate $\|w\|_{L^2(\Omega)}$. Fortunately, we can use Meyers L^p estimates to help us to overcome the difficulties (see Lemma 2.5 and Lemma 2.6). For the upper bound of $\int_{\partial\Omega}(\Lambda_D - \Lambda_\emptyset)f\bar{f}d\sigma$, see (2.4.19), we use $\|w\|_{L^2(\Omega)} \le C\|u_0\|_{W^{1,p}(D)}$ for $p \le 2$. Then we

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have

$$\int_{\partial\Omega}(\Lambda_D - \Lambda_\emptyset) f \bar{f} d\sigma \leq C \|u_0\|^2_{W^{1,2}(D)}. \tag{2.4.22}$$

By (2.4.21) and the Meyers L^p estimate $\|w\|_{L^2(\Omega)} \leq C\|u_0\|_{W^{1,p}(D)}$, we have

$$\int_{\partial\Omega}(\Lambda_D - \Lambda_\emptyset) f \bar{f} d\sigma \geq c \int_\Omega |\nabla u_0|^2 dx - c\|u_0\|^2_{W^{1,p}(D)}. \tag{2.4.23}$$

In this section, we introduce the Runge approximation property and a very useful elliptic estimate: Meyers L^p-estimates.

2.4.5 Runge approximation property

Definition 2.9. [31] Let L be a second order elliptic operator, solutions of an equation $Lu = 0$ are said to have the Runge approximation property if, whenever K and Ω are two simply connected domains with $K \subset \Omega$, any solution in K can be approximated uniformly in compact subsets of K by a sequence of solutions in Ω.

There are many applications for Runge approximation property in inverse problems. Similar results for some elliptic operators can be found in [31], [37]. The following theorem is a classical result for Runge approximation property for second order elliptic equations.

Theorem 2.10. *(Runge approximation property) Let $L_0 \cdot = \nabla \cdot (A^0(x)\nabla \cdot) + k^2 \cdot$ be a second order elliptic differential operator with $A^0(x)$ to be Lipschitz. Assume that k^2 is not a Dirichlet eigenvalue of $-\nabla(A^0(x)\nabla\cdot)$ in Ω. Let O and Ω be two open bounded domains with smooth boundary in $\mathbb{R}^3$ such that $O \Subset \Omega$ and $\Omega \backslash \overline{O}$ is connected.*

Let $u_0 \in H^1(O)$ satisfy

$$L_0 u_0 = 0 \text{ in } O.$$

Then for any compact subset $K \subset O$ and any $\epsilon > 0$, there exists $U \in H^1(\Omega)$ satisfying

$$L_0 U = 0 \text{ in } \Omega,$$

such that

$$\|u_0 - U\|_{H^1(K)} \leq \epsilon.$$

Proof. The proof is standard and it is based on the weak unique continuation property for the anisotropic second order elliptic operator L_0 and the Hahn-Banach theorem. For more details, how to derive the Runge approximation property from the weak unique continuation, we refer readers to [31] □

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2.4.6 Elliptic estimates and some identities

We need some estimates for solutions to some Dirichlet problems which will be used in next section. Recall that, for $f \in H^{1/2}(\partial\Omega)$, let u and u_0 be solutions to the Dirichlet problems (3.6.2) and (3.6.3), respectively. Note that $a_{ij}(x) = a^0_{ij}(x) + \widetilde{a_{ij}}(x)\chi_D$ and we set $w = u - u_0$, then w satisfies the Dirichlet problem

$$\begin{cases} \nabla \cdot (A(x)\nabla w) + k^2 w = -\nabla \cdot ((\widetilde{A}\chi_D)\nabla u_0) & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases} \tag{2.4.24}$$

where $A(x) = (a_{ij}(x))$, $A^0(x) = (a^0_{ij}(x))$ and $\widetilde{A}(x) = (\widetilde{a_{ij}}(x))$. Then in the following lemmas, we give some estimates for w.

Lemma 2.11. *There exists a positive constant C independent of w such that we have*

$$\|w\|_{L^2(\Omega)} \le C\|\nabla w\|_{L^p(\Omega)}$$

for $\frac{6}{5} \le p \le 2$ if $n = 3$.

Proof. The proof follows from [55] by Freidrich's inequality, see [38] p.258 and use a standard elliptic regularity. □

Lemma 2.12. *There exists $\epsilon \in (0,1)$, depending only on Ω, $A^0(x) = (a^0_{ij}(x))$ and $\widetilde{A}(x) = (\widetilde{a_{ij}}(x))$ such that*

$$\|\nabla w\|_{L^p(\Omega)} \le C\|u_0\|_{W^{1,p}(D)}$$

for $\max\{2-\epsilon, \frac{6}{5}\} < p \le 2$ if $n = 3$.

Proof. The proof also follows from [55]. Set $f := -(\widetilde{A}\chi_D)\nabla u_0$. Let w_0 be a solution of

$$\begin{cases} \nabla \cdot (A(x)\nabla w_0) = \nabla \cdot f & \text{in } \Omega, \\ w_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.4.25}$$

The following L^p-estimate of w_0, known as Meyers estimate, follows from [39],

$$\|\nabla w_0\|_{L^p(\Omega)} \le C\|f\|_{L^p(\Omega)} \tag{2.4.26}$$

for $p \in (\max\{2-\epsilon, \frac{6}{5}\}, 2]$, where $\epsilon \in (0,1)$ depends on Ω, $A^0(x) = (a^0_{ij}(x))$ and $\widetilde{A}(x) = (\widetilde{a_{ij}}(x))$. We set $W := w - w_0$, then since $w = w_0 + W$, we have

$$\|\nabla w\|_{L^p(\Omega)} \le \|\nabla w_0\|_{L^p(\Omega)} + \|\nabla W\|_{L^p(\Omega)}. \tag{2.4.27}$$

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Moreover, W satisfies

$$\begin{cases} \nabla \cdot (A(x)\nabla W) + k^2 W = -k^2 w_0 & \text{in } \Omega, \\ W = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.4.28}$$

By the standard elliptic regularity, we have

$$\|W\|_{H^1(\Omega)} \le C\|w_0\|_{L^2(\Omega)}.$$

Thus, we get for $p \le 2$,

$$\|\nabla W\|_{L^p(\Omega)} \le C\|\nabla W\|_{L^2(\Omega)} \le C\|W\|_{H^1(\Omega)} \le C\|w_0\|_{L^2(\Omega)}. \tag{2.4.29}$$

By Sobolev embedding theorem, we get

$$\|w_0\|_{L^2(\Omega)} \le C\|w_0\|_{W^{1,p}(\Omega)} \tag{2.4.30}$$

for $p \ge \frac{6}{5}$ if $n = 3$. Use Poincare's inequality in L^p spaces ($w_0|_{\partial\Omega} = 0$), we have

$$\|w_0\|_{L^2(\Omega)} \le C\|\nabla w_0\|_{L^p(\Omega)} \tag{2.4.31}$$

for $p \ge \frac{6}{5}$ if $n = 3$. Combining (2.4.26) with (2.4.27), (2.4.29) and (2.4.31), we can obtain

$$\|\nabla w\|_{L^p(\Omega)} \le C\|f\|_{L^p(\Omega)} \le C\|u_0\|_{W^{1,p}(D)}$$

for $\max\{2-\epsilon, \frac{6}{5}\} < p \le 2$ if $n = 3$. □

Recall the Dirichlet-to-Neumann map which we have defined in Section 1: $\Lambda_D f := A\nabla u \cdot \nu$ and $\Lambda_\emptyset f := A^0 \nabla u_0 \cdot \nu$, where $\nu = (\nu_1, \nu_2, \nu_3)$ is a unit outer normal on $\partial\Omega$.

We next prove some useful identities.

Lemma 2.13. *$\int_{\partial\Omega} (\Lambda_D - \Lambda_\emptyset) f\bar{f} d\sigma = Re \int_D \widetilde{A}\nabla u_0 \cdot \overline{\nabla u} dx$.*

Proof. It is clear that

$$\begin{aligned} \int_{\partial\Omega} (A\nabla u)\cdot\nu\bar{\varphi}d\sigma &= \int_\Omega \nabla\cdot(A\nabla u\bar{\varphi})dx \\ &= \int_\Omega (\nabla\cdot(A\nabla u)\bar{\varphi} + A\nabla u\cdot\overline{\nabla\varphi})dx \\ &= -k^2\int_\Omega u\bar{\varphi}dx + \int_\Omega A\nabla u\cdot\overline{\nabla\varphi}dx \end{aligned}$$

for any $\varphi \in H^1(\Omega)$. Since $u = u_0 = f$ on $\partial\Omega$, the left hand side of the identity has the same value

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whether we take $\varphi = u$ or $\varphi = u_0$, and it is equal to $\int_{\partial\Omega} \Lambda_D f \bar{f} d\sigma$. Hence we have

$$\begin{aligned}\int_{\partial\Omega} \Lambda_D f \bar{f} d\sigma &= -k^2 \int_\Omega u \overline{u_0} dx + \int_\Omega A \nabla u \cdot \overline{\nabla u_0} dx \\ &= -k^2 \int_\Omega |u|^2 dx + \int_\Omega A \nabla u \cdot \overline{\nabla u} dx.\end{aligned}$$

The right hand side of the above identity is real. Hence, by taking the real part, we have

$$\int_{\partial\Omega} \Lambda_D f \bar{f} d\sigma = -k^2 \mathrm{Re} \int_\Omega u \overline{u_0} dx + \mathrm{Re} \int_\Omega A \nabla u \cdot \overline{\nabla u_0} dx$$

and

$$\int_{\partial\Omega} \Lambda_\emptyset f \bar{f} d\sigma = -k^2 \mathrm{Re} \int_\Omega u \overline{u_0} dx + \mathrm{Re} \int_\Omega A^0 \nabla u \cdot \overline{\nabla u_0} dx.$$

Therefore, we have

$$\begin{aligned}\int_{\partial\Omega} (\Lambda_D - \Lambda_\emptyset) f \bar{f} d\sigma &= \mathrm{Re} \int_\Omega (A - A^0) \nabla u \cdot \overline{\nabla u_0} dx \\ &= \mathrm{Re} \int_D \widetilde{A} \nabla u \cdot \overline{\nabla u_0} dx.\end{aligned} \tag{2.4.32}$$

□

The estimates in the following lemma play an important role in our reconstruction algorithm.

Lemma 2.14. *We have the following identities:*

$$\begin{aligned}\int_{\partial\Omega} (\Lambda_D - \Lambda_\emptyset) f \bar{f} d\sigma &= -\int_\Omega A \nabla w \cdot \overline{\nabla w} dx + k^2 \int_\Omega |w|^2 dx \\ &\quad + \int_D \widetilde{A} \nabla u_0 \cdot \overline{\nabla u_0} dx,\end{aligned} \tag{2.4.33}$$

$$\begin{aligned}\int_{\partial\Omega} (\Lambda_D - \Lambda_\emptyset) f \bar{f} d\sigma &= \int_\Omega A^0 \nabla w \cdot \overline{\nabla w} dx - k^2 \int_\Omega |w|^2 dx \\ &\quad + \int_D \widetilde{A} \nabla u \cdot \overline{\nabla u} dx.\end{aligned} \tag{2.4.34}$$

In particular, we have

$$\int_{\partial\Omega} (\Lambda_D - \Lambda_\emptyset) f \bar{f} d\sigma \leq k^2 \int_\Omega |w|^2 dx + C \int_D |\nabla u_0|^2 dx, \tag{2.4.35}$$

$$\int_{\partial\Omega} (\Lambda_D - \Lambda_\emptyset) f \bar{f} d\sigma \geq c \int_D |\nabla u_0|^2 dx - k^2 \int_\Omega |w|^2 dx, \tag{2.4.36}$$

where $C > 0$ is a constant depending on $\widetilde{A}(x)$ and c is a constant depending on A, A^0 and $\widetilde{A}$.

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Proof. Multiplying the identity

$$\nabla\cdot(A(x)\nabla w)+k^2w+\nabla\cdot(\widetilde{A}\chi_D\nabla u_0)=0$$

by $\bar{w}$ and integrating over Ω, we get

$$\begin{aligned}
0 &= \int_\Omega \nabla\cdot(A\nabla w)\bar{w}dx+\int_\Omega \nabla\cdot(\widetilde{A}\chi_D\nabla u_0)\bar{w}dx+k^2\int_\Omega |w|^2dx \\
&= -\int_\Omega A\nabla w\cdot\overline{\nabla w}dx+\int_{\partial\Omega}(A\nabla w\cdot\nu)\bar{w}d\sigma-\int_\Omega \widetilde{A}\chi_D\nabla u_0\cdot\overline{\nabla w}dx \\
&\quad +\int_{\partial\Omega}(\widetilde{A}\chi_D\nabla u_0\cdot\nu)\bar{w}d\sigma+k^2\int_\Omega|w|^2dx \\
&= -\int_\Omega A\nabla w\cdot\overline{\nabla w}dx-\int_D \widetilde{A}\nabla u_0\cdot\overline{\nabla w}dx+k^2\int_\Omega|w|^2dx \\
&= -\int_\Omega A\nabla w\cdot\overline{\nabla w}dx-\int_D \widetilde{A}\nabla u_0\cdot\overline{\nabla u}dx+k^2\int_\Omega|w|^2dx \\
&\quad +\int_D \widetilde{A}\nabla u_0\cdot\overline{\nabla u_0}dx,
\end{aligned}$$

and use (2.4.32) we can obtain

$$\int_{\partial\Omega}(\Lambda_D-\Lambda_\emptyset)f\bar{f}d\sigma=-\int_\Omega A\nabla w\cdot\overline{\nabla w}dx+\int_D\widetilde{A}\nabla u_0\cdot\overline{\nabla u_0}dx+k^2\int_\Omega|w|^2dx.$$

Similarly, multiplying the identity

$$\nabla\cdot(\widetilde{A}\chi_D\nabla u)+\nabla\cdot(A^0\nabla w)+k^2w=0$$

by $\bar{w}$ and integrating over Ω, we get

$$\begin{aligned}
0 &= \int_\Omega \nabla\cdot(\widetilde{A}\chi_D\nabla u)\bar{w}dx+\int_\Omega\nabla\cdot(A^0\nabla w)\bar{w}dx+k^2\int_\Omega|w|^2dx \\
&= -\int_D\widetilde{A}\nabla u\cdot\overline{\nabla w}dx-\int_\Omega A^0\nabla w\cdot\overline{\nabla w}dx+k^2\int_\Omega|w|^2dx \\
&= -\int_D\widetilde{A}\nabla u\cdot\overline{\nabla u}dx+\int_D\widetilde{A}\nabla u\cdot\overline{\nabla u_0}dx+k^2\int_\Omega|w|^2dx \\
&\quad -\int_\Omega A^0\nabla w\cdot\overline{\nabla w}dx,
\end{aligned}$$

and use (2.4.32) again, we can obtain

$$\int_{\partial\Omega}(\Lambda_D-\Lambda_\emptyset)f\bar{f}d\sigma=\int_\Omega A^0\nabla w\cdot\overline{\nabla w}dx-k^2\int_\Omega|w|^2dx+\int_D\widetilde{A}\nabla u\cdot\overline{\nabla u}dx.$$

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For the remaining part, (2.4.35) is an easy consequence of (2.4.33)

$$
\begin{aligned}
\int_{\partial\Omega}(\Lambda_D-\Lambda_\emptyset)f\bar{f}d\sigma &\le k^2\int_\Omega |w|^2dx+\int_D \widetilde{A}\nabla u_0\cdot\overline{\nabla u_0}dx \\
&= k^2\int_\Omega |w|^2dx+C\int_D |\nabla u_0|^2dx,
\end{aligned}
$$

since $\widetilde{A}\in L^\infty(D)$.

Finally, for the lower bound, we use

$$
\begin{aligned}
A^0\nabla w\cdot\overline{\nabla w}+\widetilde{A}\chi_D\nabla u\cdot\overline{\nabla u} &= A\nabla u\cdot\overline{\nabla u}-2\mathrm{Re}A^0\nabla u\cdot\overline{\nabla u_0}+A^0\nabla u_0\cdot\overline{\nabla u_0} \\
&= A(\nabla u-(A)^{-1}A^0\nabla u_0)\cdot(\overline{\nabla u-(A)^{-1}A^0\nabla u_0}) \\
&\quad +(A^0-(A^0)(A)^{-1}(A^0))\nabla u_0\cdot\overline{\nabla u_0} \\
&\ge (A^0-(A^0)(A)^{-1}(A^0))\nabla u_0\cdot\overline{\nabla u_0} \\
&\ge c|\nabla u_0|^2,
\end{aligned}
$$

since $A(\nabla u-(A)^{-1}A^0\nabla u_0)\cdot(\overline{\nabla u-(A)^{-1}A^0\nabla u_0})\ge 0$ and note that $A^0-(A^0)(A)^{-1}(A^0)=A^0(A)^{-1}(A-A^0)=A^0(A)^{-1}\widetilde{A}\chi_D$ is a positive definite matrix by our previous assumptions in section 1. $\square$

Applying Lemma 3.3 to (2.4.35),

$$
\int_{\partial\Omega}(\Lambda_D-\Lambda_\emptyset)f\bar{f}d\sigma\le C\|u_0\|^2_{W^{1,2}(D)}. \tag{2.4.37}
$$

By (2.4.36) and the Meyers L^p estimate $\|w\|_{L^2(\Omega)}\le C\|u_0\|_{W^{1,p}(D)}$, we have

$$
\int_{\partial\Omega}(\Lambda_D-\Lambda_\emptyset)f\bar{f}d\sigma\ge c\int_D|\nabla u_0|^2dx-c\|u_0\|^2_{W^{1,p}(D)}. \tag{2.4.38}
$$

2.4.7 Detecting the convex hull of the unknown obstacle

We give the reconstruction algorithm in the following.

Main result

Recall that we have constructed the oscillating-decaying solutions in section 2, and note that this solution can not be defined on the whole domain, that is, the oscillating-decaying solutions $u_{\chi_t,b,t,N,\omega}(x,\tau)$ only defined on $\Omega_t(\omega)\subsetneq\Omega$. Nevertheless, with the help of the Runge approximation property, we can only determine the convex hull of the unknown obstacle D by$\Lambda_D f$ for infinitely many f.

We define B to be an open ball in $\mathbb{R}^3$ such that $\overline{\Omega}\subset B$. Assume that $\widetilde{\Omega}\subset\mathbb{R}^3$ is an open smooth

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domain with $\overline{B} \subset \widetilde{\Omega}$. As in the section 2, set $\omega \in S^2$ and $\{\eta, \zeta, \omega\}$ forms an orthonormal basis of $\mathbb{R}^3$. Suppose $t_0 = \inf_{x \in D} x \cdot \omega = x_0 \cdot \omega$, where $x_0 = x_0(\omega) \in \partial D$. For any $t \leq t_0$ and $\epsilon > 0$ small enough, we can construct

$$\begin{aligned} u_{\chi_{t-\epsilon},b,t-\epsilon,N,\omega} &= \chi_{t-\epsilon}(x')Q_{t-\epsilon}(x')e^{i\tau x\cdot\xi}e^{-\tau(x\cdot\omega-(t-\epsilon))A_{t-\epsilon}(x')}b + \gamma_{\chi_{t-\epsilon},b,t-\epsilon,N,\omega} \\ &\quad + r_{\chi_{t-\epsilon},b,t-\epsilon,N,\omega} \end{aligned}$$

to be the oscillating-decaying solution for $\nabla\cdot(A^0(x)\nabla\cdot)+k^2\cdot$ in $B_{t-\epsilon}(\omega) = B \cap \{x\cdot\omega > t-\epsilon\}$, where $\chi_{t-\epsilon}(x') \in C_0^\infty(\mathbb{R}^2)$ and $b \in \mathbb{C}$. Note that in section 2, we have assumed the leading coefficient $A^0(x) \in B^\infty(\mathbb{R}^3)$. Similarly, we have the oscillating-decaying solution

$$u_{\chi_t,b,t,N,\omega}(x,\tau) = \chi_t(x')Q_t e^{i\tau x\cdot\xi}e^{-\tau(x\cdot\omega-t)A_t(x')}b + \gamma_{\chi_t,b,t,N,\omega}(x,\tau) + r_{\chi_t,b,t,N,\omega}$$

for L_{A^0} in $B_t(\omega)$. In fact, for any τ, $u_{\chi_{t-\epsilon},b,t-\epsilon,N,\omega}(x,\tau) \to u_{\chi_t,b,t,N,\omega}(x,\tau)$ in an appropriate sense as $\epsilon \to 0$. For details, we refer readers to consult all the details and results in [47], and we list consequences in the following.

$$\chi_{t-\epsilon}(x')Q_{t-\epsilon}(x')e^{i\tau x\cdot\xi}e^{-\tau(x\cdot\omega-(t-\epsilon))A_{t-\epsilon}(x')}b \to \chi_t(x')Q_t e^{i\tau x\cdot\xi}e^{-\tau(x\cdot\omega-t)A_t(x')}b$$

in $H^2(B_t(\omega))$ as ϵ tends to 0,

$$\gamma_{\chi_{t-\epsilon},b,t-\epsilon,N,\omega} \to \gamma_{\chi_t,b,t,N,\omega}$$

in $H^2(B_t(\omega))$ as ϵ tends to 0, and finally,

$$r_{\chi_{t-\epsilon},b,t-\epsilon,N,\omega} \to r_{\chi_t,b,t,N,\omega}$$

in $H^1(B_t(\omega))$ as ϵ tends to 0.

Obviously, $B_{t-\epsilon}(\omega)$ is a convex set and $\overline{\Omega_t(\omega)} \subset B_{t-\epsilon}(\omega)$ for all $t \leq t_0$. By using the Runge approximation property, we can see that there exists a sequence of functions $\tilde{u}_{\epsilon,j}$, $j = 1, 2, \cdots$, such that

$$\tilde{u}_{\epsilon,j} \to u_{\chi_{t-\epsilon},b,t-\epsilon,N,\omega} \text{ in } H^1(\overline{\Omega_t(\omega)}),$$

where $\tilde{u}_{\epsilon,j} \in H^1(\widetilde{\Omega})$ satisfy $L_{A^0}\tilde{u}_{\epsilon,j} = 0$ in $\widetilde{\Omega}$ for all ϵ, j. Define the indicator function $I(\tau, \chi_t, b, t, \omega)$ by the formula:

$$I(\tau, \chi_t, b, t, \omega) = \lim_{\epsilon\to 0}\lim_{j\to\infty}\int_{\partial\Omega}(\Lambda_D - \Lambda_\emptyset)f_{\epsilon,j}\overline{f_{\epsilon,j}}d\sigma,$$

where $f_{\epsilon,j} = \tilde{u}_{\epsilon,j}|_{\partial\Omega}$.

Now the characterization of the convex hull of D is based on the following theorem:

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Theorem 2.15. *(1) If $t < t_0$, then for any $\chi_t \in C_0^\infty(\mathbb{R}^2)$ and $b \in \mathbb{C}^3$, we have*

$$\limsup_{\tau\to\infty} |I(\tau, \chi_t, b, t, \omega)| = 0.$$

(2) If $t = t_0$, then for any $\chi_{t_0} \in C_0^\infty(\mathbb{R}^2)$ with $x_0' = (x_0 \cdot \eta, x_0 \cdot \zeta)$ being an interior point of supp(χ_{t_0}) and $0 \neq b \in \mathbb{C}$, we have

$$\liminf_{\tau\to\infty} |I(\tau, \chi_{t_0}, b, t_0, \omega)| > 0.$$

Proof. First of all, note that we have a sequence of functions $\{\tilde{u}_{\epsilon,j}\}$ satisfies the equation $\nabla \cdot (A^0 \nabla u) + k^2 u = 0$ in Ω, as in the beginning of the section 3, let $w_{\epsilon,j} = u - \tilde{u}_{\epsilon,j}$, then $w_{\epsilon,j}$ satisfies the Dirichlet problem

$$\begin{cases} \nabla \cdot (A(x)\nabla w_{\epsilon,j}) + k^2 w_{\epsilon,j} = -\nabla \cdot (\widetilde{A}\chi_D \nabla \tilde{u}_{\epsilon,j}) & \text{in } \Omega, \\ w_{\epsilon,j} = 0 & \text{on } \partial\Omega. \end{cases}$$

So we can apply (2.4.22) directly, which means

$$\int_{\partial\Omega} (\Lambda_D - \Lambda_\emptyset) f_{\epsilon,j} \overline{f_{\epsilon,j}} d\sigma \leq C \|\tilde{u}_{\epsilon,j}\|^2_{H^1(D)} \text{ with } f_{\epsilon,j} = \tilde{u}_{\epsilon,j}|_{\partial\Omega},$$

where the last inequality obtained by the Holder's inequality.

By the Runge approximation property we have

$$\tilde{u}_{\epsilon,j} \to u_{\chi_{t-\epsilon}, b, t-\epsilon, N, \omega} \text{ in } H^1(\overline{B_t(\omega)})$$

as $j \to \infty$ and we know that the obstacle $D \subset B_t(\omega)$, so we have

$$\|\tilde{u}_{\epsilon,j} - u_{\chi_{t-\epsilon}, b, t-\epsilon, N, \omega}\|_{H^1(D)} \to 0$$

as $j \to \infty$ for all $\epsilon > 0$. Moreover, we know that $u_{\chi_{t-\epsilon}, b, t-\epsilon, N, \omega} \to u_{\chi_t, b, t, N, \omega}$ as $\epsilon \to 0$ in $H^1(B_t(\omega))$, which implies

$$\|\tilde{u}_{\epsilon,j} - u_{\chi_t, b, t, N, \omega}\|_{H^1(D)} \to 0$$

as $\epsilon \to 0$, $j \to \infty$. Now by the definition of $I(\tau, \chi_t, b, t, \omega)$, we have

$$I(\tau, \chi_t, b, t, \omega) \leq C \|u_{\chi_t, b, t, N, \omega}\|^2_{H^1(D)}.$$

Now if $t < t_0$, we substitute $u_{\chi_t, b, t, N, \omega} = w_{\chi_t, b, t, N, \omega} + r_{\chi_t, b, t, N, \omega}$ with $w_{\chi_t, b, t, N, \omega}$ being described

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by (3.7.9) into

$$I(\tau,\chi_t,b,t,\omega)\leq C(\int_D |u_{\chi_t,b,t,N,\omega}|^2dx+\int_D |\nabla u_{\chi_t,b,t,N,\omega}|^2dx)$$

and use estimates (2.4.7), (2.4.8) to obtain that

$$|I(\tau,\chi_t,b,t,\omega)|\leq C\tau^{-2N-1}$$

which finishes

$$\limsup_{\tau\to\infty}|I(\tau,\chi_t,b,t,\omega)|=0.$$

For the second part, as inequality (2.4.39), we use (2.4.23), then the similar argument follows. It is easy to get

$$I(\tau,\chi_t,b,t,\omega)\geq c\int_D |\nabla u_{\chi_t,b,t,N,\omega}|^2dx - c\|u_{\chi_t,b,t,N,\omega}\|^2_{W^{1,p}(D)}, \tag{2.4.39}$$

For $p\in(\max\{2-\epsilon,\frac{6}{5}\},2]$. For the remaining part, we need some extra estimates in the following section. □

2.4.8 End of the proof of Theorem 2.15

For further estimate of the lower bound, we need to introduce the sets $D_{j,\delta}\subset D$, $D_\delta\subset D$ as follows. Recall that $h_D(\omega)=\inf_{x\in D}x\cdot\omega$ and $t_0=h_D(\omega)=x_0\cdot\omega$ for some $x_0\in\partial D$. For any $\alpha\in\partial D\cap\{x\cdot\rho=h_D(\omega)\}:=K$, define $B(\alpha,\delta)=\{x\in\mathbb{R}^3;|x-\alpha|<\delta\}$ $(\delta>0)$. Note $K\subset\cup_{\alpha\in K}B(\alpha,\delta)$ and K is compact, so there exists $\alpha_1,\cdots,\alpha_m\in K$ such that $K\subset\cup_{j=1}^m B(\alpha_j,\delta)$. Thus, we define

$$D_{j,\delta}:=D\cap B(\alpha_j,\delta) \text{ and } D_\delta:=\cup_{j=1}^m D_{j,\delta}.$$

It is easy to see that

$$\int_{D\backslash D_\delta} e^{-p\tau(x\cdot\omega-t_0)A_{t_0}(x')}bdx=O(e^{-pa\delta\tau}),$$

because $A_{t_0}(x')\in B^\infty(\mathbb{R}^2)$ is bounded and its real part strictly greater than 0, so $\exists a>0$ such that $\mathrm{Re}A_{t_0}(x')\geq a>0$ on $D\backslash D_\delta$. Let $\alpha_j\in K$, by rotation and translation, we may assume $\alpha_j=0$ and the vector $\alpha_j-x_0=-x_0$ is parallel to $e_3=(0,0,1)$. Therefore, we consider the change of coordinates near each α_j as follows:

$$\begin{cases} y'=x' \\ y_3=x\cdot\omega-t_0, \end{cases}$$

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where $x = (x_1, x_2, x_3) = (x', x_3)$ and $y = (y_1, y_2, y_3) = (y', y_3)$. Denote the parametrization of ∂D near α_j by $l_j(y')$, then we have the following estimates.

Lemma 2.16. *For $q \leq 2$, we have*

$$\begin{aligned}\int_D |u_{\chi_{t_0},b,t_0,N,\omega}|^q dx &\leq c\tau^{-1}\sum_{j=1}^m \iint_{|y'|<\delta} e^{-aq\tau l_j(y')}dy' + O(\tau^{-1}e^{-qa\delta\tau}) \\ &\quad +O(e^{-qa\tau}) + O(\tau^{-3}) + O(\tau^{-2N-1}),\end{aligned} \tag{2.4.40}$$

$$\begin{aligned}\int_D |u_{\chi_{t_0},b,t_0,N,\omega}|^2 dx &\geq C\tau^{-1}\sum_{j=1}^m \iint_{|y'|<\delta} e^{-2a\tau l_j(y')}dy' + O(\tau^{-1}e^{-2a\delta\tau}) \\ &\quad +O(\tau^{-3}) + O(\tau^{-2N-1}),\end{aligned} \tag{2.4.41}$$

$$\begin{aligned}\int_D |\nabla u_{\chi_{t_0},b,t_0,N,\omega}|^q dx &\leq C\tau^{q-1}\sum_{j=1}^m \iint_{|y'|<\delta} e^{-qa\tau l_j(y')}dy' + O(\tau^{-1}e^{-aq\delta\tau}) \\ &\quad +O(e^{-qa\tau}) + O(\tau^{-1}) + O(\tau^{-2N-1}),\end{aligned} \tag{2.4.42}$$

and

$$\begin{aligned}\int_D |\nabla u_{\chi_{t_0},b,t_0,N,\omega}|^2 dx &\geq C\tau\sum_{j=1}^m \iint_{|y'|<\delta} e^{-2a\tau l_j(y')}dy' + O(\tau^{-1}e^{-2\delta a\tau}) \\ &\quad +O(\tau^{-1}) + O(\tau^{-2N-1}).\end{aligned} \tag{2.4.43}$$

Proof. We follow the argument in [55]. We only prove (2.4.40) and (2.4.41) and the proof of (2.4.42) and (2.4.43) are similar arguments.

For (2.4.40):

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$$\begin{aligned}
\int_D |u_{\chi_{t_0},b,t_0,N,\omega}|^q dx &\leq C\int_D e^{-qa\tau(x\cdot\omega-t_0)}dx + C_q\int_D |\gamma_{\chi_{t_0},b,t_0,N,\omega}|^q dx \\
&\quad + C_q\int_D |r_{\chi_{t_0},b,t_0,N,\omega}|^q dx \\
&\leq C\int_{D_\delta} e^{-qa\tau(x\cdot\omega-t_0)}dx + C\int_{D\setminus D_\delta} e^{-qa\tau(x\cdot\omega-t_0)}dx \\
&\quad + C\int_D |\gamma_{\chi_{t_0},b,t_0,N,\omega}|^2 dx + C\int_D |r_{\chi_{t_0},b,t_0,N,\omega}|^2 dx \\
&\leq C\sum_{j=1}^m \iint_{|y'|<\delta} dy' \int_{l_j(y')}^{\delta} e^{-qa\tau y_3}dy_3 + Ce^{-qa\delta\tau} \\
&\quad + C\|\gamma_{\chi_{t_0},b,t_0,N,\omega}\|^2_{L^2(D)} + C\|r_{\chi_{t_0},b,t_0,N,\omega}\|^2_{H^1(D)} \\
&\leq C\tau^{-1}\sum_{j=1}^m \iint_{|y'|<\delta} e^{-aq\tau l_j(y')}dy' - \frac{C}{q}\tau^{-1}e^{-qa\delta\tau} \\
&\quad + Ce^{-qa\delta\tau} + C\tau^{-3} + C\tau^{-2N-1}
\end{aligned}$$

note that $D \subset \Omega_{t_0}(\omega)$, which proves (2.4.39).

For (2.4.41):

$$\begin{aligned}
\int_D |u_{\chi_{t_0},b,t_0,N.\omega}|^2 dx &\geq C\int_D e^{-2a\tau(x\cdot\omega-t_0)}dx - C\|\gamma_{\chi_{t_0},b,t_0,N,\omega}\|^2_{L^2(\Omega_{t_0}(\omega))} \\
&\quad - C\|r_{\chi_{t_0},b,t_0,N,\omega}\|^2_{H^1(\Omega_{t_0}(\omega))} \\
&\geq C\int_{D_\delta} e^{-2a\tau(x\cdot\omega-t_0)}dx - C\tau^{-3} - C\tau^{-2N-1} \\
&= C\tau^{-1}\sum_{j=1}^m \iint_{|y'|<\delta} e^{-2a\tau l_j(y')}dy' - \frac{C}{2}\tau^{-1}e^{-2a\delta\tau} \\
&\quad - C\tau^{-3} - C\tau^{-2N-1}.
\end{aligned}$$

□

Recall that we have (2.4.39), the lower bound of $I(\tau,\chi_{t_0},b,t_0,\omega)$, so we want to compare the order (in τ) of $\|u_{\chi_{t_0},b,t_0,N,\omega}\|_{L^2(D)}$, $\|\nabla u_{\chi_{t_0},b,t_0,N,\omega}\|_{L^2(D)}$, $\|u_{\chi_{t_0},b,t_0,N,\omega}\|_{L^p(D)}$ and $\|\nabla u_{\chi_{t_0},b,t_0,N,\omega}\|_{L^p(D)}$.

Lemma 2.17. *For* $\max\{2-\epsilon,\frac{6}{5}\} < p \leq 2$*, we have the estimates as follows:*

$$\frac{\|\nabla u_{\chi_{t_0},b,t_0,N,\omega}\|^2_{L^2(D)}}{\|u_{\chi_{t_0},b,t_0,N,\omega}\|^2_{L^2(D)}} \geq C\tau^2,\ \frac{\|u_{\chi_{t_0},b,t_0,N,\omega}\|^2_{L^p(D)}}{\|u_{\chi_{t_0},b,t_0,N,\omega}\|^2_{L^2(D)}} \geq C\tau^{1-\frac{2}{p}}$$

and

$$\frac{\|\nabla u_{\chi_{t_0},b,t_0,N,\omega}\|^2_{L^p(D)}}{\|u_{\chi_{t_0},b,t_0,N,\omega}\|^2_{L^2(D)}} \geq C\tau^{3-\frac{2}{p}}$$

for $\tau \gg 1$.

Proof. The idea of the proof comes from [55], but here we still need to deal with the $\gamma_{\chi_{t_0},b,t_0,N,\omega}$

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and $r_{\chi_{t_0},b,t_0,N,\omega}$ in $D \subset \Omega_{t_0}(\omega)$. Note that if ∂D is Lipschitz, in our parametrization $l_j(y')$, we have $l_j(y') \leq C|y'|$. Hence,

$$\begin{aligned}\sum_{j=1}^{m}\iint_{|y'|<\delta} e^{-2a\tau l_j(y')}dy' &\geq C\sum_{j=1}^{m}\iint_{|y'|<\delta} e^{-2ac\tau|y'|}dy'\\ &\geq C\tau^{-1}\sum_{j=1}^{m}\iint_{|y'|<\tau\delta} e^{-2|y'|}dy'\\ &= O(\tau^{-1}).\end{aligned}$$

For simplicity, we denote $u_0 := u_{\chi_{t_0},b,t_0,N,\omega}$ in the following calculations. Using Lemma 2.16, we obtain

$$\frac{\int_D |\nabla u_0|^2dx}{\int_D |u_0|^2dx}$$

$$\begin{aligned}&\geq C\frac{\tau\sum_{j=1}^{m}\iint_{|y'|<\delta} e^{-2a\tau l_j(y')}dy' + O(\tau^{-1}e^{-2a\delta\tau}) + O(\tau^{-1}) + O(\tau^{-2N-1})}{\tau^{-1}\sum_{j=1}^{m}\iint_{|y'|<\delta} e^{-2a\tau l_j(y')}dy' + O(\tau^{-1}e^{-2a\delta\tau}) + O(\tau^{-3}) + O(\tau^{-2N-1})}\\ &\geq C\tau^2\frac{1+\frac{O(\tau^{-2}e^{-2a\delta\tau})+O(\tau^{-2})+O(\tau^{-2N-2})}{\sum_{j=1}^{m}\iint_{|y'|<\delta} e^{-2a\tau l_j(y')}dy'}}{1+\frac{O(e^{-2a\delta\tau})+O(\tau^{-2})+O(\tau^{-2N})}{\sum_{j=1}^{m}\iint_{|y'|<\delta} e^{-2a\tau l_j(y')}dy'}}\\ &= O(\tau^2)\end{aligned}$$

as $\tau \gg 1$, where

$$\lim_{\tau\to\infty}\frac{O(\tau^{-2}e^{-2a\delta\tau}) + O(\tau^{-2}) + O(\tau^{-2N-2})}{\sum_{j=1}^{m}\iint_{|y'|<\delta} e^{-2a\tau l_j(y')}dy'} = 0$$

and

$$\lim_{\tau\to\infty}\frac{O(e^{-2a\delta\tau}) + O(\tau^{-2}) + O(\tau^{-2N})}{\sum_{j=1}^{m}\iint_{|y'|<\delta} e^{-2a\tau l_j(y')}dy'} = 0.$$

Now, by using the Hölder's inequality with the exponent $q = \dfrac{2}{p} \geq 1$, we have

$$\sum_{j=1}^{m}\iint_{|y'|<\delta} e^{-pa\tau l_j(y')}dy' \leq C(\sum_{j=1}^{m}\iint_{|y'|<\delta} e^{-2a\tau l_j(y')}dy')^{\frac{p}{2}}.$$

Hence we use Lemma 2.16 again, we have

$$\frac{(\int_D |u_0|^p dx)^{\frac{2}{p}}}{\int_D |u_0|^2dx}$$

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$$\leq C\frac{\tau^{-\frac{2}{p}}(\sum_{j=1}^{m}\iint_{|y'|<\delta}e^{-pa\tau l_j(y')}dy')^{\frac{2}{p}}+O(\tau^{-\frac{2}{p}}e^{-2a\delta\tau})+O(e^{-2a\tau})}{\tau^{-1}\sum_{j=1}^{m}\iint_{|y'|<\delta}e^{-2a\tau l_j(y')}dy'+O(\tau^{-1}e^{-2a\delta\tau})+O(\tau^{-3})+O(\tau^{-2N-1})}$$
$$+\frac{O(\tau^{-\frac{6}{p}})+O(\tau^{\frac{-4N-2}{p}})}{\tau^{-1}\sum_{j=1}^{m}\iint_{|y'|<\delta}e^{-2a\tau l_j(y')}dy'+O(\tau^{-1}e^{-2a\delta\tau})+O(\tau^{-3})+O(\tau^{-2N-1})}$$

$$\leq C\tau^{-\frac{2}{p}+1}\frac{\sum_{j=1}^{m}\iint_{|y'|<\delta}e^{-2a\tau l_j(y')}dy'+O(e^{-2a\delta\tau})+O(e^{-2a\tau}\tau^{\frac{2}{p}})}{\sum_{j=1}^{m}\iint_{|y'|<\delta}e^{-2a\tau l_j(y')}dy'+O(e^{-2a\delta\tau})+O(\tau^{-2})+O(\tau^{-2N})}$$
$$+\frac{O(\tau^{-\frac{4}{p}})+O(\tau^{\frac{-4N}{p}})}{\tau^{-1}\sum_{j=1}^{m}\iint_{|y'|<\delta}e^{-2a\tau l_j(y')}dy'+O(\tau^{-1}e^{-2a\delta\tau})+O(\tau^{-3})+O(\tau^{-2N-1})}$$

$$= \tau^{-\frac{2}{p}+1}\frac{1+\frac{O(e^{-2a\delta\tau})+O(e^{-2a\tau}\tau^{\frac{2}{p}})+O(\tau^{-\frac{4}{p}})+O(\tau^{\frac{-4N}{p}})}{\sum_{j=1}^{m}\iint_{|y'|<\delta}e^{-2a\tau l_j(y')}dy'}}{1+\frac{O(e^{-2a\delta\tau})+O(\tau^{-2})+O(\tau^{-2N})}{\sum_{j=1}^{m}\iint_{|y'|<\delta}e^{-2a\tau l_j(y')}dy'}}$$
$$= O(\tau^{-\frac{2}{p}+1})$$

as $\tau \gg 1$ and

$$\frac{(\int_D |\nabla u_0|^p dx)^{\frac{2}{p}}}{\int_D |u_0|^2 dx}$$

$$\leq C\frac{\tau^{(p-1)\frac{2}{p}}(\sum_{j=1}^{m}\iint_{|y'|<\delta}e^{-pa\tau l_j(y')}dy')^{\frac{2}{p}}+O(\tau^{-\frac{2}{p}}e^{-2a\delta\tau})+O(e^{-2a\tau})}{\tau^{-1}\sum_{j=1}^{m}\iint_{|y'|<\delta}e^{-2a\tau l_j(y')}dy'+O(\tau^{-1}e^{-2a\delta\tau})+O(\tau^{-3})+O(\tau^{-2N-1})}$$
$$+C\frac{O(\tau^{-\frac{2}{p}})+O(\tau^{\frac{-4N-2}{p}})}{\tau^{-1}\sum_{j=1}^{m}\iint_{|y'|<\delta}e^{-2a\tau l_j(y')}dy'+O(\tau^{-1}e^{-2a\delta\tau})+O(\tau^{-3})+O(\tau^{-2N-1})}$$

$$\leq C\tau^{3-\frac{2}{p}}\frac{\sum_{j=1}^{m}\iint_{|y'|<\delta}e^{-2a\tau l_j(y')}dy'+O(\tau^{-1}e^{-2a\delta\tau})+O(e^{-2a\tau}\tau^{\frac{2}{p}-1})}{\sum_{j=1}^{m}\iint_{|y'|<\delta}e^{-2a\tau l_j(y')}dy'+O(e^{-2a\delta\tau})+O(\tau^{-2})+O(\tau^{-2N})}$$
$$+C\frac{O(\tau^{-1})+O(\tau^{\frac{-4N}{p}-1})}{+O(\tau^{-\frac{2}{p}})+O(\tau^{\frac{-4N-2}{p}})}$$

$$\leq C\tau^{3-\frac{2}{p}}\frac{1+\frac{O(\tau^{-1}e^{-2a\delta\tau})+O(e^{-2a\tau}\tau^{\frac{2}{p}-1})+O(\tau^{-1})+O(\tau^{\frac{-4N}{p}-1})}{\sum_{j=1}^{m}\iint_{|y'|<\delta}e^{-2a\tau l_j(y')}dy'}}{1+\frac{O(e^{-2a\delta\tau})+O(\tau^{-2})+O(\tau^{-2N})}{\sum_{j=1}^{m}\iint_{|y'|<\delta}e^{-2a\tau l_j(y')}dy'}}$$
$$= O(\tau^{3-\frac{2}{p}})$$

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as $\tau \gg 1$. By (2.4.39) and above estimates, we have

$$\begin{aligned}\frac{I(\tau,\chi_t,b,t,\omega)}{\|u_{\chi_t,b,t,N,\omega}\|^2_{L^2(D)}} &\geq C\tau^2 - C\tau^{1-\frac{2}{p}} - C\tau^{3-\frac{2}{p}} \\ &\geq C\tau^2\end{aligned}$$

for $\tau \gg 1$. On the other hand, for $\|u_{\chi_t,b,t,N,\omega}\|_{L^2(D)}$, we have

$$\begin{aligned}\int_D |u_{\chi_t,b,t,N,\omega}|^2 dx &\geq C\tau^{-1}\sum_{j=1}^m \iint_{|y'|<\delta} e^{-2a\tau l_j(y')}dy' + O(\tau^{-1}e^{-qa\delta\tau}) \\ &\quad +O(\tau^{-3}) + O(\tau^{-2N-1}) \\ &\geq C\tau^{-1}\sum_{j=1}^m \iint_{|y'|<\delta} e^{-2a\tau|y'|}dy' + O(\tau^{-1}e^{-qa\delta\tau}) \\ &\quad +O(\tau^{-3}) + O(\tau^{-2N-1}) \\ &\geq C\tau^{-2}\sum_{j=1}^m \iint_{|y'|<\tau\delta} e^{-2a|y'|}dy' + O(\tau^{-1}e^{-qa\delta\tau}) \\ &\quad +O(\tau^{-3}) + O(\tau^{-2N-1}) \\ &= O(\tau^{-2}).\end{aligned}$$

Therefore, we have

$$I(\tau,\chi_t,b,h_D(\rho),\omega) \geq C\tau^2\|u_{\chi_t,b,t,N,\omega}\|^2_{L^2(D)} \geq C > 0$$

for $\tau \gg 1$. □

In view of Theorem 2.15 and Lemma 2.16, we can give an algorithm for reconstructing the convex hull of an inclusion D by the Dirichlet-to-Neumann map Λ_D as long as $A(x)$ and D satisfy the described conditions.

The Reconstruction algorithm.

1. Give $\omega \in S^2$ and choose $\eta, \zeta, \xi \in S^2$ so that $\{\eta, \zeta, \xi\}$ forms a basis of $\mathbb{R}^3$ and ξ lies in the span of η and ζ;
2. Choose a starting t such that $\Omega \subset \{x \cdot \omega \geq t\}$;
3. Choose a ball B such that the center of B lies on $\{x \cdot \omega = s\}$ for some $s < t$ and $\Omega \subset \overline{B_t(\omega)}$ and take $0 \neq b \in \mathbb{C}$;
4. Choose $\chi_t \in C_0^\infty(\mathbb{R}^2)$ such that $\chi_t > 0$ in $\Sigma_t(\omega)$ and $\chi_t = 0$ on $\partial\Sigma_t(\omega)$;
5. Construct the oscillating-decaying solution $u_{\chi_{t-\epsilon},b,t-\epsilon,N,\omega}$ in $B_{t-\epsilon}(\omega)$ with $\chi_{t-\epsilon} = \chi_t$ and the approximation sequence $\tilde{u}_{\epsilon,j}$ in $\widetilde{\Omega}$;

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6. Compute the indicator function $I(\tau, \chi_t, b, t, \omega)$ which is determined by boundary measurements;
7. If $I(\tau, \chi_t, b, t, \omega) \to 0$ as $\tau \to \infty$, then choose $t' > t$ and repeat (iv), (v), (vi);
8. If $I(\tau, \chi_t, b, t, \omega) \nrightarrow 0$ for some $\chi_{t'}$, then $t' = t_0 = h_D(\omega)$;
9. Varying $\omega \in S^2$ and repeat (i) to (viii), we can determine the convex hull of D.

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Chapter 3

The enclosure method for the Maxwell system

We have finished to introduce the enclosure-type method for the isotropic or anisotropic second order elliptic equations. Now, our goal in this chapter is to give more enclosure methods for the isotropic or anisotropic Maxwell system model. Similar to Chapter 2, we handle the problem of reconstructing interfaces using complex geometric optics (CGO) solutions for the isotropic Maxwell system and using the oscillating-decaying (OD) solutions for the anisotropic Maxwell system. We develop an enclosure-type reconstruction scheme to identify penetrable and impenetrable obstacles in electromagnetic field with isotropic or anisotropic medium in $\mathbb{R}^3$. For the penetrable case, we model the interface by the jump discontinuity of the magnetic permeability μ. The main tool is based on the global L^p estimate for the curl of the solutions of the Maxwell system with discontinuous coefficients. For the impenetrable case, the main tool is based on the potential theory in a suitable Sobolev space, and we will give more detailed descriptions in the following.

Before stating our inverse problem, we give basic properties which will be used in the enclosure-type method for the Maxwell system.

3.1 Basic properties for the Maxwell system

The Maxwell system contains the following properties which will be used in our inverse problem. For more details, we refer readers to [27, 40].

3.1.1 Well-posedness and L^p estimate for the Maxwell system

In the following, we would list the eigenvalue property and well-posedness results of the following problem: let $\Omega \subset \mathbb{R}^3$ and $K \Subset \Omega$,

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$$
\begin{cases}
\nabla \times E = ik\mu H & \text{in } \Omega \setminus K \\
\nabla \times H = -ik\epsilon E + J & \text{in } \Omega \setminus K \\
\nu \times E = f & \text{on } \partial\Omega \\
\nu \times H = g & \text{on } \partial K,
\end{cases}
\tag{3.1.1}
$$

where μ, ϵ are symmetric and positive definite matrix-valued functions. More precisely, we assume there exist constants $\mu_0, \mu_1, \lambda_0, \Lambda_0 > 0$ such that

$$
\begin{cases}
\mu_0 I \leq \mu(x) \leq \mu_1 I, \\
\lambda_0 I \leq \epsilon(x) \leq \Lambda_0 I.
\end{cases}
\tag{3.1.2}
$$

These well-posedness for the isotropic Maxwell systems can be found in Theorem 4.18 and 4.19 of [41]. However, we have the same result under our assumption (3.1.2) following the arguments in [41]. Let

$$
X = \left\{ u \in H(curl; \Omega \setminus K) | \nu \times u = 0 \text{ on } \partial\Omega \text{ and } u_T \in L^2 \left(\partial K\right)^3 \text{ on } \partial K \right\}.
$$

Definition 3.1. We say (E, H) or E is a weak solution of (3.1.1) if $E \in X$ and satisfies

$$
\left\langle \mu^{-1} \nabla \times E, \nabla \times \phi \right\rangle_{\Omega \setminus K} - k^2 \left\langle \gamma E, \phi \right\rangle_{\Omega \setminus K} = \left\langle ikJ, \phi \right\rangle_{\Omega \setminus K} - \left\langle \mu^{-1} g, \phi_T \right\rangle_{\partial K}, \quad \forall \phi \in X, \tag{3.1.3}
$$

and $\boldsymbol{\nu} \times E = f$ on $\partial\Omega$, where $\phi_T = (\nu \times \phi) \times \nu$ and $\langle \cdot, \cdot \rangle$ denotes the standard Hermitian inner product of L^2 space. Moreover, if (3.1.3) fails to have a unique solution, then k is called an eigenvalue or a resonance of (3.1.1).

Lemma 3.2. *There is an infinite discrete set Σ of eigenvalue $k_j > 0$, $j = 1, 2, \ldots$ and corresponding eigenfunctions $E_j \in H_0(curl; \Omega)$, $E_j \neq 0$, such that (3.1.3) holds with $J = 0$ and $f = g = 0$ is satisfied.*

From the above lemma, we have the following theorem.

Theorem 3.3. *For $k \notin \Sigma$, there exists a unique weak solution $(E, H) \in H(curl; \Omega \backslash \overline{K}) \times H(curl; \Omega \backslash \overline{K})$ of (3.1.1) given any $f \in H^{-1/2}(Div; \partial\Omega)$, $g \in H^{-1/2}(Div; \partial K)$ and $J \in H^{-1}(\Omega \backslash \overline{K})$. The solution satisfies*

$$
\|E\|_{L^2(\Omega \backslash \overline{K})} + \|H\|_{L^2(\Omega \backslash K)} \leq C(\|f\|_{H^{-1/2}(Div; \partial\Omega)} + \|g\|_{H^{-1/2}(Div; \partial K)} + \|J\|_{H^{-1}(\Omega \backslash \overline{K})})
$$

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for some constant $C > 0$*, where*

$$H^{-1/2}(Div;\Gamma) := \left\{ f \in H^{-1/2}(\Gamma)^3 \;\middle|\; \nu \cdot f = 0,\ \nabla_{\partial\Omega} \cdot f \in H^{-1/2}(\Gamma) \right\},$$

$\Gamma = \partial\Omega$ *or* ∂K*.*

In the following, we state the L^p theory for the anisotropic Maxwell's system. For this purpose, we define a bilinear form

$$B_A(E,F) := \int_\Omega (A(x)\nabla \times E(x)) \cdot (\nabla \times \overline{F}(x))dx + M\int_\Omega E(x) \cdot \overline{F}(x)dx$$

for all $E \in H_0^{1,q}(curl,\Omega)$ and $F \in H_0^{1,q'}(curl,\Omega)$ with $\frac{1}{q} + \frac{1}{q'} = 1$. We only state L^p estimate in the following theorem, but we do not prove the theorem. For more details, we refer readers to read [27].

Theorem 3.4. *[27] Let* Ω *be a a smooth domain. Suppose that* $A = A(x)$ *is a real symmetric matrix with smooth entries and satisfies the uniform elliptic condition*

$$\lambda|\xi|^2 \le A(x)\xi \cdot \xi \le \Lambda|\xi|^2, \text{ for all } \xi \in \mathbb{R}^3,$$

for some constants $0 < \lambda \le \Lambda < \infty$*. Assume* q *is some number satisfying* $2 \le q < \infty$*. Under the condition*

$$\inf_{\|F\|_{1,q'}=1} \sup_{\|E\|_{1,q}=1} |B_A E, F)| \ge \frac{1}{K} > 0$$

the Maxwell's systems of the equations

$$\nabla \times (A\nabla \times E) + E = \nabla \times f + g$$

is uniquely solvable in $H_0^{1,q'}(curl,\Omega)$ *for each* $g \in L^{q'}(\Omega)$ *and* $f \in L^{q'}(\Omega)$ *and the weak solution satisfies*

$$\|E\|_{L^{q'}(\Omega)} + \|\nabla \times E\|_{L^{q'}(\Omega)} \le K\{\|f\|_{L^{q'}(\Omega)} + \|g\|_{L^{q'}(\Omega)}\},$$

where K *is a positive constant depending on* p*.*

We end up this section with the following lemma on the embedding related to the Sobolev-Besov spaces, for more details, see [40] or property 5 in the appendix of [27].

Lemma 3.5. *Let* $u \in L^p(D)$ *such that* $\nabla \cdot u \in L^p(D)$ *and* $\nabla \times u \in L^p(D)$*. If* $\nu \times u \in L^p(\partial D)$*, then also* $\nu \cdot u \in L^p(\partial D)$ *for* $p \in (1,\infty)$*. If in addition* $1 < p \le 2$*, then* $u \in B^{p,2}_{\frac{1}{p}}(D)$ *and we have*

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the estimate

$$\|u\|_{B^{p,2}_{\frac{1}{p}}(D)} \leq C\{\|u\|_{L^p(D)} + \|curlu\|_{L^p(D)} + \|\nabla \cdot u\|_{L^p(D)} + \|\nu \times u\|_{L^p(\partial D)}\}$$

where the Sobolev-Besov space $B^{p,q}_{\alpha}(D) := [L^p(D), W^{1,p}(D)]_{\alpha,q}$ *is obtained by real interpolation for* $1 < p, q < \infty$ *and* $0 < \alpha < 1$.

3.2 Enclosing unknown obstacles in the isotropic media

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a smooth boundary, and assume that $\mathbb{R}^3 \backslash \overline{\Omega}$ is connected. Let $D \Subset \Omega$ be with Lipschitz boundary and the connected complement in $\mathbb{R}^3 \backslash \overline{D}$. We are concerned with the electromagnetic wave propagation in an isotropic medium in $\mathbb{R}^3$ with the electric permittivity $\epsilon > 0$ and the magnetic permeability $\mu > 0$. We also assume that $\epsilon \in W^{1,\infty}(\Omega)$ with $\epsilon \equiv 1$ in $\Omega \backslash \overline{D}$ and $\mu(x) = 1 - \mu_D(x)\chi_D$ to be a measurable function with $\mu_D(x) \in L^{\infty}(D)$ and χ_D is the characteristic function defined on D.

For the *penetrable* (inclusion) case, we consider the boundary value problem of finding the electromagnetic fields E and H satisfying

$$\begin{cases} \nabla \times E - ik\mu H = 0 & \text{in } \Omega, \\ \nabla \times H + ik\epsilon E = 0 & \text{in } \Omega, \\ \nu \times E = f & \text{on } \partial\Omega, \end{cases} \tag{3.2.1}$$

where ν is a unit outer normal on $\partial\Omega$, $\nabla \times$ denotes the curl in $\mathbb{R}^3$ and $\times$ is the standard cross product in $\mathbb{R}^3$. For the *impenetrable* (cavity) case, we consider the following boundary value problem

$$\begin{cases} \nabla \times E - ikH = 0 & \text{in } \Omega \backslash \overline{D}, \\ \nabla \times H + ikE = 0 & \text{in } \Omega \backslash \overline{D}, \\ \nu \times E = f & \text{on } \partial\Omega, \\ \nu \times H = 0 & \text{on } \partial D, \end{cases} \tag{3.2.2}$$

where ν is a unit outer normal on $\partial\Omega \cup \partial D$. In these two boundary value problems problems, we assume that the wave number k is not an eigenvalue for the spectral problems (3.2.1) and (3.2.2), respectively. Then by using results in [40, 41], we know that (3.2.1) and (3.2.2) are well-posed in the spaces $H(curl, \Omega)$ and $H(curl, \Omega \backslash \overline{D})$, respectively, where

$$H(curl, \Omega) := \{u \in (L^2(\Omega))^3 | \nabla \times u \in (L^2(\Omega))^3\}.$$

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It is similar to the elliptic case, we can the "Dirichlet-to-Neumann" type map in the Maxwell system, we call the map to be the *impedance map.*

Impedance map: We define the impedance map $\Lambda_D : TH^{-\frac{1}{2}}(\partial\Omega) \to TH^{-\frac{1}{2}}(\partial\Omega)$ by

$$\Lambda_D(\nu \times E|_{\partial\Omega}) = (\nu \times H|_{\partial\Omega}),$$

where $TH^{-\frac{1}{2}}(\partial\Omega) := \{f \in H^{-\frac{1}{2}}(\partial\Omega) | \nu \cdot f = 0\}$. We denote by $\Lambda_\emptyset$ the impedance map for the domain without an obstacle.

Proposition 3.6. *There exists a reconstruction framework to determine the convex hull of the unknown obstacle D from the information of the impedance map Λ_D.*

The previous Proposition is similar to the elliptic reconstruction. Recall that the enclosure method contains two tools: One is special solutions (CGO solutions) and the other is the indicator function. If we can find these two tools, then we can prove the Proposition 3.6. We first introduce how to construct CGO solutions for the isotropic Maxwell system.

3.3 Constructing CGO solutions

Our goal is to construct CGO solutions for the isotropic Maxwell system

$$\begin{cases} \nabla \times E - ik\mu H = 0 & \text{in } \Omega, \\ \nabla \times H + ik\epsilon E = 0 & \text{in } \Omega, \end{cases} \tag{3.3.1}$$

where μ and ϵ are smooth positive scalar functions. The ideas for constructing CGO solutions is to transform the isotropic Maxwell system into a Schrödinger type equation, which was first introduced by [51]. Moreover, in [66], the author used the reduction technique to construct CGO solutions for the isotropic Maxwell system, let me give a brief introduction in the following.

Reduction algorithm: From Maxwell to Schrödinger

First, we define the (zero) scalar fields Φ and Ψ by

$$\Phi = \frac{i}{k}\nabla \cdot (\epsilon E), \ \Psi = \frac{i}{k}\nabla \cdot (\mu H).$$

Then the Maxwell system is equivalent to

$$\nabla \times E - \frac{1}{\epsilon}\nabla(\frac{1}{\mu}\Psi) - ik\mu H = 0, \ \nabla \times H + \frac{1}{\mu}\nabla(\frac{1}{\epsilon}\Phi) + ik\epsilon E = 0.$$

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If we set $X = (\varphi, e, h, \psi)^T$ with $e = \epsilon^{1/2}E$, $h = \mu^{1/2}H$, $\varphi = \frac{1}{\epsilon\mu^{1/2}}\Phi$ and $\psi = \frac{1}{\epsilon^{1/2}\mu}\Psi$. Then we have

$$(P(i\nabla) - \lambda + V)X = 0 \text{ in } \Omega,$$

where

$$P(i\nabla) = \begin{pmatrix} 0 & \nabla\cdot & 0 & 0 \\ \nabla & 0 & \nabla\times & 0 \\ 0 & -\nabla\times & 0 & \nabla \\ 0 & 0 & \nabla\cdot & 0 \end{pmatrix}$$

and

$$V = (\lambda - \kappa)I_8 + \left(\begin{pmatrix} 0 & \nabla\cdot & 0 & 0 \\ \nabla & 0 & -\nabla\times & 0 \\ 0 & \nabla\times & 0 & \nabla \\ 0 & 0 & \nabla\cdot & 0 \end{pmatrix} D \right) D^{-1},$$

where $D = \operatorname{diag}(\mu^{1/2}, \epsilon^{1/2}I_3, \mu^{1/2}I_3, \epsilon^{1/2})$, $\kappa = k(\epsilon\mu)^{1/2}$ and $\lambda = k(\epsilon_0\mu_0)^{1/2}$. The most important property of this operator is that we can reduce a isotropic Maxwell system to a Schrödinger matrix equation by

$$(P(i\nabla) - \lambda + V)(P(i\nabla) + \lambda - V^T) = -(\Delta + \omega^2)I_8 + Q,$$

where

$$Q = VP(i\nabla) - P(i\nabla)V^T + \omega(V + V^T) - VV^T$$

is a zeroth order matrix multiplier. If we define $X = (P(i\nabla) + \lambda - V^T)Y$, then Y satisfies

$$(-\Delta - \lambda^2 + Q)Y = 0 \text{ in } \Omega. \tag{3.3.2}$$

It is well-known that we can find CGO solutions for Schrödinger equation. By using the *Faddeev kernel* and the *Sommerfeld's radiations conditions*, one can ensure that if Y satisfies the Schrödinger equation (3.3.2), then (E, H) will satisfy the isotropic Maxwell system (3.3.1). For more details, we refer readers to [51]. Let $\rho, \rho^\perp \in \mathbb{S}^2$, given $\theta, \eta \in \mathbb{C}^3$ of the form

$$\eta := \frac{1}{|\zeta|}(-(\zeta\cdot a)\zeta - k\zeta \times b + k^2 a) \text{ and } \theta := \frac{1}{|\zeta|}(k\zeta \times a - (\zeta\cdot b)\zeta + k^2 b)$$

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where $\zeta = -i\tau\omega + \sqrt{\tau^2+k^2}\omega^{\perp}$ and $a \in \mathbb{R}^3$, $b \in \mathbb{C}^3$. Then the authors constructed the CGO solutions in the following form:

$$\begin{cases} E_0 = \eta e^{\tau(x\cdot\rho - t)+i\sqrt{\tau^2+k^2}x\cdot\rho^{\perp}}, \\ H_0 = \theta e^{\tau(x\cdot\rho - t)+i\sqrt{\tau^2+k^2}x\cdot\rho^{\perp}}, \end{cases}$$

with (E_0, H_0) satisfies

$$\begin{cases} \nabla \times E_0 - ikH_0 = 0 \quad \text{in } \Omega, \\ \nabla \times H_0 + ikE_0 = 0 \quad \text{in } \Omega. \end{cases} \tag{3.3.3}$$

For the application on the reconstruction, we need to use two different types CGO solutions:

1. For the *penetrable* case, we choose $a \perp \rho$, $a \perp \rho^{\perp}$ and $b = \overline{\hat{\zeta}}$ with $\hat{\zeta} = \dfrac{\zeta}{|\zeta|}$ such that $\eta = O(1)$ and $\theta = O(\tau)$ for all $\tau \gg 1$..
2. For the *impenetrable* case, we choose $b \perp \rho$, $b \perp \rho^{\perp}$ and $a = \sqrt{2}\rho^{\perp}$ such that $\eta = O(\tau)$ and $\theta = O(1)$ for all $\tau \gg 1$..

Indicator function for the Maxwell system

Recall that $\Lambda_D : \nu \times E|_{\partial\Omega} \to \nu \times H|_{\partial\Omega}$ is the impedance map for the Maxwell system, then we can define the indicator function in the following.

Definition 3.7. For $\rho \in \mathbb{S}^2$, $\tau > 0$ and $t > 0$, we define the indicator function

$$I_\rho(\tau, t) := ik\tau \int_{\partial\Omega} (\nu \times E_0) \cdot \overline{((\Lambda_D - \Lambda_\emptyset)(\nu \times E_0) \times \nu)} dS,$$

where E_0 is the CGO solution of the Maxwell system given above.

Similarly, we can define the support function

$$h_D(\rho) := \sup_{x\in D} x \cdot \rho,$$

then we have the following result.

Theorem 3.8. *Let $\rho \in \mathbb{S}^2$. For the penetrable (or impenetrable) obstacle case, we have the following characterization of $h_D(\rho)$.*

$$\begin{cases} \lim_{\tau\to\infty} |I_\rho(\tau, t)| = 0 \text{ when } t > h_D(\rho), \\ \liminf_{\tau\to\infty} |I_\rho(\tau, h_D(\rho))| > 0, \\ \lim_{\tau\to\infty} |I_\rho(\tau, t)| = 0 \text{ when } t < h_D(\rho). \end{cases}$$

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The theorem shows that we can reconstruct the convex hull of the unknown obstacle D from the impedance map Λ_D, combining with CGO solutions and the indicator function, we can give a reconstruction algorithm of D.

3.4 Proof of the Theorem 3.3

3.4.1 Penetrable Case

We give key points of proof for the penetrable case of theorem 4.3. For more details, we refer readers to [27] First, we give the proof for the penetrable case. Recall that the model of the penetrable problem is

$$\begin{cases} \nabla \times E - ik\mu H = 0 & \text{in } \Omega, \\ \nabla \times H + ik\epsilon E = 0 & \text{in } \Omega, \\ \nu \times E = f & \text{on } \partial\Omega, \end{cases}$$

and CGO solutions of (3.2.1) are

$$\begin{cases} E_0 = \eta e^{\tau(x\cdot\rho - t) + i\sqrt{\tau^2+k^2}x\cdot\rho^\perp}, \\ H_0 = \theta e^{\tau(x\cdot\rho - t) + i\sqrt{\tau^2+k^2}x\cdot\rho^\perp}. \end{cases}$$

where $\eta = O(1)$ and $\theta = O(\tau)$ for all $\tau \gg 1$. Let $\widetilde{E} := E - E_0$ be the reflected solution, where E satisfies (3.2.1) and E_0 satisfies (3.3.3). $\widetilde{E}$ satisfies the (zero) boundary value problem

$$\begin{cases} \nabla \times (\frac{1}{\mu(x)} \nabla \times \widetilde{E}) - k^2\epsilon(x)\widetilde{E} = -\nabla \times (\frac{1}{\mu(x)} - 1)\nabla \times E_0 + k^2(\epsilon(x) - 1)E_0 & \text{in } \Omega, \\ \nu \times \widetilde{E} = 0 & \text{on } \partial\Omega. \end{cases}$$

We state the following useful estimates without any proofs, all proofs can be found in [27].

Lemma 3.9. *For $1 - \mu(x) > 0$, we have*

$$-\tau^{-1}I_\rho(\tau, t) \geq \int_D (1 - \mu(x))|\nabla \times E_0|^2 dx - k^2 \int_\Omega |\widetilde{E}(x)|^2 dx - k^2 \int_D (\epsilon(x) - 1)|E_0(x)|^2 dx.$$

For $\mu(x) - 1 > 0$, we have

$$\tau^{-1}I_\rho(\tau, t) \geq \int_D (1 - \frac{1}{\mu(x)})|\nabla \times E_0|^2 dx - k^2 \int_\Omega \epsilon(x)|\widetilde{E}(x)|^2 dx + k^2 \int_D (\epsilon(x) - 1)|E_0(x)|^2 dx.$$

Similar to the elliptic case, there is no need to worry about the upper bound of the indicator function $\tau^{-1}I_\rho(\tau, t)$. The difficulty is to estimates the lower order term $\widetilde{E}(x)$. In fact, we have the

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following key L^p estimates.

Proposition 3.10. *Suppose that* Ω *is a* C^1 *domain and* $D \Subset \Omega$. *Then there exists a constant* $C > 0$ *independent of* $\widetilde{E}$ *and* E_0 *such that we have*

$$\int_\Omega |\widetilde{E}|^2 dx \leq C\{\left(\int_D |\nabla \times E_0|^p\right)^{2/p} + \int_D |E_0|^2 dx\},$$

for all $p \in (\frac{4}{3}, 2]$.

Remark 3.11. Note that there is no need to assume the regularity on D, we only need ∂D is Lipschitz.

In view of the lower bound, we need to introduce the sets $D_{j,\delta} \subset D$, $D_\delta \subset D$ in the following. Recall that $h_D(\rho) = \sup_{x\in D} x \cdot \rho$ and $t_0 = h_D(\rho) = x_0 \cdot \rho$ for some $x_0 \in \partial D$. $\forall \alpha \in \partial D \cap \{x \cdot \rho = h_D(\rho)\} := K$, define $B(\alpha, \delta) = \{x \in \mathbb{R}^3; |x - \alpha| < \delta\}$ $(\delta > 0)$. Note $K \subset \cup_{\alpha\in K} B(\alpha, \delta)$ and K is compact, so there exists $\alpha_1, \cdots, \alpha_m \in K$ such that $K \subset \cup_{j=1}^m B(\alpha_j, \delta)$. Thus, we define

$$D_{j,\delta} := D \cap B(\alpha_j, \delta) \text{ and } D_\delta := \cup_{j=1}^m D_{j,\delta}.$$

It is easy to see that

$$\int_{D\setminus D_\delta} e^{-p\tau(h_D(\rho)-x\cdot\rho)} dx = O(e^{-pc\tau}) \text{ as } \tau \to \infty,$$

where c is a positive constant. Let $\alpha_j \in K$, by using rotation and translation, then we can assume $\alpha_j = 0$ and the vector $\alpha_j - x_0 = -x_0$ is parallel to $e_3 = (0, 0, 1)$. Therefore, we consider the change of coordinates near each α_j as follows:

$$\begin{cases} y' = x' \\ y_3 = x \cdot \rho - t_0, \end{cases}$$

where $x = (x_1, x_2, x_3) = (x', x_3)$ and $y = (y_1, y_2, y_3) = (y', y_3)$. Denote the parametrization of ∂D near α_j by $l_j(y')$, then we have the following estimates. Note that the oscillating-decaying solutions are well-defined in D.

Lemma 3.12. *For* $1 \leq q < \infty$, $\tau \gg 1$, *we have the following estimates.*

1.

$$\int_D |E_0(x)|^q dx \leq C\tau^{-1} \sum_{j=1}^m \iint_{|y'|<\delta} e^{-q\tau l_j(y')} dy' - \frac{C}{q}\tau^{-1} e^{-q\delta\tau} + Ce^{-qc\tau}$$

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2.

$$\int_D |E_0|^2 dx \quad \geq \quad C\tau^{-1}\sum_{j=1}^{m}\iint_{|y'|<\delta} e^{-2\tau l_j(y')}dy' - \frac{C}{2}\tau^{-1}e^{-2\delta\tau}$$

3.

$$\int_D |H_0(x)|^q dx \quad \leq \quad C\tau^{q-1}\sum_{j=1}^{m}\iint_{|y'|<\delta} e^{-q\tau l_j(y')}dy' - \frac{C}{q}\tau^{q-1}e^{-q\delta\tau} + C\tau^q e^{-qc\tau}$$

4.

$$\int_D |H_0|^2 dx \quad \geq \quad C\tau\sum_{j=1}^{m}\iint_{|y'|<\delta} e^{-2\tau l_j(y')}dy' - \frac{C}{2}\tau e^{-2\delta\tau}.$$

Lemma 3.13. *For $\tau \gg 1$ and $p \in (\frac{4}{3}, 2]$, we have*

$$\frac{\|H_0\|^2_{L^2(D)}}{\|E_0\|^2_{L^2(D)}} \geq O(\tau^2),$$

$$\frac{\|\widetilde{E}\|^2_{L^2(\Omega)}}{\|\nabla \times E_0\|^2_{L^2(D)}} \leq C\tau^{1-\frac{2}{p}}.$$

Moreover, when $t = h_D(\rho)$, the following estimate holds

$$\liminf_{\tau\to\infty}\int_D \tau|\nabla \times E_0|^2 dx \geq C.$$

End the proof

For $1 - \mu(x) \geq C > 0$, by Lemma 3.9, we have

$$\begin{aligned} -I_\rho(\tau, t) \quad &\geq \quad \tau\int_D (1-\mu(x))|\nabla \times E_0|^2 dx - \tau C\int_\Omega |\widetilde{E}(x)|^2 dx - \tau C\int_D |E_0(x)|^2 dx \\ &\geq \quad \tau C\int_D |\nabla \times E_0|^2 dx - \tau C\int_\Omega |\widetilde{E}(x)|^2 dx - \tau C\int_D |E_0(x)|^2 dx. \end{aligned}$$

Using above lemmas, we have

$$\frac{-I_\rho(\tau, t)}{\|\nabla \times E_0\|^2_{L^2(D)}} \geq C\tau\{1 - \tau^{1-\frac{2}{p}} - 2\tau^{-2}\},$$

and we get

$$|I_\rho(\tau, h_D(\rho))| \geq C > 0.$$

It is similar to the case $\mu(x) - 1 \geq C > 0$.

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3.4.2 Impenetrable Case

Recall that the mathematical model of the impenetrable case is

$$\begin{cases} \nabla \times E - ikH = 0 & \text{in } \Omega \backslash \overline{D}, \\ \nabla \times H + ikE = 0 & \text{in } \Omega \backslash \overline{D}, \\ \nu \times E = f & \text{on } \partial\Omega, \\ \nu \times H = 0 & \text{on } \partial D, \end{cases}$$

and CGO solutions of (3.2.2) are

$$\begin{cases} E_0 = \eta e^{\tau(x\cdot\rho - t) + i\sqrt{\tau^2+k^2}x\cdot\rho^{\perp}}, \\ H_0 = \theta e^{\tau(x\cdot\rho - t) + i\sqrt{\tau^2+k^2}x\cdot\rho^{\perp}}. \end{cases}$$

where $\eta = O(\tau)$ and $\theta = O(1)$ for all $\tau \gg 1$.

For the impenetrable case, the situation is quite different from the penetrable case. We start by the following lemma.

Lemma 3.14. *Assume* $(E, H) \in (H(curl, \Omega\backslash\overline{D}))^2$ *satisfies*

$$\begin{cases} \nabla \times E - ikH = 0 & \textit{in } \Omega \backslash \overline{D}, \\ \nabla \times H + ikE = 0 & \textit{in } \Omega \backslash \overline{D}, \\ \nu \times E = f & \textit{on } \partial\Omega, \\ \nu \times H = 0 & \textit{on } \partial D, \end{cases}$$

with $f = \nu \times E_0|_{\partial\Omega} \in TH^{-1/2}(\partial\Omega)$. *The the following identity holds*

$$\begin{aligned} -\frac{1}{\tau} I_\rho(\tau, t) &= -\int_D \{|\nabla \times E_0(x)|^2 - k^2|E_0(x)|^2\}dx - \int_{\Omega\backslash\overline{D}} \{|\nabla \times \widetilde{E}(x)|^2 - k^2|\widetilde{E}(x)|^2\}dx \\ &= \int_D \{|\nabla \times H_0(x)|^2 - k^2|H_0(x)|^2\}dx + \int_{\Omega\backslash\overline{D}} \{|\nabla \times \widetilde{H}(x)|^2 - k^2|\widetilde{H}(x)|^2\}dx, \end{aligned}$$

and the inequality

$$-\frac{1}{\tau} I_\rho(\tau, t) \geq \int_D \{|\nabla \times H_0(x)|^2 - k^2|H_0(x)|^2\}dx - k^2 \int_{\Omega\backslash\overline{D}} |\widetilde{H}(x)|^2 dx,$$

where $\widetilde{E} = E - E_0$ *and* $\widetilde{H} = H - H_0$.

The remaining task is to estimate the lower order term $\widetilde{H}$. In [27], the authors used the potential theory to prove the following key estimate.

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Proposition 3.15. *Let Ω be a C^1 domain and $D \Subset \Omega$ be Lipschitz. Then for all $p \in (\frac{4}{3}, 2]$ and $s \in (0,1]$, we have*

$$\int_{\Omega\backslash\overline{D}} |\widetilde{H}(x)|^2 dx \leq C\{\|\nabla \times H_0\|^2_{L^p(D)} + \|H_0\|^2_{H^{s+1/2}(D)}\},$$

where $C > 0$ is independent of $(\widetilde{E}, \widetilde{H})$ and (E_0, H_0).

Proof. (Sketch) Let (E^{ex}, H^{ex}) be the solution of the exterior problem in the following

$$\begin{cases} \nabla \times E^{ex} - ikH^{ex} = 0 & \text{in } \mathbb{R}^3\backslash\overline{D}, \\ \nabla \times H^{ex} + ikE^{ex} = 0 & \text{in } \mathbb{R}^3\backslash\overline{D}, \\ \nu \times H^{ex} = -\nu \times H_0 & \text{on } \partial D, \end{cases}$$

and (E^{ex}, H^{ex}) satisfies the Silver-Müller radiation condition. Let $\Phi_k(x,y) := -\dfrac{e^{ik|x-y|}}{4\pi|x-y|}$, $x \neq y \in \mathbb{R}^3$, then we can write E^{ex}, H^{ex} to be

$$\begin{aligned} H^{ex}(x) &:= \nabla \times \int_{\partial D} \Phi_k(x,y) f(y) dS(y), \\ E^{ex}(x) &:= -\frac{1}{ik} \nabla \times H^{ex}(x), \; x \in \mathbb{R}^3\backslash\partial D, \end{aligned}$$

where f is the density. We refer readers to [40, 41, 10, 11] for more details about the layer potential theory for the Maxwell system. By properties of the layer potential potential theory and for $p \in (\frac{4}{3}, 2]$, we have

$$\|H^{ex}\|_{L^2(\Omega\backslash\overline{D})} \leq C\{\|\nu \times H_0\|_{L^p(\partial D)} + \|\nabla \times H_0\|_{L^p(D)}\}.$$

Define $\mathcal{E} := \widetilde{E} - E^{ex}$ and $\mathcal{H} := \widetilde{H} - H^{ex}$, then $(\mathcal{E}, \mathcal{H})$ satisfies

$$\begin{cases} \nabla \times \mathcal{E} - ik\mathcal{H} = 0 & \text{in } \Omega\backslash\overline{D}, \\ \nabla \times \mathcal{H} + ik\mathcal{E} = 0 & \text{in } \Omega\backslash\overline{D}, \\ \nu \times \mathcal{H} = 0 & \text{on } \partial D, \\ \nu \times \mathcal{E} = -\nu \times E^{ex} & \text{on } \partial\Omega. \end{cases}$$

We apply the well-posedness theory for the Maxwell system, then we have

$$\|\mathcal{H}\|_{L^2(\Omega\backslash\overline{D})} \leq \|\mathcal{E}\|_{H(curl,\Omega\backslash\overline{D})} \leq C\|\nu \times \mathcal{E}\|_{H^{-1/2}(\partial\Omega)} \leq C\|\nu \times E^{ex}\|_{H^{-1/2}(\partial\Omega)}.$$

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Moreover, by $E^{ex}(x) := -\frac{1}{ik}\nabla \times H^{ex}(x)$, we can obtain

$$\|\nu \times E^{ex}\|_{H^{-1/2}(\partial\Omega)} \le C\|f\|_{L^p(\partial D)},\ \forall p \ge 1.$$

From the above inequalities and the vector potential theories, we can obtain

$$\|\mathcal{H}\|_{L^2(\Omega\setminus\overline{D})} \le C\{\|\nu \times H_0\|^2_{L^p(\partial D)} + \|\nabla \times H_0\|^2_{L^p(D)}\}$$

and use $\mathcal{H} = \widetilde{H} - H^{ex}$, then we have

$$\int_{\Omega\setminus\overline{D}} |\widetilde{H}(x)|^2 dx \le C\{\|\nu \times H_0\|^2_{L^p(\partial D)} + \|\nabla \times H_0\|^2_{L^p(D)}\},$$

for $p \in (\frac{4}{3}, 2]$. Since $p \le 2$, we use the Hölder's inequality and the trace theorem(see [6]), then we have for all $s \in (0,1]$,

$$\|\nu \times H_0\|_{L^p(\partial D)} \le C\|H_0\|_{H^s(\partial D)} \le C\|H_0\|_{H^{s+1/2}(D)},$$

which proves the result. □

Remark 3.16. The hardest part is to estimate $\widetilde{H}$ in $\Omega\setminus\overline{D}$ in terms of some suitable norm of H_0 in D. Moreover, for the anisotropic Maxwell system, it is more complicated than the previous proposition.

In order to prove the Theorem 4.3, we use similar arguments for the penetrable case as before. We use the impenetrable-type CGO solutions, then we have the following estimates. Let $l_j(y')$ be the functions described as before for $j = 1, 2, \cdots, m$.

Lemma 3.17. *For $1 \le q < \infty$, $\tau \gg 1$, we have the following estimates.*

1.

$$\int_D |H_0(x)|^q dx \quad \le C \quad \tau^{-1}\sum_{j=1}^m \iint_{|y'|<\delta} e^{-q\tau l_j(y')}dy' - \frac{C}{q}\tau^{-1}e^{-q\delta\tau} + Ce^{-qc\tau}$$

2.

$$\int_D |H_0|^2 dx \quad \ge \quad C\tau^{-1}\sum_{j=1}^m \iint_{|y'|<\delta} e^{-2\tau l_j(y')}dy' - \frac{C}{2}\tau^{-1}e^{-2\delta\tau}$$

3.

$$\int_D |\nabla \times H_0(x)|^q dx \quad \le \quad C\tau^{q-1}\sum_{j=1}^m \iint_{|y'|<\delta} e^{-q\tau l_j(y')}dy' - \frac{C}{q}\tau^{q-1}e^{-q\delta\tau} + C\tau^q e^{-qc\tau}$$

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4.

$$\int_D |\nabla \times H_0|^2 dx \quad \geq \quad C\tau \sum_{j=1}^{m} \iint_{|y'|<\delta} e^{-2\tau l_j(y')} dy' - \frac{C}{2}\tau e^{-2\delta\tau}.$$

Similarly, we have the following estimates.

For $\tau \gg 1$, for $p < 2$, the following estimates hold:

$$\frac{\|H_0\|^2_{L^2(D)}}{\|\nabla \times H_0\|^2_{L^2(D)}} \leq O(\tau^{-2})$$

and

$$\frac{\|\nabla \times H_0\|^2_{L^p(D)}}{\|\nabla \times H_0\|^2_{L^2(D)}} \leq C\tau^{1-\frac{2}{p}}.$$

Moreover, when $t = h_D(\rho)$, the following estimate holds

$$\liminf_{\tau \to \infty} \int_D \tau |\nabla \times H_0|^2 dx \geq C.$$

End the proof

Finally, we use the above estimates and similar method to obtain the lower bound of the indicator function,

3.5 Enclosing unknown obstacles in the anisotropic media

In this chapter, we develop an enclosure-type reconstruction scheme to identify penetrable and impenetrable obstacles in electromagnetic field with anisotropic medium in $\mathbb{R}^3$. The main difficulty in treating this problem lies in the fact that there are so far no complex geometrical optics solutions available for the Maxwell's equation with anisotropic medium in $\mathbb{R}^3$. Instead, we derive and use another type of special solutions called oscillating-decaying solutions. To justify this scheme, we use Meyers' L^p estimate, for the Maxwell system, to compare the integrals coming from oscillating-decaying solutions and those from the reflected solutions.

3.5.1 Problem descriptions and main results

Let Ω be a bounded C^∞-smooth domain in $\mathbb{R}^3$ with connected complement $\mathbb{R}^3 \setminus \overline{\Omega}$ and D be a subset of Ω with Lipschitz boundary. We are concerned with the electromagnetic wave propagation in an anisotropic medium in $\mathbb{R}^3$ with the electric permittivity $\epsilon = (\epsilon_{ij}(x))$ a 3×3 positive definite matrix and $\epsilon(x) = \epsilon_0(x)$ in $\Omega \backslash \bar{D}$. We also assume that $\epsilon(x) = \epsilon_0(x) - \epsilon_D(x)\chi_D(x)$ with $\epsilon_0 \in C^\infty(\Omega)$

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a positive definite 3×3 symmetric matrix and $\epsilon_D(x)$ is a positive 3×3 symmetric matrix and μ a smooth scalar function defined on Ω such that there exist $\mu_c > 0$ and $\epsilon_c > 0$ verifying

$$\mu(x) \geq \mu_c > 0 \text{ and } \sum_{i.j=1}^{3} \epsilon_{ij}(x)\xi_i\xi_j \geq \epsilon_c|\xi|^2 \ \forall \xi \in \mathbb{R}^3, \ \forall x \in \Omega. \tag{3.5.1}$$

If we denote by E and H the electric and the magnetic fields respectively, then the electromagnetic wave propagation by a penetrable obstacle problem reads as

$$\begin{cases} \nabla \times E - ik\mu H = 0 & \text{in } \Omega, \\ \nabla \times H + ik\epsilon E = 0 & \text{in } \Omega, \\ \nu \times E = f & \text{on } \partial\Omega, \end{cases} \tag{3.5.2}$$

with $\epsilon = \epsilon_0 - \epsilon_D \chi_D$, and the one by the impenetrable obstacle as

$$\begin{cases} \nabla \times E - ik\mu H = 0 & \text{in } \Omega \backslash \bar{D}, \\ \nabla \times H + ik\epsilon E = 0 & \text{in } \Omega \backslash \bar{D}, \\ \nu \times E = f & \text{on } \partial\Omega, \\ \nu \times H = 0 & \text{on } \partial D, \end{cases} \tag{3.5.3}$$

where ν is the unit outer normal vector on $\partial\Omega \cup \partial D$ and $k > 0$ is the wave number. In this paper, we assume that k is not an eigenvalue for (3.5.2) and (3.5.3).

Impedance Map: We define the impedance map $\Lambda_D : TH^{-\frac{1}{2}}(\partial\Omega) \to TH^{-\frac{1}{2}}(\partial\Omega)$ by

$$\Lambda_D(\nu \times H|_{\partial\Omega}) = (\nu \times E|_{\partial\Omega}),$$

where $TH^{-\frac{1}{2}}(\partial\Omega) := \{f \in H^{-\frac{1}{2}}(\partial\Omega) | \nu \cdot f = 0\}$ and $\times$ is the standard cross product in $\mathbb{R}^3$. We denote by $\Lambda_\emptyset$ the impedance map for the domain without an obstacle.

Consider the anisotropic Maxwell system

$$\begin{cases} \nabla \times E - ik\mu H = 0 & \text{in } \Omega, \\ \nabla \times H + ik\epsilon E = 0 & \text{in } \Omega, \end{cases} \tag{3.5.4}$$

where μ and ϵ satisfy (3.5.1). Inspired by [51] and [48], our idea is to reduce (3.5.4) to an elliptic systems and then use the results in [48] to construct oscillating-decaying type solutions to the anisotropic Maxwell system. Precisely, we can decompose the equation (3.5.4) into two decoupled strongly elliptic systems. The main difference between the construction of the oscillating-decaying

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solutions in [48] and ours is about the higher derivatives of oscillating-decaying solutions.

One of the main differences between the CGOs and the oscillating-decaying solutions is that, roughly speaking, given a hyperplane, an oscillating decaying solution is oscillating very rapidly along this plane and decaying exponentially in the direction transversely to the same plane. Oscillating-decaying solutions are special solutions with the phase function having nonnegative imaginary part. In addition, these oscillating decaying solutions are only defined on a half plane. To use them as inputs for our detection algorithm, we need to extend them to the whole domain Ω. One way to do the extension is to use the Runge approximation property for the anisotropic Maxwell's equation. The Runge approximation property will help us to find a sequence of approximated solutions which are defined on Ω, satisfy (3.5.4) and their limit is the oscillating-decaying solution. Note that it was first recognized by Lax [31] that the Runge approximation property is a consequence of the weak unique continuation property. In [33], the authors already proved the unique continuation property and based on it we derive the Runge approximation property for the anisotropic Maxwell's equation.

To be more precise, let ω be a unit vector in $\mathbb{R}^3$, denote $\Omega_t(\omega) = \Omega \cap \{x|x\cdot\omega > t\}$, $\Sigma_t(\omega) = \Omega\cap\{x|x\cdot\omega = t\}$ and set (E_t, H_t) to be the oscillating-decaying solution for the anisotropic Maxwell's equation in $\Omega_t(\omega)$.

Support function: For $\rho \in \mathbb{S}^2$, we define the support function of D by $h_D(\rho) = \inf_{x\in D} x\cdot\rho$.

When $t = h_D(\omega)$, which means $\Sigma_t(\omega)$ touches ∂D, we cannot apply the Runge approximation property to (E_t, H_t) in $\Omega_t(\omega)$. Therefore, we need to enlarge the domain $\Omega_t(\omega)$ such that the OD solutions exist and the Runge approximation property works. Let η be a positive real number, denote $\Omega_{t-\eta}(\omega)$ and $\Sigma_{t-\eta}(\omega)$ and note that $\Omega_{t-\eta}(\omega) \supset \Omega_t(\omega)\ \forall\eta > 0$. We can find $(E_{t-\eta}, H_{t-\eta})$ to be the OD solution in $\Omega_{t-\eta}(\omega)$. By the Runge approximation property, there exists a sequence of functions $\{(E_{\eta,\ell}, H_{\eta,\ell})\}$ satisfying the Maxwell system in Ω such that $(E_{\eta,\ell}, H_{\eta,\ell})$ converges to $(E_{t-\eta}, H_{t-\eta})$ as $\ell\to\infty$ in $L^2(\Omega_{t-\eta}(\omega))$ and in $H(curl, D)$ by interior estimates since $D \Subset \Omega_{t-\eta}(\omega)$. In addition we show that $(E_{t-\eta}, H_{t-\eta})$ converges to (E_t, H_t) in $H(curl, D)$ as $\eta \to 0$. Then we can define the indicator function as follows.

Indicator function: For $\rho \in S^2$, $\tau > 0$ and $t > 0$ we define the indicator function

$$I_\rho(\tau, t) := \lim_{\eta\to 0}\lim_{\ell\to\infty} I_\rho^{\eta,\ell}(\tau, t),$$

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where

$$I_\rho^{\eta,\ell}(\tau,t) := ik\tau \int_{\partial\Omega} (\nu \times H_{\eta,\ell}) \cdot (\overline{(\Lambda_D - \Lambda_\emptyset)(\nu \times H_{\eta,\ell})} \times \nu) dS.$$

Goal: We want to characterize the convex hull of the obstacle D from the impedance map Λ_D.

The answer to this goal is the following theorem.

Theorem 3.18. *Let $\rho \in \mathbb{S}^2$. For the penetrable (or impenetrable) obstacle case, we have the following characterization of $h_D(\rho)$.*

$$\begin{cases} \lim_{\tau\to\infty} |I_\rho(\tau,t)| = 0 \ \textit{when}\ t < h_D(\rho), \\ \liminf_{\tau\to\infty} |I_\rho(\tau, h_D(\rho))| > 0, \end{cases}$$

To prove Theorem 3.18, for the penetrable obstacle case, we need an appropriate L^p estimate of the corresponding reflected solution. We follow the idea in [27] to prove a global L^p estimate for the curl of the solutions of the anisotropic Maxwell's equation, for p near 2 and $p \leq 2$.

To prove Theorem 3.18, in the impenetrable obstacle case, we use layer potential arguments as in [27] coupled with appropriate L^p estimates. Precisely, first, we use the well-posedness for an exterior isotropic Maxwell's system with the Silver-Müller radiation condition and, in particular, the layer potential theory to find a suitable estimate for the solution of this exterior problem. Second, we decompose the reflected solution into two functions, one satisfies the reflected Maxwell's equation with a zero boundary data, the other satisfies the original anisotropic Maxwell's equation with the same boundary conditions which come from the reflected equation. For the first decomposed function, we use the L^p estimates, and for the second function, we will use the well-posedness, in L^2, for the anisotropic Maxwell's system. Combining these two steps, we derive the full estimate for the reflected solution in the impenetrable obstacle case.

This paper is organized as follows. In the section 2, we give decompose the anisotropic Maxwell system into two strongly elliptic systems. In section 3, we use the elliptic systems derived in the section 2 to build the oscillating-decaying solutions for the Maxwell system. Then, we give the Runge approximation for the anisotropic Maxwell equation in section 4. In section 5, we prove the Theorem 3.18 for both penetrable and impenetrable obstacle case. Finally, in the last section, as an appendix, we provide some technical details which we postponed in the main text and recall some useful estimates for solutions of the Maxwell system. Before closing this introduction, let us mention that in the whole text whenever we use the word smooth it means C^∞-smooth.

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3.6 A new reduction method: From the anisotropic Maxwell system to the second order strongly elliptic system

Our goal is to construct the oscillating-decaying (OD) solution for the following anisotropic time-harmonic Maxwell's system

$$\begin{cases} \nabla \times E = ik\mu H \\ \nabla \times H = -ik\epsilon E \\ \operatorname{div}(\epsilon E) = 0 \\ \operatorname{div}(\mu H) = 0, \end{cases} . \tag{3.6.1}$$

where E, H denote the electric and magnetic field intensity respectively, and μ denotes the positive scalar permeability, ϵ denotes the permittivity, which is a real, symmetric, positive definite 3×3 matrix.

Inspired by [51], the first step of constructing OD solutions is to reduce (3.6.1) to a strongly elliptic system. In fact, we reduce the anisotropic Maxwell's system (3.6.1) to two separate strongly elliptic equations (3.6.3), while in [51] the isotropic Maxwell's system is reduced to an elliptic (a single Schrödinger) system with coupled zero-th order term. The following theorem is our reduction result.

Theorem 3.19. *We set E and H of the following forms*

$$\begin{cases} E = -\frac{i}{k}\epsilon^{-1}\nabla \times (\mu^{-1}(\nabla \times B)) - \epsilon^{-1}(\nabla \times A) \\ H = \frac{i}{k}\mu^{-1}\nabla \times (\epsilon^{-1}(\nabla \times A)) - \mu^{-1}(\nabla \times B) \end{cases} \tag{3.6.2}$$

with A, B satisfying the strongly elliptic systems

$$\begin{cases} \mu\nabla tr(M^A\nabla A) - \nabla \times (\epsilon^{-1}(\nabla \times A)) + k^2\mu A = 0 \\ \epsilon\nabla tr(M^B\nabla B) - \nabla \times (\mu^{-1}(\nabla \times B)) + k^2\epsilon B = 0 \end{cases} , \tag{3.6.3}$$

where M^A, M^B are introduced in Theorem 3.22, then E and H satisfy (3.6.1).

Remark 3.20. Theorem 2.1 shows that, if we can find solutions of (3.6.3), then we can find solutions of (3.6.1).

Proof. In this proof, we will show the process of the reduction. And the proof that the systems (3.6.3) are strongly elliptic systems will be postponed to Theorem 3.22.

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As in [51], we set the following two auxiliary functions which are similar to what they used:

$$\Phi = \frac{i}{k}\mathrm{div}(\epsilon E)$$

and

$$\Psi = \frac{i}{k}\mathrm{div}(\mu H).$$

Note that Φ and Ψ are actually zero by the Maxwell's equation. We consider the following first-order matrix differential operator P

$$P = \begin{pmatrix} 0 & \mathrm{div}(\epsilon(\cdot)) & 0 & 0 \\ \mu^{-1}\nabla & 0 & \nabla\times & 0 \\ 0 & -\nabla\times & 0 & \epsilon^{-1}\nabla \\ 0 & 0 & \mathrm{div}(\mu(\cdot)) & 0 \end{pmatrix}.$$

Note that P is a 8×8 matrix. Let

$$Y = \begin{pmatrix} \Phi \\ E \\ H \\ \Psi \end{pmatrix}$$

Then the problem (3.6.1) can be rewritten as follows:

$$PY = -ikVY,$$

where

$$V = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \epsilon & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Thus, the Maxwell's system (3.6.1) implies

$$(P + ikV)Y = 0 \text{ and } \Phi = \Psi = 0. \tag{3.6.4}$$

It is easy to see that conversely (3.6.4) implies the Maxwell's system, and hence they are equivalent.

The first idea of the reducing process is to construct a suitable $\widetilde{Q}$, which can make $(P + ikV)\widetilde{Q}$

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a "good" second-order differential operator. Then, a solution **X** for the problem

$$(P + ikV)\widetilde{Q}X = 0 \tag{3.6.5}$$

will give rise to a solution $Y = \widetilde{Q}X$ for

$$(P + ikV)Y = 0.$$

Moreover, if we find the solution X such that the first and the last component of $Y = \widetilde{Q}X$ are zero, then we obtain solutions for the Maxwell's system.

We try the matrix differential operator $\widetilde{Q} = Q - ikI$, where

$$Q = \begin{pmatrix} 0 & \mathrm{div}(\epsilon(\cdot)) & 0 & 0 \\ \nabla & 0 & \epsilon^{-1}(\nabla \times (\cdot)) & 0 \\ 0 & -\mu^{-1}(\nabla \times (\cdot)) & 0 & \nabla \\ 0 & 0 & \mathrm{div}(\mu(\cdot)) & 0 \end{pmatrix}. \tag{3.6.6}$$

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Then

$$
\begin{aligned}
&(P+ikV)\widetilde{Q} \\
&= (P+ikV)(Q-ikI) \\
&= PQ - ikP + ikVQ + k^2V \\
&= \begin{pmatrix} \mathrm{div}(\epsilon\nabla) & 0 & 0 & 0 \\ 0 & L_1 & 0 & 0 \\ 0 & 0 & L_2 & 0 \\ 0 & 0 & 0 & \mathrm{div}(\mu\nabla) \end{pmatrix} \\
&+ \begin{pmatrix} 0 & -ik\mathrm{div}(\epsilon(\cdot)) & 0 & 0 \\ -ik\mu^{-1}\nabla & 0 & -ik\nabla\times & 0 \\ 0 & ik\nabla\times & 0 & -ik\epsilon^{-1}\nabla \\ 0 & 0 & -ik\mathrm{div}(\mu(\cdot)) & 0 \end{pmatrix} \\
&+ \begin{pmatrix} 0 & ik\mathrm{div}(\epsilon(\cdot)) & 0 & 0 \\ ik\epsilon\nabla & 0 & ik\nabla\times & 0 \\ 0 & -ik\nabla\times & 0 & ik\mu\nabla \\ 0 & 0 & ik\mathrm{div}(\mu(\cdot)) & 0 \end{pmatrix} \\
&+ \begin{pmatrix} k^2 & 0 & 0 & 0 \\ 0 & k^2\epsilon & 0 & 0 \\ 0 & 0 & k^2\mu & 0 \\ 0 & 0 & 0 & k^2 \end{pmatrix} \\
&= \begin{pmatrix} \mathrm{div}(\epsilon\nabla)+k^2 & 0 & 0 & 0 \\ ik(\epsilon-\mu^{-1})\nabla & L_1+k^2\epsilon & 0 & 0 \\ 0 & 0 & L_2+k^2\mu & ik(\mu-\epsilon^{-1})\nabla \\ 0 & 0 & 0 & \mathrm{div}(\mu\nabla)+k^2 \end{pmatrix},
\end{aligned}
$$

where

$$L_1 = \mu^{-1}\nabla(\mathrm{div}(\epsilon(\cdot))) - \nabla\times(\mu^{-1}(\nabla\times(\cdot))) \tag{3.6.7}$$

$$L_2 = \epsilon^{-1}\nabla(\mathrm{div}(\mu(\cdot))) - \nabla\times(\epsilon^{-1}(\nabla\times(\cdot))). \tag{3.6.8}$$

A prominent feature of the above operator is that it decomposes the original eight-component

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system into two four-component systems. Precisely, Set

$$X = \begin{pmatrix} \varphi \\ e \\ h \\ \psi \end{pmatrix},$$

then (3.6.5) can be separated into two systems:

$$\begin{cases} \operatorname{div}(\epsilon\nabla\varphi) + k^2\varphi = 0 \\ L_1 e + k^2\epsilon e + ik(\epsilon - \mu^{-1})\nabla\varphi = 0. \end{cases}$$

and

$$\begin{cases} \operatorname{div}(\mu\nabla\psi) + k^2\psi = 0 \\ L_2 h + k^2\mu h + ik(\mu - \epsilon^{-1})\nabla\psi = 0. \end{cases}$$

Moreover,

$$\begin{aligned} Y &= \widetilde{Q}X \\ &= \left[\begin{pmatrix} 0 & \operatorname{div}(\epsilon(\cdot)) & 0 & 0 \\ \nabla & 0 & \epsilon^{-1}(\nabla\times(\cdot)) & 0 \\ 0 & -\mu^{-1}(\nabla\times(\cdot)) & 0 & \nabla \\ 0 & 0 & \operatorname{div}(\mu(\cdot)) & 0 \end{pmatrix} - ikI \right] X \\ &= \begin{pmatrix} \operatorname{div}(\epsilon e) - ik\varphi \\ \nabla\varphi + \epsilon^{-1}(\nabla\times h) - ike \\ -\mu^{-1}(\nabla\times \mathbf{e}) + \nabla\psi - ikh \\ \operatorname{div}(\mu h) - ik\psi \end{pmatrix}. \end{aligned}$$

Therefore, the problem of finding the solutions X of

$$(P + ikV)\widetilde{Q}X = 0 \text{ with the first and last component of } \widetilde{Q}X \text{ being } 0 \qquad (3.6.9)$$

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is equivalent to the problem of finding solutions of the following two separate systems:

$$\begin{cases} \operatorname{div}(\epsilon e) - ik\varphi = 0, \\ \operatorname{div}(\epsilon\nabla\varphi) + k^2\varphi = 0, \\ \mu^{-1}\nabla(\operatorname{div}(\epsilon e)) - \nabla\times(\mu^{-1}(\nabla\times e)) + k^2\epsilon e + ik(\epsilon-\mu^{-1})\nabla\varphi = 0, \end{cases} \tag{3.6.10}$$

and

$$\begin{cases} \operatorname{div}(\mu h) - ik\psi = 0, \\ \operatorname{div}(\mu\nabla\psi) + k^2\psi = 0, \\ \epsilon^{-1}\nabla(\operatorname{div}(\mu h)) - \nabla\times(\epsilon^{-1}(\nabla\times h)) + k^2\mu h + ik(\mu-\epsilon^{-1})\nabla\psi = 0. \end{cases} \tag{3.6.11}$$

Notice that if we set e in the following form

$$e = -\frac{i}{k}(\nabla\varphi + \epsilon^{-1}(\nabla\times A)), \tag{3.6.12}$$

then the first equation of (3.6.10) becomes the same as the second one. For the third equation, we have

$$\begin{aligned} &\mu^{-1}\nabla\big(\operatorname{div}(\epsilon e)\big) - \nabla\times\big(\mu^{-1}(\nabla\times e)\big) + k^2\epsilon e + ik(\epsilon-\mu^{-1})\nabla\varphi \\ &= -\frac{i}{k}\mu^{-1}\nabla\left(\operatorname{div}(\epsilon\nabla\varphi)\right) + \frac{i}{k}\nabla\times\Big(\mu^{-1}\big[\nabla\times(\epsilon^{-1}(\nabla\times A))\big]\Big) \\ &\qquad - ik\epsilon\nabla\varphi - ik(\nabla\times A) + ik\epsilon\nabla\varphi - \frac{i}{k}\mu^{-1}\nabla\Big(k^2\varphi\Big) \\ &= -\frac{i}{k}\mu^{-1}\nabla\left(\operatorname{div}(\epsilon\nabla\varphi) + k^2\varphi\right) + \frac{i}{k}\nabla\times\Big(\mu^{-1}\big[\nabla\times(\epsilon^{-1}(\nabla\times A))\big]\Big) - ik\nabla\times A \\ &= 0 + \frac{i}{k}\nabla\times\Big(\mu^{-1}\big[\nabla\times(\epsilon^{-1}(\nabla\times A))\big]\Big) - ik\nabla\times A, \end{aligned}$$

by the second equation of (3.6.10). Thus, by letting e be of the form (3.6.12), the system (3.6.10) reduces to

$$\begin{cases} \operatorname{div}\big(\gamma\nabla\varphi\big) + k^2\varphi = 0, \\ \nabla\times\Big(\mu^{-1}\big[\nabla\times(\epsilon^{-1}(\nabla\times A))\big] - k^2 A\Big) = 0. \end{cases} \tag{3.6.13}$$

Similarly, by letting

$$h = -\frac{i}{k}(\nabla\psi + \mu^{-1}(\nabla\times B))$$

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for some vector field B, we can reduce (3.6.11) to the following system:

$$\begin{cases} \operatorname{div}\big(\mu\nabla\psi\big) + k^2\psi = 0, \\ \nabla \times \left(\epsilon^{-1}\big[\nabla \times (\mu^{-1}(\nabla \times B))\big] - k^2 B\right) = 0. \end{cases} \tag{3.6.14}$$

To resume, if we can find solutions φ, A, ψ and B of (3.6.13) and (3.6.14), we can find solutions of the problem (3.6.9) and therefore the original problem (3.6.1).

Now let us focus on (3.6.13) and (3.6.14). The goal is to find special solutions (e.g. oscillating-decaying solutions) of (3.6.13) and (3.6.14). The idea of doing that is to subtract zero terms of the form $\nabla \times \big(\nabla tr(M^A \nabla A)\big)$ and $\nabla \times \big(\nabla tr(M^B \nabla B)\big)$ from the second equations of (3.6.13) and (3.6.14) for some matrices M^A, M^B, so that they become $\nabla \times (\mathcal{L}^A A) = 0$ and $\nabla \times (\mathcal{L}^B B) = 0$ with $\mathcal{L}^A$ and $\mathcal{L}^B$ being strongly elliptic operators. Precisely, we want to find suitable matrices M^A and M^B such that

$$\mu\nabla tr(M^A\nabla A) - \nabla \times \big(\epsilon^{-1}(\nabla \times A)\big) + k^2\mu A = 0 \tag{3.6.15}$$

and

$$\epsilon\nabla tr(M^B\nabla B) - \nabla \times \big(\mu^{-1}(\nabla \times B)\big) + k^2\epsilon B = 0 \tag{3.6.16}$$

are strongly elliptic systems. In fact, by letting $M^A = m\mu^{-1}I$ and $M^B = m\mu^{-1}\epsilon$, we can show that (3.6.15) and (3.6.16) are strong elliptic systems for arbitrary positive constant m. The proof are given in Theorem 3.22. □

To prove Theorem 3.22, we start with the following computational lemma.

Lemma 3.21. *Let M be a matrix-valued function with smooth entries and $\mathbf{F}$ be a vector field. Then the i-th component of the vector $\nabla \times \big(M(\nabla \times \mathbf{F})\big)$ is given by*

$$\big(\nabla \times \big(M(\nabla \times \mathbf{F})\big)\big)_i = \sum_{j,k,\ell} \widetilde{C}_{ijk\ell}\partial_{j\ell} f_k + \widetilde{R}_i, \tag{3.6.17}$$

where

$$\widetilde{C}_{ijk\ell} = \delta_{j\ell}M_{ki} + \delta_{ik}M_{\ell j} - \delta_{jk}M_{\ell i} - \delta_{i\ell}M_{kj} + \big(\delta_{i\ell}\delta_{jk} - \delta_{ik}\delta_{j\ell}\big)tr(M),$$

and $\widetilde{R}_i$ contains the lower order terms. Here, δ_{ij} is the Kronecker delta, M_{ij} is the ij-th entry of M, and $\mathbf{F} = (f_1, f_2, f_3)^T$.

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Proof. We prove it by direct computations. For any vectors $\mathbf{a}, \mathbf{b}$, letting $\mathbf{c} = \mathbf{a} \times \mathbf{b}$, we have

$$c_m = \sum_{k\ell} \varepsilon_{m\ell k} a_\ell b_k,$$

where $\mathbf{a} = (a_1, a_2, a_3)^T$, $\mathbf{b} = (b_1, b_2, b_3)^T$, $\mathbf{c} = (c_1, c_2, c_3)^T$ and $\varepsilon_{m\ell k}$ denotes the Levi-Civita symbol. Therefore, we obtain the m-th component of $\nabla \times \mathbf{F}$:

$$\Big(\nabla \times \mathbf{F}\Big)_m = \sum_{k\ell} \varepsilon_{m\ell k} \partial_\ell f_k.$$

Then, the n-th component of $M(\nabla \times \mathbf{F})$ is

$$\Big(M(\nabla \times \mathbf{F})\Big)_n = \sum_{m,k,\ell} M_{nm} \varepsilon_{m\ell k} \partial_\ell f_k.$$

Finally, taking the curl operator on the vector $M(\nabla \times \mathbf{F})$, the i-th component of the resulted vector is

$$\begin{aligned}\Big(\nabla \times \big(M(\nabla \times \mathbf{F})\big)\Big)_i &= \sum_{j,n,m,k,\ell} \varepsilon_{ijn} \partial_j \big(M_{nm} \varepsilon_{m\ell k} \partial_\ell f_k\big) \\ &= \sum_{j,n,m,k,\ell} \varepsilon_{ijn} \varepsilon_{m\ell k} \big((\partial_j M_{nm}) \partial_\ell f_k + M_{nm} \partial_{j\ell} f_k\big)\end{aligned}$$

Thus

$$\Big(\nabla \times \big(M(\nabla \times \mathbf{F})\big)\Big)_i = \sum_{j,k,\ell} \widetilde{C}_{ijk\ell} \partial_{j\ell} f_k + \widetilde{R}_i,$$

where

$$\widetilde{C}_{ijk\ell} := \sum_{m,n} \varepsilon_{ijn} \varepsilon_{m\ell k} M_{nm}, \qquad \widetilde{R}_i := \sum_{j,m,n,k,\ell} \varepsilon_{ijn} \varepsilon_{m\ell k} (\partial_j M_{nm}) \partial_\ell f_k.$$

Since

$$\begin{aligned}\varepsilon_{ijn} \varepsilon_{m\ell k} &= \begin{vmatrix} \delta_{im} & \delta_{i\ell} & \delta_{ik} \\ \delta_{jm} & \delta_{j\ell} & \delta_{jk} \\ \delta_{nm} & \delta_{n\ell} & \delta_{nk} \end{vmatrix} \\ &= \delta_{im}\big(\delta_{j\ell}\delta_{nk} - \delta_{n\ell}\delta_{jk}\big) - \delta_{i\ell}\big(\delta_{jm}\delta_{nk} - \delta_{nm}\delta_{jk}\big) + \delta_{ik}\big(\delta_{jm}\delta_{n\ell} - \delta_{nm}\delta_{j\ell}\big),\end{aligned}$$

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we can obtain

$$\begin{aligned}\widetilde{C}_{ijk\ell} &= \sum_{mn}\Big(\delta_{im}\big(\delta_{j\ell}\delta_{nk}-\delta_{n\ell}\delta_{jk}\big)-\delta_{i\ell}\big(\delta_{jm}\delta_{nk}-\delta_{nm}\delta_{jk}\big)\\ &\qquad +\delta_{ik}\big(\delta_{jm}\delta_{n\ell}-\delta_{nm}\delta_{j\ell}\big)\Big)M_{nm}\\ &= \big(\delta_{j\ell}M_{ki}-\delta_{jk}M_{\ell i}\big)-\delta_{i\ell}M_{kj}+\delta_{i\ell}\delta_{jk}tr(M)+\delta_{ik}M_{\ell j}-\delta_{ik}\delta_{j\ell}tr(M)\\ &= \delta_{j\ell}M_{ki}+\delta_{ik}M_{\ell j}-\delta_{jk}M_{\ell i}-\delta_{i\ell}M_{kj}+\big(\delta_{i\ell}\delta_{jk}-\delta_{ik}\delta_{j\ell}\big)tr(M).\end{aligned}$$

□

Theorem 3.22. *Assume that μ is a smooth, positive scalar function and ϵ is a symmetric, positive definite matrix-valued function with smooth entries. The eigenvalues of ϵ are denoted by $\lambda_1(x), \lambda_2(x)$ and $\lambda_3(x)$. Assume there exist positive constants μ_0, Λ, λ such that for all $x \in \Omega$*

$$0 < \mu(x) \le \mu_0$$

$$0 < \lambda \le \lambda_1(x) \le \lambda_2(x) \le \lambda_3(x) \le \Lambda. \tag{3.6.18}$$

Then (3.6.15) and (3.6.16) are uniformly strongly elliptic by letting $M^A = m\mu^{-1}I$ and $M^B = m\mu^{-1}\epsilon$, for arbitrary positive constant m. Here I denotes the 3×3 identity matrix.

Proof. To see whether (3.6.15) and (3.6.16) are strongly elliptic, we only have to check the leading order terms of (3.6.15) and (3.6.16). We divide this proof into two parts, Part A and Part B, to deal with the equation (3.6.15) for A and the equation (3.6.16) for B respectively.

Part A. By Lemma 3.21,

$$\begin{aligned}&\Big(\mu\nabla tr\big(M^A\nabla A\big)-\nabla\times\big(\epsilon^{-1}(\nabla\times A)\big)\Big)_i\\ &\quad=\sum_{jk\ell}\mu\delta_{ij}\partial_j\big(M^A_{\ell k}\partial_\ell A_k\big)-\sum_{jk\ell}\widetilde{C}^A_{ijk\ell}\partial_{j\ell}A_k-\widetilde{R}^A_i\\ &\quad=\sum_{jk\ell}\big(\mu\delta_{ij}M^A_{\ell k}-\widetilde{C}^A_{ijk\ell}\big)\partial_{j\ell}A_k+\sum_{jk\ell}\mu\delta_{ij}(\partial_j M^A_{\ell k})\partial_\ell A_k-\widetilde{R}^A_i\\ &\quad=\sum_{jk\ell}C^A_{ijk\ell}\partial_{j\ell}A_k+\sum_{jk\ell}\mu\delta_{ij}(\partial_j M^A_{\ell k})\partial_\ell A_k-\widetilde{R}^A_i,\end{aligned}$$

where $C^A_{ijk\ell}=\mu\delta_{ij}M^A_{\ell k}-\widetilde{C}^A_{ijk\ell}$ are the coefficients of the leading order terms of (3.6.15) and

$$\widetilde{C}^A_{ijk\ell}=\delta_{j\ell}(\epsilon^{-1})_{ki}+\delta_{ik}(\epsilon^{-1})_{\ell j}-\delta_{jk}(\epsilon^{-1})_{\ell i}-\delta_{i\ell}(\epsilon^{-1})_{kj}+\big(\delta_{i\ell}\delta_{jk}-\delta_{ik}\delta_{j\ell}\big)tr(\epsilon^{-1}).$$

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Recall that (3.6.15) is called uniformly strongly elliptic in some domain Ω if there exists a positive $c_0 > 0$ independent of $x \in \Omega$ such that

$$\sum_{ijk\ell} C^A_{ijk\ell}(x) a_i a_k b_j b_\ell \geq c_0 |\mathbf{a}|^2 |\mathbf{b}|^2 \tag{3.6.19}$$

for any $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ and for all $x \in \Omega$. Now

$$\begin{aligned}
\sum_{ijk\ell} C^A_{ijk\ell} a_i a_k b_j b_\ell &= \sum_{ijk\ell} \big(\mu \delta_{ij} M^A_{\ell k} - \widetilde{C}^A_{ijk\ell}\big) a_i a_k b_j b_\ell \\
&= \mu(\mathbf{a}\cdot\mathbf{b})(\mathbf{b}^T M^A \mathbf{a}) \\
&\quad - \sum_{ijk\ell} \Bigg(\delta_{j\ell}(\epsilon^{-1})_{ki} + \delta_{ik}(\epsilon^{-1})_{\ell j} - \delta_{jk}(\epsilon^{-1})_{\ell i} \\
&\qquad\quad - \delta_{i\ell}(\epsilon^{-1})_{kj} + \big(\delta_{i\ell}\delta_{jk} - \delta_{ik}\delta_{j\ell}\big) tr(\epsilon^{-1}) \Bigg) a_i a_k b_j b_\ell \\
&= \mu(\mathbf{a}\cdot\mathbf{b})(\mathbf{b}^T M^A \mathbf{a}) \\
&\quad - \Bigg(|\mathbf{b}|^2 (\mathbf{a}^T \epsilon^{-1} \mathbf{a}) + |\mathbf{a}|^2 (\mathbf{b}^T \epsilon^{-1} \mathbf{b}) - (\mathbf{a}\cdot\mathbf{b})(\mathbf{b}^T \epsilon^{-1} \mathbf{a}) \\
&\qquad\quad - (\mathbf{a}\cdot\mathbf{b})(\mathbf{a}^T \epsilon^{-1} \mathbf{b}) + tr(\epsilon^{-1})(\mathbf{a}\cdot\mathbf{b})^2 - tr(\epsilon^{-1}) |\mathbf{a}|^2 |\mathbf{b}|^2 \Bigg) \\
&= tr(\epsilon^{-1}) |\mathbf{a}|^2 |\mathbf{b}|^2 - |\mathbf{a}|^2 (\mathbf{b}^T \epsilon^{-1} \mathbf{b}) - |\mathbf{b}|^2 (\mathbf{a}^T \epsilon^{-1} \mathbf{a}) - tr(\epsilon^{-1})(\mathbf{a}\cdot\mathbf{b})^2 \\
&\quad + 2(\mathbf{a}\cdot\mathbf{b})\big(\mathbf{b}^T \epsilon^{-1} \mathbf{a}\big) + \mu(\mathbf{a}\cdot\mathbf{b})\big(\mathbf{b}^T M^A \mathbf{a}\big)
\end{aligned}$$

since ϵ (and hence ϵ^{-1}) is symmetric. Let S be the orthogonal matrix such that $\epsilon = S^T D S$, where $D = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$. Thus $\epsilon^{-1} = S^T D^{-1} S$. Also let $M^A = S^T N^A S$. By letting $v = S\mathbf{a}/|\mathbf{a}|$ and $w = S\mathbf{b}/|\mathbf{b}|$, it's easy to see that (3.6.19) holds for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ iff

$$\begin{aligned}
tr(\epsilon^{-1}) - (\mathbf{w}^T D^{-1} \mathbf{w}) - (v^T D^{-1} \mathbf{v}) - tr(\epsilon^{-1})(\mathbf{v}\cdot\mathbf{w})^2 \\
+ 2(\mathbf{v}\cdot\mathbf{w})\big(\mathbf{w}^T D^{-1} \mathbf{v}\big) + \mu(\mathbf{v}\cdot\mathbf{w})\big(\mathbf{w}^T N^A \mathbf{v}\big) \geq c_0
\end{aligned}$$

for all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ such that $|\mathbf{v}| = |\mathbf{w}| = 1$. Note that $tr(\epsilon^{-1}) = tr(D^{-1}) = \lambda_1^{-1} + \lambda_2^{-1} + \lambda_3^{-1}$. In summary, we find that (3.6.15) is uniformly strongly elliptic on Ω iff

$$\inf_{\mathbf{x}\in\Omega} \left(\min_{|\mathbf{v}|=|\mathbf{w}|=1} F(\mathbf{v}, \mathbf{w}) \right) > 0, \tag{3.6.20}$$

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where

$$
\begin{aligned}
F(\mathbf{v},\mathbf{w}) &= \Big(tr(D^{-1}) - (\mathbf{w}^T D^{-1}\mathbf{w}) - (\mathbf{v}^T D^{-1}\mathbf{v}) - tr(D^{-1})(\mathbf{v}\cdot\mathbf{w})^2 \\
&\quad + 2(\mathbf{v}\cdot\mathbf{w})\big(\mathbf{w}^T D^{-1}\mathbf{v}\big)\Big) + \mu(\mathbf{v}\cdot\mathbf{w})\big(\mathbf{w}^T N^A \mathbf{v}\big) \\
&=: G(\mathbf{v},\mathbf{w}) + \mu(\mathbf{v}\cdot\mathbf{w})\big(\mathbf{w}^T N^A \mathbf{v}\big).
\end{aligned}
$$

We will show that

$$
G(\mathbf{v},\mathbf{w}) \geq \lambda_3^{-1}\big(1-(\mathbf{v}\cdot\mathbf{w})^2\big) \tag{3.6.21}
$$

under the constraints $|\mathbf{v}| = |\mathbf{w}| = 1$. Then, by choosing $M^A = m\mu^{-1}I$ for some positive constant m, we also have $N^A = m\mu^{-1}I$, and

$$
\begin{aligned}
F(\mathbf{v},\mathbf{w}) &= G(\mathbf{v},\mathbf{w}) + m(\mathbf{v}\cdot\mathbf{w})^2 \\
&\geq \lambda_3^{-1}\big(1-(\mathbf{v}\cdot\mathbf{w})^2\big) + m(\mathbf{v}\cdot\mathbf{w})^2 \\
&= \lambda_3^{-1} + (m-\lambda_3^{-1})(\mathbf{v}\cdot\mathbf{w})^2.
\end{aligned}
$$

Now since $0 \leq (\mathbf{v}\cdot\mathbf{w})^2 \leq 1$, if $m \geq \lambda_3^{-1}$, we have $F(\mathbf{v},\mathbf{w}) \geq \lambda_3^{-1}$, while if $m < \lambda_3^{-1}$, we have $F(\mathbf{v},\mathbf{w}) \geq \lambda_3^{-1} + (m-\lambda_3^{-1}) = m$. Remember that $\lambda_3^{-1}(x) \geq \Lambda^{-1}$ on Ω, we conclude that $F(\mathbf{v},\mathbf{w}) \geq \min(\Lambda^{-1}, m)$ for all $|\mathbf{v}| = |\mathbf{w}| = 1$ and all $x \in \Omega$.

It remains to show (3.6.21). For this, note that

$$
G(\mathbf{v},\mathbf{w}) = \sum_{j=1,2,3} \lambda_j^{-1}\Big(1 - w_j^2 - v_j^2 - (\mathbf{v}\cdot\mathbf{w})^2 + 2(\mathbf{v}\cdot\mathbf{w})v_j w_j\Big) =: \sum_j \lambda_j^{-1} K_j.
$$

We can prove $K_j \geq 0$ as follows: Since $(\mathbf{v}\cdot\mathbf{w}) - v_1 w_1 = v_2 w_2 + v_3 w_3$, by Schwarz inequality we have

$$
|(\mathbf{v}\cdot\mathbf{w}) - v_1 w_1| \leq \sqrt{v_2^2 + v_3^2}\sqrt{w_2^2 + w_3^2} = \sqrt{1-v_1^2}\sqrt{1-w_1^2}.
$$

Taking square, we obtain

$$
(\mathbf{v}\cdot\mathbf{w})^2 - 2(\mathbf{v}\cdot\mathbf{w})v_1 w_1 + v_1^2 w_1^2 \leq 1 - v_1^2 - w_1^2 + v_1^2 w_1^2,
$$

which means $K_1 \geq 0$. Similarly $K_2, K_3 \geq 0$. As a consequence, since $\lambda_1^{-1} \geq \lambda_2^{-1} \geq \lambda_3^{-1}$, we

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have

$$G(\mathbf{v},\mathbf{w}) \geq \lambda_3^{-1}(K_1+K_2+K_3) = \lambda_3^{-1}\big(1-(\mathbf{v}\cdot\mathbf{w})^2\big),$$

which completes the proof of Part A.

Part B. For (3.6.16), we have

$$\begin{aligned}
&\Big(\epsilon\nabla tr\big(M^B\nabla\mathbf{B}\big) - \nabla\times\big(\mu^{-1}(\nabla\times\mathbf{B})\big)\Big)_i \\
&\quad= \sum_{jk\ell}\epsilon_{ij}\partial_j\big(M^B_{\ell k}\partial_\ell B_k\big) - \sum_{jk\ell}\widetilde{C}^B_{ijk\ell}\partial_{j\ell}B_k - \widetilde{R}^B_i \\
&\quad= \sum_{jk\ell}\big(\epsilon_{ij}M^B_{\ell k} - \widetilde{C}^B_{ijk\ell}\big)\partial_{j\ell}B_k + \sum_{jk\ell}\epsilon_{ij}(\partial_j M^B_{\ell k})\partial_\ell B_k - \widetilde{R}^B_i,
\end{aligned} \tag{3.6.22}$$

where

$$\begin{aligned}
\widetilde{C}^B_{ijk\ell} &= \delta_{j\ell}\mu^{-1}\delta_{ki} + \delta_{ik}\mu^{-1}\delta_{\ell j} - \delta_{jk}\mu^{-1}\delta_{\ell i} \\
&\qquad - \delta_{i\ell}\mu^{-1}\delta_{kj} + \big(\delta_{i\ell}\delta_{jk} - \delta_{ik}\delta_{j\ell}\big)tr(\mu^{-1}I) \\
&= \mu^{-1}\Big(\delta_{i\ell}\delta_{jk} - \delta_{ik}\delta_{j\ell}\Big).
\end{aligned}$$

Denote the coefficients of the leading order terms of (3.6.22) by $C^B_{ijk\ell}$, we have

$$C^B_{ijk\ell} = \epsilon_{ij}M^B_{\ell k} - \widetilde{C}^B_{ijk\ell} = \epsilon_{ij}M^B_{\ell k} - \mu^{-1}\Big(\delta_{i\ell}\delta_{jk} - \delta_{ik}\delta_{j\ell}\Big).$$

By choosing $M^B = m\mu^{-1}\epsilon$ we obtain

$$\sum_{ijk\ell} C^B_{ijk\ell}a_i a_k b_j b_\ell = \mu^{-1}\Big(m(\mathbf{a}^T\gamma\mathbf{b})^2 - \Big((\mathbf{a}\cdot\mathbf{b})^2 - |\mathbf{a}|^2|\mathbf{b}|^2\Big)\Big)$$

for all $\mathbf{a},\mathbf{b}\in\mathbb{R}^3$. Remember that $\epsilon = S^TDS$. Since we have assumed $\mu^{-1}\geq\mu_0$ for some positive constant μ_0, by letting $\mathbf{v} = S\mathbf{a}/|\mathbf{a}|$ and $\mathbf{w} = S\mathbf{b}/|\mathbf{b}|$ for $\mathbf{a},\mathbf{b}\neq 0$, we see to prove $C^B_{ijk\ell}a_i a_k b_j b_\ell \geq c_0|\mathbf{a}|^2|\mathbf{b}|^2$ for some constant $c_0>0$ is equivalent to prove

$$\inf_{\mathbf{x}\in\Omega}\min_{|\mathbf{v}|=|\mathbf{w}|=1} H(\mathbf{v},\mathbf{w}) > 0, \tag{3.6.23}$$

where $H(\mathbf{v},\mathbf{w}) = m(\mathbf{v}^TD\mathbf{w})^2 + \big(1-(\mathbf{v}\cdot\mathbf{w})^2\big)$. Although (3.6.23) looks simpler than (3.6.20), we fail to find a simple method as before to get a clear lower bound. Nevertheless, it is also easy to see that (3.6.23) is true by continuity, as follows: If $(\mathbf{v}\cdot\mathbf{w})^2=1$, then $\mathbf{v}=\pm\mathbf{w}$, and

$$m(\mathbf{v}^TD\mathbf{w})^2 = m(\lambda_1 v_1^2 + \lambda_2 v_2^2 + \lambda_3 v_3^2)^2 \geq m\lambda_1^2.$$

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By continuity, there exists $\varepsilon > 0$ such that for $0 \le 1 - (\mathbf{v}\cdot\mathbf{w})^2 \le \varepsilon$ we have $m(\mathbf{v}^T D\mathbf{w})^2 \ge m\lambda_1^2/2$. Thus for $0 \le 1 - (\mathbf{v}\cdot\mathbf{w})^2 \le \varepsilon$ we have $H(\mathbf{v},\mathbf{w}) \ge m\lambda_1^2/2$. While for $1 - (\mathbf{v}\cdot\mathbf{w})^2 > \varepsilon$, $H(\mathbf{v},\mathbf{w}) > \varepsilon$. Thus under the constraints $|\mathbf{v}| = |\mathbf{w}| = 1$ we obtain

$$H(\mathbf{v},\mathbf{w}) \ge \min(m\lambda_1^2/2, \varepsilon) \ge \min(m\lambda^2/2, \varepsilon),$$

where recall that λ is the lower bound of $\lambda_1(x)$ on Ω. This completes the proof of Part B.

□

Remark 3.23. One can check that the $\widetilde{C}^A$ and $\widetilde{C}^B$ satisfy $\widetilde{C}^A_{ijk\ell} = \widetilde{C}^A_{k\ell ij}$ and $\widetilde{C}^B_{ijk\ell} = \widetilde{C}^B_{k\ell ij}$. And, by choosing $M^A = m\mu^{-1}I$ and $M^B = m\mu^{-1}\epsilon$ as above, the C^A and C^B also satisfy such symmetry. This additional property is useful in the next section.

3.7 Construction of oscillating-decaying solutions for the anisotropic Maxwell system

In this section, we will use the reduction results in section 4.2 to construct oscillating-decaying solutions of (3.6.1). From now on, we suppose that $\mu > 0$ is a C^∞ scalar function and ϵ is a 3×3 real positive definite matrix-valued smooth functions (i.e. every entry is a real C^∞ function) and E , H satisfy

$$\begin{cases} \nabla\times E - ik\mu H = 0 & \text{in } \Omega, \\ \nabla\times H + ik\epsilon E = 0 & \text{in } \Omega. \end{cases}$$

In order to obtain the oscillating-decaying solutions of E and H, we have to construct the oscillating-decaying solutions for A and B. We follow the proof in [48] to construct the oscillating-decaying solutions for A and B, but here we need to derive higher derivatives for A and B.

From [48], we borrow several notations as follows. Assume that $\Omega \subset \mathbb{R}^3$ is an open set with smooth boundary and $\omega \in S^2$ is given. Let $\eta \in S^2$ and $\zeta \in S^2$ be chosen so that $\{\eta, \zeta, \omega\}$ forms an orthonormal system of $\mathbb{R}^3$. We then denote $x' = (x\cdot\eta, x\cdot\zeta)$. Let $t \in \mathbb{R}$, $\Omega_t(\omega) = \Omega \cap \{x\cdot\omega > t\}$ and $\Sigma_t(\omega) = \Omega \cap \{x\cdot\omega = t\}$ be a non-empty open set.

3.7.1 Construction of the oscillating-decaying solutions A and B

In this subsection, we show how the scheme in [48] can be used to derive the oscillating-decaying solutions A and B. Recall that E and H satisfy equation (3.6.2), therefore we need to derive estimates of the higher derivatives for A and B. Note that the main term of $w^A_{\chi_t,b,t,N,\omega}$ (resp. $w^B_{\chi_t,b,t,N,\omega}$) is $\chi_t(x')Q_t e^{i\tau x\cdot\xi} e^{-\tau(x\cdot\omega - t)A^A_t(x')} b$ (resp. $\chi_t(x')Q_t e^{i\tau x\cdot\xi} e^{-\tau(x\cdot\omega - t)A^B_t(x')} b$), which can be

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directly differentiated term by term since it is a multiplication of smooth functions. So we can calculate E and H directly. For convenience, we denote $w = w_{\chi_t,b,t,N,\omega}$ $\gamma = \gamma_{\chi_t,b,t,N,\omega}(x,\tau)$. Without loss of generality, we can use the change of coordinates to assume $t = 0$, $\omega = (0,0,1)$ and $\eta = (1,0,0)$, $\zeta = (0,1,0)$. Define

$$\widetilde{Q_A} := e^{-i\tau x'\cdot\xi'} L_A(e^{i\tau x'\cdot\xi'}\cdot),\ \widetilde{Q_B} := e^{-i\tau x'\cdot\xi'} L_B(e^{i\tau x'\cdot\xi'}\cdot)$$

where $x' = (x_1, x_2)$, $\xi' = (\xi_1, \xi_2)$ with $|\xi'| = 1$ and L_A, L_B have been defined by (3.7.7) and (3.7.8). In the following, we will give all the details for the higher derivatives of E and H.

In [48], the authors used the phase plane method to get a first order ODE system and we want to decouple the equation in order to solve it by direct calculations. The method of construction the oscillating-decaying solution is decomposed into several steps:

Step 1. As mentioned before, we set $\widetilde{Q_A} = e^{-i\tau x'\cdot\xi'} L_A(e^{i\tau x'\cdot\xi'}\cdot)$, $\widetilde{Q_B} := e^{-i\tau x'\cdot\xi'} L_B(e^{i\tau x'\cdot\xi'}\cdot)$ and solve $\widetilde{Q_A} v_A = 0$, $\widetilde{Q_B} v_B = 0$. In the following calculations, we only need to consider $\widetilde{Q_A} v_A = 0$ since $\widetilde{Q_B} v_B = 0$ will follow the same calculations. Let $Q_A = C_A \widetilde{Q_A}$ be the operator which satisfies the leading coefficient of ∂_3^2 is 1 and the existence of C_A is given by the strong ellipticity of L_A and we need to solve $Q_A v_A = 0$ (the same reason for the operator $\widetilde{Q_B}$ and Q_B). Now, We introduce the concept of the order in the following manner. We consider τ, ∂_3 are of order 1, ∂_1, ∂_2 are of order 0 and x_3 is of order -1.

Step 2. Use the Taylor expansion with respect to x_3, we have

$$\begin{aligned} Q_A(x', x_3) &= Q_A(x', 0) + \cdots + \frac{x_3^{N-1}}{(N-1)!}\partial_3^{N-1} Q_A(x', 0) + R \\ &= Q_A^2 + Q_A^1 + \cdots + Q_A^{-N+1} + R \end{aligned}$$

where $\mathrm{ord}(Q_A^j) = j$ and $\mathrm{ord}(R) = -N$. Since we hope that $Q_A v_A = 0$, we have

$$Q_A^2 v_A = -(Q_A^2 + Q_A^1 + \cdots + Q_A^{-N+1} + R) v_A := f.$$

Step 3. Following the paper [48], we denote $D_3 = -i\partial_3$, $\rho = (\xi_1, \xi_2, 0)$ and $\langle a, b\rangle = (\langle a, b\rangle_{ik})$ for $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$, where $\langle a, b\rangle_{ik} = \sum_{jl} C^A_{ijkl} a_j b_l$ with C^A_{ijkl} being the leading coefficient of the second order strongly elliptic operator L_A. If we set $W = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$, where

$$\begin{cases} w_1 = v_A \\ w_2 = -\tau^{-1}\langle e_3, e_3\rangle_{x_3=0} D_3 v_A - \langle e_3, \rho\rangle_{x_3=0} v_A \end{cases},$$

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and use $f = -(Q_A^2 + Q_A^1 + \cdots + Q_A^{-N+1} + R)v_A$, then W will satisfy

$$\begin{aligned} D_3 W &= \tau K^A W + \begin{bmatrix} 0 \\ \tau^{-1} \langle e_3, e_3 \rangle_{x_3=0} f \end{bmatrix} \\ &= (\tau K^A + K_0^A + \cdots + K_{-N}^A + S)W \end{aligned}$$

where K^A is a matrix in depending of x_3 which can be diagonlizable by the property of the strong ellipticity of L_A. Note that each K_j^A's only involves the x' derivatives with $\mathrm{ord}(K_j^A) = j$, $\mathrm{ord}(S) = -N-1$. It is worth to mention that with the help of such special W, then we can solve the ODE system explicitly.

Step 4. Decompose K^A such that

$$\widetilde{K^A} = \tilde{Q}^{-1} K^A \tilde{Q} = \begin{bmatrix} \widetilde{K_+^A} & 0 \\ 0 & \widetilde{K_-^A} \end{bmatrix},$$

where $\mathrm{spec}(\widetilde{K_\pm^A}) \subset \mathbb{C}_\pm := \{\pm Im\lambda > 0\}$ (the existence of $\widetilde{K^A}$ and $\tilde{Q}$ were showed in [48]). If we set $\widehat{W} = \tilde{Q}^{-1} W$, then

$$D_3 \widehat{W} = (\tau \widetilde{K^A} + \widehat{K}_0 + \cdots + \widehat{K}_{-N} + \widehat{S})\widehat{W},$$

Step 5. If we write $\widehat{W} = (I + x_3 A^{(0)} + B^{(0)})\widetilde{W}^{(0)}$ with $A^{(0)}, B^{(0)}$ being differential operators in $\partial_{x'}$ (their coefficients independent of x_3), then

$$\begin{aligned} D_3 \widetilde{W}^{(0)} &= \{\tau \widetilde{K^A} + (\widehat{K}_0 - \tau x_3 A^{(0)} \widetilde{K^A} + \tau x_3 \widetilde{K^A} A^{(0)} - B^{(0)} \widetilde{K^A} \\ &\quad + \widetilde{K^A} B^{(0)} + i A^{(0)}) + \widehat{K'}_{-1} + \cdots\} \widetilde{W}^{(0)} \\ &:= (\tau \widetilde{K^A} + \widetilde{K}_0 + \widehat{K'}_{-1} + \cdots) \widetilde{W}^{(0)} \end{aligned}$$

where $\mathrm{ord}(\widehat{K'}_{-1}) = -1$ and the remainders are at most -2. We choose $A^{(0)}, B^{(0)}$ to be suitable operators and use the same calculations in [48], then we will get

$$\widetilde{K}_0 = \begin{bmatrix} \widetilde{K}_0(1,1) & 0 \\ 0 & \widetilde{K}_0(2,2) \end{bmatrix}$$

to be a diagonal form (here we omit all the details).

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Step 6. Finally, following step 5, we can write

$$\begin{aligned}\widehat{W} &= (I + x_3 A^{(0)} + \tau^{-1} B^{(0)})(I + x_3^2 A^{(1)} + \tau^{-1} x_3 B^{(1)} + \tau^{-2} C^{(1)}) \cdots \\ &\quad \times (I + x_3^{N+1} A^{(N)} + \tau^{-1} x_3^N B^{(N)} + \tau^{-2} x_3^{N-1} C^{(N)}) \widetilde{W}^{(N)}\end{aligned}$$

with suitable $A^{(j)}, B^{(j)}$ and $C^{(j)}$ for $j = 0, 1, 2, \cdots, N$ ($C^{(0)} = 0$), then $\widetilde{W}^{(N)}$ satisfies

$$D_3 \widetilde{W}^{(N)} = \{\tau \widetilde{K^A} + \widetilde{K}_0 + \cdots + \widetilde{K}_{-N} + \widetilde{S}\} \widetilde{W}^{(N)},$$

with all $\widetilde{K}_{-j}$ are decoupled for $0 \le j \le N$ and $\text{ord}(\widetilde{S}) = -N - 1$. If we omit the term $\widetilde{S}$, we can find an approximated solution of the form

$$\hat{v}_A^{(N)} = \sum_{j=0}^{N+1} \hat{v}_{-j,A}^{(N)}$$

satisfying

$$D_3 \hat{v}_A^{(N)} = \{\tau \widetilde{K_+^A} + \widetilde{K}_0(1,1) + \cdots + \widetilde{K}_{-N}(1,1)\} \hat{v}_A^{(N)}$$

and each $\hat{v}_{-j,A}^{(N)}$ has to satisfy

$$\begin{cases} D_3 \hat{v}_{0,A}^{(N)} = \tau \widetilde{K_+^A} \hat{v}_{0,A}^{(N)}, & \hat{v}_{0,A}^{(N)}|_{x_3=0} = \chi_t(x')b, \\ D_3 \hat{v}_{-1,A}^{(N)} = \tau \widetilde{K_+^A} \hat{v}_{-1,A}^{(N)} + \widetilde{K}_0(1,1) \hat{v}_{0,A}^{(N)}, & \hat{v}_{-1,A}^{(N)}|_{x_3=0} = 0, \\ \vdots & \\ D_3 \hat{v}_{-N-1,A}^{(N)} = \tau \widetilde{K_+^A} \hat{v}_{-N-1,A}^{(N)} + \sum_{j=0}^{N} \widetilde{K}_{-j}(1,1) \hat{v}_{-j,A}^{(N)}, & \hat{v}_{-N-1,A}^{(N)}|_{x_3=0} = 0, \end{cases}$$

where $\chi_t(x') \in C_0^\infty(\mathbb{R}^2)$ and $b \in \mathbb{C}^3$. Thus, by solving this ODE system we can get the following estimates:

$$\|x_3^\beta \partial_{x'}^\alpha (\hat{v}_{-j,A}^{(N)})\|_{L^2(\mathbb{R}_+^3)} \le c\tau^{-\beta-j-1/2} \tag{3.7.1}$$

for $0 \le j \le N+1$. Moreover, if we set $\hat{V}_A^{(N)} = \begin{bmatrix} \hat{v}_A^{(N)} \\ 0 \end{bmatrix}$, then it satisfies

$$\begin{cases} \hat{V}_A^{(N)} - \{\tau \widetilde{K^A} + \widetilde{K}_0 + \cdots + \widetilde{K}_{-N}\} \hat{V}_A^{(N)} = \tilde{R}, \\ \hat{V}_A^{(N)}|_{x_3=0} = \begin{bmatrix} \chi_t(x')b \\ 0 \end{bmatrix}, \end{cases}$$

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where

$$\|\tilde{R}\|_{L^2(\mathbb{R}^3_+)} \leq c\tau^{-N-3/2}.$$

Step 7. Finally, if we define the function $\tilde{v}_A = \begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \\ \tilde{v}_3 \end{bmatrix}$, with $\tilde{v}_j$ being the jth component of the vector $\tilde{Q}(I+x_3A^{(0)}+\tau^{-1}B^{(0)})(I+x_3^2A^{(1)}+\tau^{-1}x_3B^{(1)}+\tau^{-2}C^{(1)})\cdots(I+x_3^{N+1}A^{(N)}+\tau^{-1}x_3^NB^{(N)}+\tau^{-2}x_3^{N-1}C^{(N)})\hat{V}_A^{(N)}$ and set $w_A = \exp(i\tau x'\cdot\xi')\tilde{v}_A$, we will get that

$$\begin{aligned} w_A &= Q\exp(i\tau x'\cdot\xi')\exp(i\tau x_3\widetilde{K_+^A}(x'))\chi_t(x')b + \exp(i\tau x'\cdot\xi')\tilde{\Gamma}(x,\tau) \\ &= Q\exp(i\tau x'\cdot\xi')\exp(-i\tau x_3(-\widetilde{K_+^A}(x')))\chi_t(x')b + \Gamma(x,\tau) \end{aligned}$$

and

$$w_A|_{x_3=0} = \exp(i\tau x'\cdot\xi')(\chi_t(x')Qb + \beta_0(x',\tau),$$

where $\beta_0(x',\tau) = \tilde{\Gamma}(x',0,\tau)$ is supported in $\mathrm{supp}(\chi_t)$. Note that the function $\tilde{\gamma}$ comes from the combination of $\hat{v}_{-j,A}^{(N)}$'s, for $j = 1,2,\cdots,N+1$. Now, we derive higher derivative estimates for the oscillating-decaying solutions, back to see all the $\hat{v}_{-j,A}^{(N)}$'s separately. In fact, only need to see $\hat{v}_{-1,A}^{(N)}$. From the estimate (3.7.1), we know that the estimate is independent of the derivative of x' variables, all we need to concern is the ∂_3 derivative. From the equation

$$D_3\hat{v}_{-1,A}^{(N)} = \tau\widetilde{K_+^A}\hat{v}_{-1,A}^{(N)} + \widetilde{K}_0(1,1)\hat{v}_{0,A}^{(N)} \tag{3.7.2}$$

and the standard regularity theory of ODEs(ordinary differential equations), we know that $\hat{v}_{-1,A}^{(N)} \in C^\infty$ if all the coefficients are smooth. Moreover, note that $\widetilde{K}_+$ independent of x_3, then we can differentiate (3.7.2) directly, to get

$$\begin{aligned} D_3^2\hat{v}_{-1,A}^{(N)} &= D_3[\tau\widetilde{K_+^A}\hat{v}_{-1,A}^{(N)} + \widetilde{K}_0(1,1)\hat{v}_{0,A}^{(N)}] \\ &= \tau\widetilde{K_+^A}(D_3\hat{v}_{-1,A}^{(N)}) + (D_3\widetilde{K}_0(1,1))\hat{v}_{0,A}^{(N)} + \widetilde{K}_0(1,1)D_3\hat{v}_{0,A}^{(N)} \\ &= \tau^2(\widetilde{K_+^A})^2\hat{v}_{-1,A}^{(N)} + \tau\widetilde{K_+^A}\widetilde{K}_0(1,1)\hat{v}_{0,A}^{(N)} + (D_3\widetilde{K}_0(1,1))\hat{v}_{0,A}^{(N)} \\ &\quad +\tau\widetilde{K}_0(1,1)\widetilde{K_+^A}\hat{v}_{0,A}^{(N)}. \end{aligned}$$

Thus, we can obtain that

$$\|x_3^\beta\partial_{x'}^\alpha\partial_3^\eta(\hat{v}_{-1,A}^{(N)})\|_{L^2(\mathbb{R}^3_+)} \leq c\tau^{-\beta+\eta-3/2},$$

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for all $\eta \leq 2$. Inductively, we have

$$\|x_3^\beta \partial_{x'}^\alpha \partial_3^\eta (\hat{v}_{-1,A}^{(N)})\|_{L^2(\mathbb{R}_+^3)} \leq c\tau^{-\beta+\eta-3/2},$$

for all $\eta \in \mathbb{N}$. Similarly, for other $\hat{v}_{-j,A}^{(N)}$ with $2 \leq j \leq N+1$, we can get similar estimate in the following:

$$\|x_3^\beta \partial_{x'}^\alpha \partial_3^\eta (\hat{v}_{-j,A}^{(N)})\|_{L^2(\mathbb{R}_+^3)} \leq c\tau^{\eta-\beta-j-1/2}$$

$\forall \eta \in \mathbb{N} \cup \{0\}$. Therefore, Γ satisfies

$$\|\partial_x^\alpha \Gamma\|_{L^2(\Omega_s)} \leq c\tau^{|\alpha|-3/2} e^{-\tau(s-t)\lambda}$$

on $\Omega_s := \{x_3 > s\} \cap \Omega$ for $s \geq 0$ and $\forall |\alpha| \in \mathbb{N} \cup \{0\}$. Note that since each $\hat{v}_{-j,A}^{(N)}$'s are smooth, we can get the smoothness of $\tilde{R}$ and

$$\|\partial_x^\alpha \tilde{R}\|_{L^2(\mathbb{R}_+^3)} \leq c\tau^{|\alpha|-N-3/2}$$

for all $|\alpha| \in \mathbb{N} \cup \{0\}$. Furthermore, we have that

$$\|\partial_x^\alpha (Q_A \widetilde{v_A})\|_{L^2(\Omega_0)} \leq c\tau^{|\alpha|-N-1/2}.$$

Step 8. Now let $u = w + r = e^{i\tau x'\cdot\xi'}\tilde{v} + r$ and r be the solution to the boundary value problem

$$\begin{cases} L_A r = -e^{i\tau x'\cdot\xi'} \widetilde{Q_A} \widetilde{v_A} & \text{in } \Omega_0 \\ r = 0 & \text{on } \partial\Omega_0 \end{cases}.$$

However, note that $\Omega_0 = \{x_3 > 0\} \cap \Omega$ is not a smooth domain since $\partial\Omega_0 = (\{x_3 = 0\} \cap \Omega) \cup (\{x_3 > 0\} \cap \partial\Omega)$. Note that the oscillating-decaying solution exists in the half space, from the construction, we know that the solution is independent of the domain Ω. Let $\widetilde{\Omega} \subset \mathbb{R}_+^3$ be a open bounded smooth domain containing Ω with $\{x_3 = 0\} \cap \Omega \subset \partial\widetilde{\Omega}$, from the construction, it is easy to see the form of oscillating-decaying solution does not depend on the domain Ω, then we can extend r to be defined on $\widetilde{\Omega}$ and call it $\tilde{r}(x)$. Here we can also extend $\widetilde{v_A}$ to be defined on $\widetilde{\Omega}$, still denote $\widetilde{v_A}$ and all the decaying estimates will hold since our estimates were considered in $\mathbb{R}_+^3$, then we have

$$\begin{cases} L_A \tilde{r} = -e^{i\tau x'\cdot\xi'} \widetilde{Q_A} \widetilde{v_A} & \text{in } \widetilde{\Omega}, \\ \tilde{r} = 0 & \text{on } \partial\widetilde{\Omega}. \end{cases}$$

Note that all the coefficients are smooth, we apply a well-known elliptic regularity theorem (The-

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orem2.3, [12]), then we will get $\tilde{r} \in C^k(\Omega)$ $\forall k$ (recall that $\partial\Omega \in C^\infty$) and

$$\|\tilde{r}\|_{H^{k+1}(\Omega;\mathbb{R}^3)} \leq c\|\widetilde{Q_A}\widetilde{v_A}\|_{H^k(\Omega;\mathbb{R}^3)}.$$

Hence $\|\partial_x^\alpha r\|_{L^2(\Omega_0)} \leq \|\partial_x^\alpha \tilde{r}\|_{L^2(\widetilde{\Omega})} \leq c\tau^{|\alpha|-N+1/2}$ for all $|\alpha| \leq k$, $\forall k \in \mathbb{N}$. Similarly, we can construct the oscillating decaying solution for $L_B B = 0$. Then we represent A and B to be two oscillating-decaying solution in the following form:

$$\begin{cases} A = & w^A_{\chi_t,b,t,N,\omega} + r^A_{\chi_t,b,t,N,\omega} \text{ in } \Omega_t(\omega), \\ A = & e^{i\tau x\cdot\xi}\{\chi_t(x')Q_t(x')b + \beta^A_{\chi_t,t,b,N,\omega}\} \text{ on } \Sigma_t(\omega), \\ B = & w^B_{\chi_t,b,t,N,\omega} + r^B_{\chi_t,b,t,N,\omega} \text{ in } \Omega_t(\omega), \\ B = & e^{i\tau x\cdot\xi}\{\chi_t(x')Q_t(x')b + \beta^B_{\chi_t,t,b,N,\omega}\} \text{ on } \Sigma_t(\omega), \end{cases}$$

where

$$\begin{cases} w^A_{\chi_t,b,t,N,\omega} = \chi_t(x')Q_t e^{i\tau x\cdot\xi}e^{-\tau(x\cdot\omega-t)A_t^A(x')}b + \gamma^A_{\chi_t,b,t,N,\omega}(x,\tau), \\ w^B_{\chi_t,b,t,N,\omega} = \chi_t(x')Q_t e^{i\tau x\cdot\xi}e^{-\tau(x\cdot\omega-t)A_t^B(x')}b + \gamma^B_{\chi_t,b,t,N,\omega}(x,\tau), \end{cases}$$

$\gamma^A_{\chi_t,b,t,N,\omega}$ and $\gamma^B_{\chi_t,b,t,N,\omega}$ satisfy (3.7.10) and (3.7.11).

3.7.2 Construct oscillating-decaying solutions for E and H

We can construct oscillating-decaying solutions for E and H in the following.

Theorem 3.24. *Given $\{\eta, \zeta, \omega\}$ an orthonormal system of $\mathbb{R}^3$, $x' = (x\cdot\eta, x\cdot\zeta)$ and $t \in \mathbb{R}$. We set $\Omega_t(\omega) = \Omega \cap \{x\cdot\omega > t\}$ and $\Sigma_t(\omega) = \Omega \cap \{x\cdot\omega = t\}$, then We can construct two types OD solutions for the Maxwell system in $\Omega_t(\omega)$ which can be useful for penetrable and impenetrable obstacles respectively. There exist two solutions of (3.7.7) of the forms. The first one is*

$$\begin{cases} E = F_A^1(x)e^{i\tau x\cdot\xi}e^{-\tau(x\cdot\omega-t)A_t^A(x')}b + \Gamma^{A,1}_{\chi_t,b,t,N,\omega}(x,\tau) + r^{A,1}_{\chi_t,b,t,N,\omega}(x,\tau) \quad \text{in } \Omega_t(\omega), \\ H = F_A^2(x)e^{i\tau x\cdot\xi}e^{-\tau(x\cdot\omega-t)A_t^A(x')}b + \Gamma^{A,2}_{\chi_t,b,t,N,\omega}(x,\tau) + r^{A,2}_{\chi_t,b,t,N,\omega}(x,\tau) \quad \text{in } \Omega_t(\omega), \end{cases} \tag{3.7.3}$$

where $F_A^1(x) = O(\tau)$, $F_A^2(x) = O(\tau^2)$ are some smooth functions and for $|\alpha| = j$, $j = 1, 2$, we have

$$\begin{cases} \|\Gamma^{A,j}_{\chi_t,b,t,N,\omega}(x,\tau)\|_{L^2(\Omega_t(\omega))} \leq c\tau^{|\alpha|-3/2}e^{-\tau(s-t)a_A}, \\ \|r^{A,j}_{\chi_t,b,t,N,\omega}(x,\tau)\|_{L^2(\Omega_t(\omega))} \leq c\tau^{j-N+1/2}, \end{cases} \tag{3.7.4}$$

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for some positive constants a_A and c. The second one has the form

$$\begin{cases} E = G_B^2(x)e^{i\tau x\cdot\xi}e^{-\tau(x\cdot\omega-t)A_t^B(x')}b + \Gamma^{B,2}_{\chi_t,b,t,N,\omega}(x,\tau) + r^{B,2}_{\chi_t,b,t,N,\omega}(x,\tau) & \text{in } \Omega_t(\omega), \\ H = G_B^1(x)e^{i\tau x\cdot\xi}e^{-\tau(x\cdot\omega-t)A_t^B(x')}b + \Gamma^{B,1}_{\chi_t,b,t,N,\omega}(x,\tau) + r^{B,1}_{\chi_t,b,t,N,\omega}(x,\tau) & \text{in } \Omega_t(\omega), \end{cases} \tag{3.7.5}$$

where $G_B^1(x) = O(\tau)$,$G_B^2(x) = O(\tau^2)$ are some smooth functions and for $|\alpha| = j$, $j = 1, 2$, we have

$$\begin{cases} \|\Gamma^{B,j}_{\chi_t,b,t,N,\omega}(x,\tau)\|_{L^2(\Omega_t(\omega))} \le c\tau^{|\alpha|-3/2}e^{-\tau(s-t)a_B}, \\ \|r^{B,j}_{\chi_t,b,t,N,\omega}(x,\tau)\|_{L^2(\Omega_t(\omega))} \le c\tau^{j-N+1/2}, \end{cases} \tag{3.7.6}$$

for some positive constants a_B and c.

Proof. We want to find special solutions $A, B \in (C^\infty(\overline{\Omega_t(\omega)}\backslash\partial\Sigma_t(\omega)) \cap C^0(\overline{\Omega_t(\omega)}))^3$ with $\tau \gg 1$ satisfying Dirichlet boundary problems

$$\begin{cases} L_A A := \mu\nabla tr(M^A\nabla A) - \nabla\times(\epsilon^{-1}(\nabla\times A)) + k^2\mu A = 0 & \text{in } \Omega_t(\omega) \\ A = e^{i\tau x\cdot\xi}\left\{\chi_t(x')Q_t(x')b + \beta^A_{\chi_t,t,b,N,\omega}\right\} & \text{on } \Sigma_t(\omega), \end{cases} \tag{3.7.7}$$

and

$$\begin{cases} L_B B := \epsilon\nabla tr(M^B\nabla B) - \nabla\times(\mu^{-1}(\nabla\times B)) + k^2\epsilon B = 0 & \text{in } \Omega_t(\omega) \\ B = e^{i\tau x\cdot\xi}\left\{\chi_t(x')Q_t(x')b + \beta^B_{\chi_t,t,b,N,\omega}\right\} & \text{on } \Sigma_t(\omega), \end{cases} \tag{3.7.8}$$

where $\xi \in S^2$ lying in the span of $\{\eta, \zeta\}$ is chosen and fixed, $\chi_t(x') \in C_0^\infty(\mathbb{R}^2)$ with $\text{supp}(\chi_t) \subset \Sigma_t(\omega)$, $Q_t(x')$ is a nonzero smooth function and $0 \neq b \in \mathbb{C}^3$ and N is some large nature number. Moreover, $\beta^A_{\chi_t,b,t,N,\omega}(x',\tau), \beta^B_{\chi_t,b,t,N,\omega}(x',\tau)$ are smooth functions supported in $\text{supp}(\chi_t)$ satisfying:

$$\|\beta^A_{\chi_t,b,t,N,\omega}(\cdot,\tau)\|_{L^2(\mathbb{R}^2)} \le c\tau^{-1},\ \|\beta^B_{\chi_t,b,t,N,\omega}(\cdot,\tau)\|_{L^2(\mathbb{R}^2)} \le c\tau^{-1}$$

for some constant $c > 0$. From now on, we use c to denote a general positive constant whose value may vary from line to line. As in [48], A, B satisfy second order strongly elliptic equations, then it can be written as

$$\begin{cases} A = A_{\chi_t,b,t,N,\omega} = w^A_{\chi_t,b,t,N,\omega} + r^A_{\chi_t,b,t,N,\omega} \\ B = B_{\chi_t,b,t,N,\omega} = w^B_{\chi_t,b,t,N,\omega} + r^B_{\chi_t,b,t,N,\omega} \end{cases}$$

with

$$\begin{cases} w^A_{\chi_t,b,t,N,\omega} = \chi_t(x')Q_t e^{i\tau x\cdot\xi}e^{-\tau(x\cdot\omega-t)A_t^A(x')}b + \Gamma^A_{\chi_t,b,t,N,\omega}(x,\tau) \\ w^B_{\chi_t,b,t,N,\omega} = \chi_t(x')Q_t e^{i\tau x\cdot\xi}e^{-\tau(x\cdot\omega-t)A_t^B(x')}b + \Gamma^B_{\chi_t,b,t,N,\omega}(x,\tau) \end{cases} \tag{3.7.9}$$

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and $r^A_{\chi_t b,t,N,\omega}, r^B_{\chi_t b,t,N,\omega}$ satisfying

$$\|r^A_{\chi_t,b,t,N,\omega}\|_{H^k(\Omega_t(\omega))} \leq c\tau^{k-N+1/2}, \ \|r^B_{\chi_t,b,t,N,\omega}\|_{H^k(\Omega_t(\omega))} \leq c\tau^{k-N+1/2}, \tag{3.7.10}$$

where $A^A_t(\cdot)$, $A^B_t(\cdot)$ are smooth matrix functions with its real part $\mathrm{Re}A^A_t(x') > 0$, $\mathrm{Re}A^B_t(x') > 0$ and $\Gamma^A_{\chi_t,b,t,N,\omega}$, $\Gamma^B_{\chi_t,b,t,N,\omega}$ are a smooth functions supported in $\mathrm{supp}(\chi_t)$ satisfying

$$\begin{cases} \|\partial^\alpha_x \Gamma^A_{\chi_t,b,t,N,\omega}\|_{L^2(\Omega_s(\omega))} \leq c\tau^{|\alpha|-3/2} e^{-\tau(s-t)a_A} \\ \|\partial^\alpha_x \Gamma^B_{\chi_t,b,t,N,\omega}\|_{L^2(\Omega_s(\omega))} \leq c\tau^{|\alpha|-3/2} e^{-\tau(s-t)a_B} \end{cases} \tag{3.7.11}$$

for $|\alpha| \in \mathbb{N} \cup \{0\}$ and $s \geq t$, where $a_A, a_B > 0$ are some constants depending on $A^A_t(x')$ and $A^B_t(x')$ respectively. We give details of the construction of A and B with the estimates (3.7.10) and (3.7.11).

We have derived the explicit representation of A and B. Recall that E and H are represented in terms of A and B as follows

$$\begin{cases} E = -\dfrac{i}{k}\epsilon^{-1}\nabla \times (\mu^{-1}(\nabla \times B)) - \epsilon^{-1}(\nabla \times A), \\ H = \dfrac{i}{k}\mu^{-1}\nabla \times (\epsilon^{-1}(\nabla \times A)) - \mu^{-1}(\nabla \times B). \end{cases} \tag{3.7.12}$$

Now, we can show that (E, H) satisfies (3.7.3), (3.7.4) and we will use this form to prove Theorem 3.18 for the penetrable case. Similarly, we can show that (E, H) satisfies (3.7.5), (3.7.6) in order to prove Theorem 3.18 for the impenetrable case. All we need to do is to differentiate A and B term by term componentwisely. For the main terms of A and B, we can differentiate $\chi_t(x')Q_t e^{i\tau x\cdot\xi} e^{-\tau(x\cdot\omega - t)A^A_t(x')} b$ and $\chi_t(x')Q_t e^{i\tau x\cdot\xi} e^{-\tau(x\cdot\omega - t)A^B_t(x')} b$ directly and it is easy to see that

$$\begin{cases} \nabla \times A = \tau\widetilde{F_A}(x) e^{i\tau x\cdot\xi} e^{-\tau(x\cdot\omega - t)A^A_t(x')} b + \nabla \times \Gamma^A_{\chi_t,b,t,N,\omega}(x,\tau) + \nabla \times r^A_{\chi_t,b,t,N,\omega}, \\ \nabla \times B = \tau\widetilde{F_B}(x) e^{i\tau x\cdot\xi} e^{-\tau(x\cdot\omega - t)A^B_t(x')} b + \nabla \times \Gamma^B_{\chi_t,b,t,N,\omega}(x,\tau) + \nabla \times r^B_{\chi_t,b,t,N,\omega}, \end{cases}$$

where $\widetilde{F_A}(x)$ and $\widetilde{F_B}(x)$ are smooth matrix-valued functions and support in $\mathrm{supp}(\chi_t(x'))$. For the penetrable obstacle case, we choose $A = w^A_{\chi_t,b,t,N,\omega} + r^A_{\chi_t,b,t,N,\omega}$ to be the oscillating-decaying solution satisfies $L_A A = 0$ and $B \equiv 0$ (also satisfies $L_B 0 = 0$) in $\Omega_t(\omega)$, then (3.7.12) will become to

$$\begin{cases} E = -\epsilon^{-1}(\nabla \times A), \\ H = \dfrac{i}{k}\mu^{-1}\nabla \times (\epsilon^{-1}(\nabla \times A)), \end{cases}$$

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which means

$$\begin{cases} E = F_A^1(x)e^{i\tau x\cdot\xi}e^{-\tau(x\cdot\omega-t)A_t^A(x')}b + \Gamma^{A,1}_{\chi_t,b,t,N,\omega}(x,\tau) + r^{A,1}_{\chi_t,b,t,N,\omega}(x,\tau), \\ H = F_A^2(x)e^{i\tau x\cdot\xi}e^{-\tau(x\cdot\omega-t)A_t^A(x')}b + \Gamma^{A,2}_{\chi_t,b,t,N,\omega}(x,\tau) + r^{A,2}_{\chi_t,b,t,N,\omega}(x,\tau), \end{cases}$$

where $F_A^1(x)$, $F_A^2(x)$ are smooth functions consisting $\mu(x)$, $\epsilon(x)$, $Q_t(x')$, $A_t^A(x')$ and their curls (it can be seen by directly calculation). Moreover, by suitable choice of b (for example, we can choose $b \neq 0$ is not parallel to ξ), we will get $F_A^1(x) = O(\tau)$ and $F_A^2(x) = O(\tau^2)$. Moreover, $\Gamma^{A,1}_{\chi_t,b,t,N,\omega}$ and $\Gamma^{A,2}_{\chi_t,b,t,N,\omega}$ satisfy (3.7.11) for $|\alpha| = 1$ and $|\alpha| = 2$, respectively, $r^{A,1}_{\chi_t,b,t,N,\omega}$ and $r^{A,1}_{\chi_t,b,t,N,\omega}$ satisfy (3.7.10) for $k = 1$ and $k = 2$, respectively. Similarly, for the impenetrable obstacle case, we choose $A = 0$ and $B = w^B_{\chi_t,b,t,N,\omega} + r^B_{\chi_t,b,t,N,\omega}$ in $\Omega_t(\omega)$, then

$$\begin{cases} E = G_B^2(x)e^{i\tau x\cdot\xi}e^{-\tau(x\cdot\omega-t)A_t^B(x')}b + \Gamma^{B,2}_{\chi_t,b,t,N,\omega}(x,\tau) + r'^{B,2}_{\chi_t,b,t,N,\omega}(x,\tau), \\ H = G_B^1(x)e^{i\tau x\cdot\xi}e^{-\tau(x\cdot\omega-t)A_t^B(x')}b + \Gamma^{B,1}_{\chi_t,b,t,N,\omega}(x,\tau) + r^{B,1}_{\chi_t,b,t,N,\omega}(x,\tau), \end{cases}$$

where $G_B^1(x) = O(\tau)$ and $G_B^2(x) = O(\tau^2)$ and $\Gamma^{B,j}_{\chi_t,b,t,N,\omega}$ satisfies (3.7.11) for $|\alpha| = j$ and $r^{B,j}_{\chi_t,b,t,N,\omega}$ satisfies (3.7.11) for $k = j$. □

3.8 Proof of Theorem 3.13

Recall that we have constructed the oscillating-decaying (OD) solutions in the previous section and note that OD solutions only exists on a half space. Similar to the anisotropic elliptic case, we need to use the *Runge approximation property* for the anisotropic Maxwell system, which means that we can find a sequence of solutions satisfying the anisotropic Maxwell system and approximates to the OD solution on the unknown obstacle.

3.8.1 Runge approximation property: Maxwell version

We derive the Runge approximation property for the following anisotropic Maxwell equation

$$\begin{cases} \nabla \times E - ik\mu H = 0 \\ \nabla \times H + ik\epsilon E = 0 \end{cases} \quad \text{in } \Omega,$$

where μ is a smooth scalar function defined on Ω and ϵ is a 3×3 smooth positive definite matrix. Recall that

$$\mu(x) \geq \mu_0 > 0 \text{ and } \sum_{i.j=1}^{3} \epsilon_{ij}(x)\xi_i\xi_j \geq \epsilon_0|\xi|^2 \ \forall \xi \in \mathbb{R}^3.$$

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If we set $u = \begin{pmatrix} H \\ E \end{pmatrix}$ and

$$L := i \begin{pmatrix} \epsilon^{-1} & 0 \\ 0 & \mu^{-1} I_3 \end{pmatrix} \begin{pmatrix} 0 & \nabla\times \\ -\nabla\times & 0 \end{pmatrix} + kI_6, \tag{3.8.1}$$

then we have

$$Lu = 0, \tag{3.8.2}$$

where I_j means $j \times j$ identity matrix for $j = 3, 6$.

Theorem 3.25. *Let D and Ω be two open bounded domains with C^∞ boundary in $\mathbb{R}^3$ with $D \Subset \Omega$. If $u \in (H(curl, D))^2$ satisfies*

$$Lu = 0 \text{ in } D.$$

Given any compact subset $K \subset D$ and any $\epsilon > 0$, there exists $U \in (H(curl, \Omega))^2$ such that

$$LU = 0 \text{ in } \Omega,$$

and $\|U - u\|_{H(curl,K)} < \epsilon$, where $\|f\|_{H(curl,\Omega)} = \left(\|f\|_{L^2(\Omega)} + \|curl f\|_{L^2(\Omega)}\right)$.

Proof. The proof is standard and it is based on weak unique continuation property for the anisotropic Maxwell system L in (3.8.1) and the Hahn-Banach theorem. The unique continuation property of the system L is proved in [33]. For more details, how to derive the Runge approximation property from the weak unique continuation, we refer readers to [31]. □

Now, we start to prove Theorem 3.13 by using the Runge approximation property and the OD solutions to prove Theorem 3.18. We define B to be an open ball in $\mathbb{R}^3$ such that $\overline{\Omega} \subset B$. Assume that $\widetilde{\Omega} \subset \mathbb{R}^3$ is an open Lipschitz domain with $\overline{B} \subset \widetilde{\Omega}$. Recall we have set $\omega \in S^2$ and $\{\eta, \zeta, \omega\}$ forms an orthonormal basis of $\mathbb{R}^3$ and $t_0 = \inf_{x\in D} x \cdot \omega = x_0 \cdot \omega$, where $x_0 = x_0(\omega) \in \partial D$. The proof is divided in the following two cases: the penetrable case and the impenetrable case.

3.8.2 Penetrable Case

For the anisotropic Maxwell's equation

$$\begin{cases} \nabla \times E = ik\mu H \\ \nabla \times H = -ik\epsilon E \\ \operatorname{div}(\epsilon E) = 0 \\ \operatorname{div}(\mu H) = 0, \end{cases} . \tag{3.8.3}$$

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for any $t \le t_0$ and $\eta > 0$ small enough, in section 3, we have constructed

$$\begin{cases} E_{t-\eta} = F_A^1(x) e^{i\tau x\cdot\xi} e^{-\tau(x\cdot\omega-(t-\eta))A_t^A(x')} b + \Gamma^{A,1}_{\chi_t,b,t-\eta,N,\omega}(x,\tau) + r^{A,1}_{\chi_t,b,t-\eta,N,\omega}(x,\tau), \\ H_{t-\eta} = F_A^2(x) e^{i\tau x\cdot\xi} e^{-\tau(x\cdot\omega-(t-\eta))A_t^A(x')} b + \Gamma^{A,2}_{\chi_t,b,t-\eta,N,\omega}(x,\tau) + r^{A,2}_{\chi_t,b,t-\eta,N,\omega}(x,\tau), \end{cases}$$

to be the oscillating-decaying solutions satisfying (3.8.3) in $B_{t-\eta}(\omega) = B \cap \{x | x\cdot\omega > t-\eta\}$, where $F_A^1(x) = O(\tau)$ and $F_A^2(x) = O(\tau^2)$. Moreover, $\Gamma^{A,1}_{\chi_t,b,t-\eta,N,\omega}$ and $\Gamma^{A,2}_{\chi_t,b,t-\eta,N,\omega}$ satisfy (3.7.11) for $|\alpha| = 1$ and $|\alpha| = 2$, respectively, $r^{A,1}_{\chi_t,b,t-\eta,N,\omega}$ and $r^{A,1}_{\chi_t,b,t-\eta,N,\omega}$ satisfy (3.7.10) for $k = 1$ and $k = 2$, respectively. Similarly, we have

$$\begin{cases} E_t = F_A^1(x) e^{i\tau x\cdot\xi} e^{-\tau(x\cdot\omega-t)A_t^A(x')} b + \Gamma^{A,1}_{\chi_t,b,t,N,\omega}(x,\tau) + r^{A,1}_{\chi_t,b,t,N,\omega}(x,\tau), \\ H_t = F_A^2(x) e^{i\tau x\cdot\xi} e^{-\tau(x\cdot\omega-t)A_t^A(x')} b + \Gamma^{A,2}_{\chi_t,b,t,N,\omega}(x,\tau) + r^{A,2}_{\chi_t,b,t,N,\omega}(x,\tau), \end{cases}$$

so be the oscillating-decaying solutions satisfying (3.8.3) in $B_t(\omega) = B \cap \{x | x\cdot\omega > t\}$, where $\Gamma^{A,1}_{\chi_t,b,t,N,\omega}$ and $\Gamma^{A,2}_{\chi_t,b,t,N,\omega}$ satisfy (3.7.11) for $|\alpha| = 1$ and $|\alpha| = 2$, respectively, $r^{A,1}_{\chi_t,b,t,N,\omega}$ and $r^{A,1}_{\chi_t,b,t,N,\omega}$ satisfy (3.7.10) for $k = 1$ and $k = 2$, respectively. In fact, from the construction the oscillating-decaying solutions and the property of continuous dependence on parameters in ordinary differential equations in section 3, it is not hard to see that for any τ,

$$\begin{cases} E_{t-\eta} \to E_t \\ H_{t-\eta} \to H_t \end{cases}$$

in $H^2(B_t(\omega))$ as η tends to 0.

Note that $\overline{\Omega_t(\omega)} \subset B_{t-\eta}(\omega)$ for all $t \le t_0$. By using the Runge approximation property, we can see that there exists a sequence of functions $(E_{\eta,\ell}, H_{\eta,\ell})$, $\ell = 1, 2, \cdots$, such that

$$\begin{cases} E_{\eta,\ell} \to E_{t-\eta} \\ H_{\eta,\ell} \to H_{t-\eta} \end{cases} \quad \text{in } H(curl, B_t(\omega)),$$

as $\ell \to \infty$, where $(E_{\eta,\ell}, H_{\eta,\ell})$ satisfy (3.8.3) in $\widetilde{\Omega}$ for all $\eta > 0, \ell \in \mathbb{N}$. Recall that the indicator function $I_\rho(\tau, t)$ was defined by the formula:

$$I_\rho(\tau, t) := \lim_{\eta\to 0} \lim_{\ell\to\infty} I_\rho^{\epsilon,\ell}(\tau, t),$$

where

$$I_\rho^{\eta,\ell}(\tau, t) := ik\tau \int_{\partial\Omega} (\nu \times H_{\eta,\ell}) \cdot (\overline{(\Lambda_D - \Lambda_\emptyset)(\nu \times H_{\eta,\ell})} \times \nu) dS.$$

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We prove the Theorem 1.1 for the penetrable obstacle case. For the anisotropic penetrable obstacle problem

$$\begin{cases} \nabla \times E - ik\mu H = 0 & \text{in } \Omega, \\ \nabla \times H + ik\epsilon E = 0 & \text{in } \Omega, \\ \nu \times H = f & \text{on } \partial\Omega, \end{cases} \tag{3.8.4}$$

where k is not an eigenvalue of (3.8.4). Moreover, we assume μ is a positive smooth scalar function, $\epsilon = \epsilon_0(x) - \chi_D \epsilon_D(x)$, where ϵ_0 is symmetric positive definite smooth matrix, $\epsilon_D(x)$ is a symmetric smooth matrix with $\det \epsilon_D(x) \neq 0 \ \forall x \in D$ and $\chi_D = \begin{cases} 1 & x \in D \\ 0 & \text{otherwise} \end{cases}$. Moreover, we need $\epsilon = \epsilon(x)$ is a positive definite matrix satisfying the uniform elliptic condition. Recall that when $\epsilon(x) = \epsilon_0(x)$, we have constructed E_t and H_t which are oscillating-decaying solutions defined on the half space for the anisotropic Maxwell's equation

$$\begin{cases} \nabla \times E - ik\mu H = 0 & \text{in } \Omega, \\ \nabla \times H + ik\epsilon E = 0 & \text{in } \Omega, \end{cases} \tag{3.8.5}$$

and $\{(E_{\eta,\ell}, H_{\eta,\ell})\}$ are sequence of functions satisfying (3.8.5) defined on the whole Ω. Therefore, we can define the boundary data $f_{\eta,\ell} = \nu \times H_{\eta,\ell}$ on $\partial\Omega$ and solve (E, H) satisfies (3.8.4). Let $\widetilde{H_{\eta,\ell}} = H - H_{\eta,\ell}$ be the reflected solution, then $\widetilde{H_{\eta,\ell}}$ satisfies

$$\begin{cases} \nabla \times (\epsilon^{-1} \nabla \times \widetilde{H_{\eta,\ell}}) - k^2 \mu \widetilde{H_{\eta,\ell}} = -\nabla \times ((\epsilon^{-1}(x) - \epsilon_0^{-1}(x)) \nabla \times H_{\eta,\ell}) \text{ in } \Omega, \\ \nu \times \widetilde{H_{\eta,\ell}} = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.8.6}$$

Lemma 3.26. *We have the following estimates*

1.

$$-\tau^{-1} I_\rho^{\eta,\ell} \geq \int_D [\epsilon(\epsilon^{-1} - \epsilon_0^{-1})^{-1} \epsilon_0^{-1} \nabla \times H_{\eta,\ell}] \cdot (\nabla \times \overline{H_{\eta,\ell}}) dx - k^2 \int_\Omega \mu |\widetilde{H_{\eta,\ell}}|^2 dx.$$

2.

$$\tau^{-1} I_\rho^{\eta,\ell}(\tau, t) \geq \int_D ((\epsilon_0^{-1} - \epsilon^{-1}) \nabla \times H_{\eta,\ell}) \cdot (\nabla \times \overline{H_{\eta,\ell}}) dx - k^2 \int_\Omega \mu |\widetilde{H_{\eta,\ell}}|^2 dx.$$

Proof. First, we need to prove the following identity

$$\begin{aligned} -\tau^{-1} I_\rho^{\eta,\ell}(\tau, t) &= \int_\Omega \big((\epsilon^{-1} - \epsilon_0^{-1}) \nabla \times H_{\eta,\ell}\big) \cdot (\nabla \times \overline{H_{\eta,\ell}}) dx \\ &\quad - \int_\Omega (\epsilon^{-1} \nabla \times \widetilde{H_{\eta,\ell}}) \cdot (\nabla \times \overline{\widetilde{H_{\eta,\ell}}}) dx - k^2 \int_\Omega \mu |\widetilde{H_{\eta,\ell}}|^2 dx. \end{aligned} \tag{3.8.7}$$

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Multiplying $\overline{\widetilde{H_{\eta,l}}}$ in the equation (3.8.6) and integrating by parts we have

$$\int_\Omega (\epsilon^{-1}\nabla \times \widetilde{H_{\eta,\ell}}) \cdot (\nabla \times \overline{\widetilde{H_{\eta,\ell}}}) dx - k^2 \int_\Omega \mu |\widetilde{H_{\eta,\ell}}|^2 dx$$
$$+ \int_\Omega ((\epsilon^{-1} - \epsilon_0^{-1})\nabla \times H_{\eta,\ell}) \cdot (\nabla \times \overline{\widetilde{H_{\eta,\ell}}}) dx = 0,$$

$$\int_\Omega (\epsilon^{-1}\nabla \times \widetilde{H_{\eta,\ell}}) \cdot (\nabla \times \overline{\widetilde{H_{\eta,\ell}}}) dx - k^2 \int_\Omega \mu |\widetilde{H_{\eta,\ell}}|^2 dx$$
$$- \int_\Omega (\epsilon^{-1} - \epsilon_0^{-1})\nabla \times H_{\eta,\ell}) \cdot (\nabla \times \overline{H_{\eta,\ell}}) dx \tag{3.8.8}$$
$$= - \int_\Omega ((\epsilon^{-1} - \epsilon_0^{-1})\nabla \times H_{\eta,\ell}) \cdot (\nabla \times \overline{H}) dx. \tag{3.8.9}$$

On the other hand, $H(x)$ satisfies

$$\nabla \times (\epsilon^{-1}(x)\nabla \times H(x)) - k^2 \mu H(x) = 0, \tag{3.8.10}$$

then multiply by $H_{\eta,l}(x)$ in the equation (3.8.10) and integrating by parts we have

$$\int_\Omega ((\epsilon^{-1} - \epsilon_0^{-1})\nabla \times H_{\eta,\ell}) \cdot (\nabla \times \overline{H}) dx \quad = \quad \int_{\partial\Omega} (\epsilon^{-1}\nabla \times \overline{H}) \cdot (\nu \times H_{\eta,\ell}) ds$$
$$- \int_{\partial\Omega} (\epsilon_0^{-1}\nabla \times H_{\eta,\ell}) \cdot (\nu \times \overline{H}) ds \tag{3.8.11}$$

Thus, combine (3.8.8), (3.8.11) and $\int_{\partial\Omega} (\nu \times \overline{H_{\eta,\ell}}) \cdot (\epsilon_0^{-1}\nabla \times H_{\eta,\ell}) ds$ is real, then we have

$$\int_\Omega (\epsilon^{-1}\nabla \times \widetilde{H_{\eta,\ell}}) \cdot (\nabla \times \overline{\widetilde{H_{\eta,\ell}}}) dx - k^2 \int_\Omega \mu |\widetilde{H_{\eta,\ell}}|^2 dx$$
$$- \int_\Omega ((\epsilon^{-1} - \epsilon_0^{-1})\nabla \times H_{\eta,\ell}) \cdot (\nabla \times \overline{H_{\eta,\ell}}) dx \tag{3.8.12}$$
$$= \int_{\partial\Omega} (\nu \times H_{\eta,\ell}) \cdot (\epsilon^{-1}\nabla \times \overline{H}) ds - \int_{\partial\Omega} (\nu \times \overline{H}) \cdot (\epsilon_0^{-1}\nabla \times H_{\eta,\ell}) ds$$
$$= \int_{\partial\Omega} (\nu \times H_{\eta,\ell}) \cdot (\epsilon^{-1}\nabla \times \overline{H}) ds - \int_{\partial\Omega} (\nu \times \overline{H_{\eta,\ell}}) \cdot (\epsilon_0^{-1}\nabla \times H_{\eta,\ell}) ds$$
$$= \int_{\partial\Omega} (\nu \times H_{\eta,\ell}) \cdot (\epsilon^{-1}\overline{\nabla \times H}) ds - \int_{\partial\Omega} (\nu \times H_{\eta,\ell}) \cdot (\epsilon_0^{-1}\overline{\nabla \times H_{\eta,\ell}}) ds$$
$$= \int_{\partial\Omega} (\nu \times H_{\eta,\ell}) \cdot [\overline{-ikE + ikE_{\eta,\ell}}] ds$$
$$= ik \int_{\partial\Omega} (\nu \times H_{\eta,\ell}) \cdot [\overline{(\Lambda_D - \Lambda_\emptyset)(\nu \times H_{\eta,\ell})} \times \nu] ds$$
$$= \tau^{-1} I_\rho^{\eta,\ell}. \tag{3.8.13}$$

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Second, we show the following identity

$$\begin{aligned}
&\int_\Omega (\epsilon_0^{-1}\nabla \times \widetilde{H_{\eta,\ell}}) \cdot (\nabla \times \overline{\widetilde{H_{\eta,\ell}}})dx - k^2 \int_\Omega \mu|\widetilde{H_{\eta,\ell}}|^2 dx \\
&+ \int_\Omega ((\epsilon^{-1}(x) - \epsilon_0^{-1}(x))\nabla \times H) \cdot (\nabla \times \overline{H})dx \\
=& -\tau^{-1} I_\rho^{\eta,\ell}.
\end{aligned} \tag{3.8.14}$$

Replacing $H_{\eta,\ell}(x)$ by $H(x) - \widetilde{H_{\eta,\ell}}(x)$ in the equation (3.8.6), then we have

$$\nabla \times \left((\epsilon^{-1} - \epsilon_0^{-1})\nabla \times H\right) + \nabla \times \left(\epsilon_0^{-1}\nabla \times \widetilde{H_{\eta,\ell}}\right) - k^2 \mu \widetilde{H_{\eta,\ell}} = 0 \text{ in } \Omega. \tag{3.8.15}$$

Multiplying $\overline{\widetilde{H_{\eta,l}}}(x)$ in the equation (3.8.15) and using integration by parts we have

$$\begin{aligned}
&\int_\Omega \left((\epsilon^{-1} - \epsilon_0^{-1})\nabla \times H\right) \cdot \left(\nabla \times \overline{\widetilde{H_{\eta,\ell}}}\right) dx \\
&+ \int_\Omega \left(\epsilon_0^{-1}\nabla \times \widetilde{H_{\eta,\ell}}\right) \cdot \left(\nabla \times \overline{\widetilde{H_{\eta,\ell}}}\right) dx - k^2 \int_\Omega \mu \left|\widetilde{H_{\eta,\ell}}\right|^2 dx = 0,
\end{aligned} \tag{3.8.16}$$

since $\nu \times \widetilde{H_{\eta,l}} = 0$ on $\partial\Omega$. Then we can write equation (3.8.16) to be

$$\begin{aligned}
&\int_\Omega \left(\epsilon_0^{-1}\nabla \times \widetilde{H_{\eta,\ell}}\right) \cdot \left(\nabla \times \overline{\widetilde{H_{\eta,\ell}}}\right) dx - k^2 \int_\Omega \mu \left|\widetilde{H_{\eta,\ell}}\right|^2 dx \\
&+ \int_\Omega \left((\epsilon^{-1} - \epsilon_0^{-1})\nabla \times H\right) \cdot (\nabla \times \overline{H})dx \\
=& \int_\Omega \left((\epsilon^{-1} - \epsilon_0^{-1})\nabla \times H\right) \cdot (\nabla \times \overline{H_{\eta,\ell}})dx.
\end{aligned} \tag{3.8.17}$$

Eliminating $H(x)$ by $\widetilde{H_{\eta,l}}(x) + H_{\eta,l}(x)$ in (3.8.17) we have

$$\begin{aligned}
&\int_\Omega \left(\epsilon_0^{-1}\nabla \times \widetilde{H_{\eta,\ell}}\right) \cdot \left(\nabla \times \overline{\widetilde{H_{\eta,\ell}}}\right) dx - k^2 \int_\Omega \mu \left|\widetilde{H_{\eta,\ell}}\right|^2 dx \\
&+ \int_\Omega \left((\epsilon^{-1} - \epsilon_0^{-1})\nabla \times H\right) \cdot (\nabla \times \overline{H})dx \\
=& \int_\Omega ((\epsilon^{-1}(x) - \epsilon_0^{-1}(x))\nabla \times H_{\eta,\ell}) \cdot (\nabla \times \overline{H_{\eta,\ell}})dx \\
&+ \int_\Omega ((\epsilon^{-1}(x) - \epsilon_0^{-1}(x))\nabla \times \widetilde{H_{\eta,\ell}}) \cdot (\nabla \times \overline{H_{\eta,\ell}})dx
\end{aligned} \tag{3.8.18}$$

Again from (3.8.6) and by taking the complex conjugate, we can write

$$\nabla \times (\epsilon^{-1}\nabla \times \overline{\widetilde{H_{\eta,\ell}}}) - k^2 \mu \overline{\widetilde{H_{\eta,\ell}}} + \nabla \times ((\epsilon^{-1}(x) - \epsilon_0^{-1}(x))\nabla \times \overline{H_{\eta,\ell}}) = 0. \tag{3.8.19}$$

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Multiplying by $\widetilde{H_{\eta,l}}(x)$ in the equation (3.8.19) and using integration by parts we have

$$\int_\Omega (\epsilon^{-1}\nabla \times \overline{\widetilde{H_{\eta,\ell}}}) \cdot (\nabla \times \widetilde{H_{\eta,\ell}}) dx - k^2 \int_\Omega \mu |\widetilde{H_{\eta,\ell}}|^2 dx$$
$$+ \int_\Omega ((\epsilon^{-1}(x) - \epsilon_0^{-1}(x))\nabla \times \overline{H_{\eta,\ell}}) \cdot (\nabla \times \widetilde{H_{\eta,\ell}}) dx = 0. \tag{3.8.20}$$

Then from the equations (3.8.18), (3.8.20) and the first identity (3.8.7), we can obtain

$$\begin{aligned}
&\int_\Omega (\epsilon_0^{-1}\nabla \times \widetilde{H_{\eta,\ell}}) \cdot (\nabla \times \overline{\widetilde{H_{\eta,\ell}}}) dx - k^2 \int_\Omega \mu |\widetilde{H_{\eta,\ell}}|^2 dx \\
&\quad + \int_\Omega ((\epsilon^{-1}(x) - \epsilon_0^{-1}(x))\nabla \times H) \cdot (\nabla \times \overline{H}) dx \\
&= \int_\Omega ((\epsilon^{-1}(x) - \epsilon_0^{-1}(x))\nabla \times H_{\eta,\ell}) \cdot (\nabla \times \overline{H_{\eta,\ell}}) dx \\
&\quad - \int_\Omega (\epsilon^{-1}\nabla \times \overline{\widetilde{H_{\eta,\ell}}}) \cdot (\nabla \times \widetilde{H_{\eta,\ell}}) dx + k^2 \int_\Omega \mu |\widetilde{H_{\eta,\ell}}|^2 dx \\
&= -\tau^{-1} I_\rho^{\eta,\ell}.
\end{aligned} \tag{3.8.21}$$

Combine (3.8.21) with the formula

$$\begin{aligned}
&(\epsilon_0^{-1}\nabla \times \widetilde{H_{\eta,\ell}}) \cdot (\nabla \times \overline{\widetilde{H_{\eta,\ell}}}) + ((\epsilon^{-1} - \epsilon_0^{-1})\nabla \times H) \cdot (\nabla \times \overline{H}) \\
=&((\epsilon^{-1} - \epsilon_0^{-1})\nabla \times H) \cdot \nabla \times \overline{H} + \epsilon_0^{-1}(\nabla \times H) \cdot (\nabla \times \overline{H}) \\
&- 2\mathrm{Re}\left\{\epsilon_0^{-1}\nabla \times H \cdot \nabla \times \overline{H_{\eta,\ell}}\right\} + \epsilon_0^{-1}\nabla \times H_{\eta,\ell} \cdot \nabla \times \overline{H_{\eta,\ell}} \\
=&\epsilon^{-1}(\nabla \times H) \cdot (\nabla \times \overline{H}) - 2\mathrm{Re}\left\{\epsilon_0^{-1}\nabla \times H \cdot \nabla \times \overline{H_{\epsilon,l}}\right\} + \epsilon_0^{-1}\nabla \times H_{\epsilon,\ell} \cdot \nabla \times \overline{H_{\epsilon,\ell}} \\
=&\left[\epsilon^{-\frac{1}{2}}\nabla \times H - \epsilon^{\frac{1}{2}}\epsilon_0^{-1}\left(\nabla \times \overline{H_{\eta,\ell}}\right)\right] \cdot \overline{\left[\epsilon^{-\frac{1}{2}}\nabla \times H - \epsilon^{\frac{1}{2}}\epsilon_0^{-1}\left(\nabla \times \overline{H_{\eta,\ell}}\right)\right]} \\
&- \left[\epsilon^{\frac{1}{2}}\epsilon_0^{-1}\left(\nabla \times \overline{H_{\eta,\ell}}\right)\right] \cdot \overline{\left[\epsilon^{\frac{1}{2}}\epsilon_0^{-1}\left(\nabla \times \overline{H_{\eta,\ell}}\right)\right]} + \epsilon_0^{-1}\nabla \times H_{\eta,\ell} \cdot \nabla \times \overline{H_{\eta,\ell}} \\
=&\left[\epsilon^{-\frac{1}{2}}\nabla \times H - \epsilon^{\frac{1}{2}}\epsilon_0^{-1}\left(\nabla \times \overline{H_{\eta,\ell}}\right)\right] \cdot \overline{\left[\epsilon^{-\frac{1}{2}}\nabla \times H - \epsilon^{\frac{1}{2}}\epsilon_0^{-1}\left(\nabla \times \overline{H_{\eta,\ell}}\right)\right]} \\
&+ \left(\epsilon_0^{-1} - \epsilon\epsilon_0^{-2}\right)(\nabla \times H_{\eta,\ell}) \cdot \left(\nabla \times \overline{H_{\eta,\ell}}\right) \\
\geq&[\left(I - \epsilon\epsilon_0^{-1}\right)\epsilon_0^{-1}\nabla \times H_{\eta,\ell}] \cdot (\nabla \times \overline{H_{\eta,\ell}}) \\
\geq&[\epsilon(\epsilon^{-1} - \epsilon_0^{-1})^{-1}\epsilon_0^{-1}\nabla \times H_{\eta,\ell}] \cdot (\nabla \times \overline{H_{\eta,\ell}})
\end{aligned}$$

and note that

$$\left[\epsilon^{-\frac{1}{2}}\nabla \times H - \epsilon^{\frac{1}{2}}\epsilon_0^{-1}\left(\nabla \times \overline{H_{\eta,\ell}}\right)\right] \cdot \overline{\left[\epsilon^{-\frac{1}{2}}\nabla \times H - \epsilon^{\frac{1}{2}}\epsilon_0^{-1}\left(\nabla \times \overline{H_{\eta,\ell}}\right)\right]} \geq 0.$$

Therefore, we get

$$-\tau^{-1} I_\rho^{\eta,\ell} \geq \int_D [\epsilon(\epsilon^{-1} - \epsilon_0^{-1})^{-1}\epsilon_0^{-1}\nabla \times H_{\eta,\ell}] \cdot (\nabla \times \overline{H_{\eta,\ell}}) dx - k^2 \int_\Omega \mu |\widetilde{H_{\eta,\ell}}|^2 dx$$

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which finished the part 1 of Lemma 3.26. Finally, again from (3.8.13), we have

$$\tau^{-1}I_\rho^{\eta,\ell} \geq \int_\Omega ((\epsilon_0^{-1}-\epsilon^{-1})\nabla\times H_{\eta,\ell})\cdot(\nabla\times\overline{H_{\eta,\ell}})dx - k^2\int_\Omega \mu|\widetilde{H_{\eta,\ell}}|^2 dx.$$

□

Remark 3.27. The first inequality will be used when $\left(\epsilon^{-1}-\epsilon_0^{-1}\right)$ is strictly positive definite, i.e.

$$\xi\cdot(\epsilon^{-1}-\epsilon_0^{-1})\xi \geq \Lambda|\xi|^2 \text{ for all } \xi\in\mathbb{R}^3 \text{ and for some } \Lambda>0;$$

and the second inequality will be used when $\left(\epsilon_0^{-1}-\epsilon^{-1}\right)$ is strictly positive definite, i.e.

$$\xi\cdot(\epsilon_0^{-1}-\epsilon^{-1})\xi \geq \lambda|\xi|^2 \text{ for all } \xi\in\mathbb{R}^3 \text{ and for some } \lambda>0.$$

Now, our work is to estimate the lower order term $\widetilde{H_{\eta,\ell}}$.

3.8.2.1 Estimate of the lower order term $\widetilde{H_{\eta,\ell}}$

Proposition 3.28. *Assume Ω is a smooth domain and $D\Subset\Omega$. Then there exist a positive constant C and $\delta>0$ such that*

$$\|\widetilde{H_{\eta,\ell}}\|_{L^2(\Omega)} \leq C\|\nabla\times H_{\eta,\ell}\|_{L^p(D)}$$

for every $p\in(\max\{\frac{4}{3},\frac{2+\delta}{1+\delta}\},2]$.

Proof. We follow the proof of the Proposition 3.2 in [27]. Fix $l\in\mathbb{N}$ and we set $f:=-(\epsilon^{-1}-\epsilon_0^{-1})(\nabla\times H_{\eta,\ell})$, $g=0$. Note that, $\epsilon^{-1}-\epsilon_0^{-1}=\epsilon^{-1}(\epsilon_D\chi_D)\epsilon_0^{-1}$ is supported in D. Then the reflected solution $\widetilde{H_{\eta,\ell}}$ satisfies

$$\begin{cases}\nabla\times(\epsilon^{-1}\nabla\times\widetilde{H_{\eta,\ell}})-k^2\mu\widetilde{H_{\eta,\ell}} = -\nabla\times((\epsilon^{-1}(x)-\epsilon_0^{-1}(x))\nabla\times H_{\eta,\ell}) \text{ in } \Omega,\\ \nu\times\widetilde{H_{\eta,\ell}}=0 \text{ on } \partial\Omega.\end{cases} \tag{3.8.22}$$

From the L^p estimate for the Maxwell type system, if we consider the following problem

$$\begin{cases}\nabla\times(\epsilon^{-1}\nabla\times U)+\epsilon_{\max}^{-1}U=\nabla\times f & \text{in } \Omega,\\ \nu\times U=0 & \text{on } \partial\Omega,\end{cases}$$

has a unique solution in $H_0^{1,q}(curl,\Omega)$, where $\epsilon_{\max}^{-1}$ is the maximum value among all eigenvalues of the matrix $\epsilon^{-1}(x)$ in the region $\overline{\Omega}$. Moreover, we have the estimate

$$\|U\|_{L^p(\Omega)}+\|\nabla\times U\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)} \tag{3.8.23}$$

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for $p \in (\frac{2+\delta}{1+\delta}, 2]$ for some $\delta > 0$ which depends only on Ω. Now, we set $\Pi_{\eta,\ell} = \widetilde{H_{\eta,\ell}} - U$, then $\Pi_{\eta,\ell}$ satisfies

$$\begin{cases} \nabla \times (\epsilon^{-1} \nabla \times \Pi_{\eta,\ell}) - k^2 \mu \Pi_{\eta,\ell} = (k^2\mu + \epsilon_{\max}^{-1}) U \text{ in } \Omega, \\ \nu \times \Pi_{\eta,\ell} = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.8.24}$$

By the well-posedness of (3.8.24) in $H(curl, \Omega)$ for the anisotropic Maxwell's equation (see Appendix), we have

$$\|\Pi_{\eta,\ell}\|_{L^2(\Omega)} + \|\nabla \times \Pi_{\eta,\ell}\|_{L^2(\Omega)} \leq C\|U\|_{L^2(\Omega)} \tag{3.8.25}$$

if k is not an eigenvalue. Moreover, for $p \leq 2$, it is to see that

$$\|\Pi_{\eta,\ell}\|_{L^p(\Omega)} + \|\nabla \times \Pi_{\eta,\ell}\|_{L^p(\Omega)} \leq C\|U\|_{L^2(\Omega)}.$$

Following the proof in the Proposition 3.2 in [27] again, we denote $B^{p,2}_{\frac{1}{p}}(\Omega)$ to be the Sobolev-Besov space, then we have $U \in B^{p,2}_{\frac{1}{p}}(\Omega)$ and the inclusion map $B^{p,2}_{\frac{1}{p}}(\Omega) \to L^2(\Omega)$ is continuous for $p \in (\frac{4}{3}, 2]$. Moreover, since $\nabla \cdot U = 0$ and $\nu \times U = 0$ on $\partial\Omega$ and use Lemma 3.5, we have the estimate

$$\|U\|_{L^2(\Omega)} \leq C\|U\|_{B^{p,2}_{\frac{1}{p}}(\Omega)} \leq C\{\|U\|_{L^p(\Omega)} + \|\nabla \times U\|_{L^p(\Omega)}\} \tag{3.8.26}$$

for $p \in (\frac{4}{3}, 2]$. Combining (3.8.23), (3.8.25) and (3.8.26), we obtain

$$\|\Pi_{\eta,\ell}\|_{L^p(\Omega)} + \|\nabla \times \Pi_{\eta,\ell}\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)} \tag{3.8.27}$$

for $p \in (\max\{\frac{4}{3}, \frac{2+\delta}{1+\delta}\}, 2]$. Since $\widetilde{H_{\eta,\ell}} = \Pi_{\eta,\ell} + U$, by using (3.8.23) and (3.8.27), we have

$$\|\widetilde{H_{\eta,\ell}}\|_{L^p(\Omega)} + \|\nabla \times \widetilde{H_{\eta,\ell}}\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)}. \tag{3.8.28}$$

Since $\nu \times \widetilde{H_{\eta,l}} = 0$ on $\partial\Omega$, we use the Lemma 3.5 again, then we can obtain

$$\begin{aligned} \|\widetilde{H_{\eta,\ell}}\|_{L^2(\Omega)} &\leq C\|\widetilde{H_{\eta,\ell}}\|_{B^{p,2}_{\frac{1}{p}}(\Omega)} \\ &\leq C\{\|\widetilde{H_{\eta,\ell}}\|_{L^p(\Omega)} + \|\nabla \times \widetilde{H_{\eta,\ell}}\|_{L^p(\Omega)} + \|\nabla \cdot \widetilde{H_{\eta,\ell}}\|_{L^p(\Omega)}\}. \end{aligned} \tag{3.8.29}$$

In addition, from (3.8.22), it is easy to see $0 = \nabla \cdot (\mu\widetilde{H_{\eta,\ell}}) = \nabla\mu \cdot \widetilde{H_{\eta,\ell}} + \mu(\nabla \cdot \widetilde{H_{\eta,\ell}})$, then we have

$$\|\nabla \cdot \widetilde{H_{\eta,\ell}}\|_{L^p(\Omega)} \leq \frac{\|\nabla\mu\|_{L^\infty(\Omega)}}{\|\mu\|_{L^\infty(\Omega)}}\|\widetilde{H_{\eta,\ell}}\|_{L^p(\Omega)}. \tag{3.8.30}$$

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Finally, use (3.8.28), (3.8.29) and (3.8.30), we will get

$$\begin{aligned}\|\widetilde{H_{\eta,\ell}}\|_{L^2(\Omega)} &\leq C\{\|\widetilde{H_{\eta,\ell}}\|_{L^p(\Omega)} + \|\nabla \times \widetilde{H_{\eta,\ell}}\|_{L^p(\Omega)}\} \\ &\leq C\|f\|_{L^p(\Omega)} \\ &\leq C\|\nabla \times H_{\eta,\ell}\|_{L^p(D)}. \end{aligned} \tag{3.8.31}$$

□

Remark 3.29. In the reconstruction scheme, we need to take $\limsup_{\ell\to\infty}$ for (3.8.31) on both sides and $H_{t-\eta} \to H_t$ in $H(curl, \Omega_t(\omega))$ as $\eta \to 0$, then we have

$$\lim_{\eta\to 0}\limsup_{\ell\to\infty} \|\widetilde{H_{\eta,\ell}}\|_{L^2(\Omega)} \leq C\|\nabla \times H_t\|_{L^p(D)},$$

for $p \in (\frac{4}{3}, 2]$.

In view of the lower bound, we need to introduce the sets $D_{j,\delta} \subset D$, $D_\delta \subset D$ in the following. Recall that $h_D(\rho) = \inf_{x\in D} x \cdot \rho$ and $t_0 = h_D(\rho) = x_0 \cdot \rho$ for some $x_0 \in \partial D$. $\forall \alpha \in \partial D \cap \{x \cdot \rho = h_D(\rho)\} := K$, define $B(\alpha, \delta) = \{x \in \mathbb{R}^3; |x - \alpha| < \delta\}$ ($\delta > 0$). Note $K \subset \cup_{\alpha\in K} B(\alpha, \delta)$ and K is compact, so there exists $\alpha_1, \cdots, \alpha_m \in K$ such that $K \subset \cup_{j=1}^m B(\alpha_j, \delta)$. Thus, we define

$$D_{j,\delta} := D \cap B(\alpha_j, \delta) \text{ and } D_\delta := \cup_{j=1}^m D_{j,\delta}.$$

It is easy to see that

$$\begin{cases} \int_{D\setminus D_\delta} e^{-p\tau(x\cdot\omega - t_0)A_{t_0}^A(x')} b dx = O(e^{-pa\tau}) \\ \int_{D\setminus D_\delta} e^{-p\tau(x\cdot\omega - t_0)A_{t_0}^B(x')} b dx = O(e^{-pa\tau}) \end{cases}$$

where $A_{t_0}^A(x'), A_{t_0}^B(x')$ are smooth matrix-valued functions with bounded entries and their real part strictly greater than 0. so $\exists a > 0$ such that $\mathrm{Re}A_{t_0}^A(x') \geq a > 0$ and $\mathrm{Re}A_{t_0}^B(x') \geq a > 0$. Let $\alpha_j \in K$, by rotation and translation, we may assume $\alpha_j = 0$ and the vector $\alpha_j - x_0 = -x_0$ is parallel to $e_3 = (0, 0, 1)$. Therefore, we consider the change of coordinates near each α_j as follows:

$$\begin{cases} y' = x' \\ y_3 = x \cdot \rho - t_0, \end{cases}$$

where $x = (x_1, x_2, x_3) = (x', x_3)$ and $y = (y_1, y_2, y_3) = (y', y_3)$. Denote the parametrization of ∂D near α_j by $l_j(y')$, then we have the following estimates. Note that the oscillating-decaying solutions are well-defined in D.

Lemma 3.30. *For $q \leq 2$, $\tau \gg 1$, we have the following estimates.*

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1.

$$\int_D |H_t(x)|^q dx \leq \tau^{2q-1}\sum_{j=1}^{m}\iint_{|y'|<\delta} e^{-aq\tau l_j(y')}dy' + O(\tau^{2q-1}e^{-qa\delta\tau})$$
$$+O(\tau^{2q}e^{-qa\tau}) + O(\tau e^{-c\tau}) + O(\tau^{-2N+5})$$

2.

$$\int_D |H_t|^2 dx \geq C\tau^3\sum_{j=1}^{m}\iint_{|y'|<\delta} e^{-2a\tau l_j(y')}dy' - C\tau^3 e^{-2a\delta\tau}$$
$$-C\tau e^{-2c\tau} - C\tau^{-2N+5}$$

3.

$$\int_D |E_t(x)|^q dx \leq \tau^{q-1}\sum_{j=1}^{m}\iint_{|y'|<\delta} e^{-aq\tau l_j(y')}dy' + O(\tau^{q-1}e^{-qa\delta\tau})$$
$$+O(\tau^{q}e^{-qa\tau}) + O(\tau^{-1}) + O(\tau^{-2N+3})$$

4.

$$\int_D |E_t|^2 dx \geq C\tau\sum_{j=1}^{m}\iint_{|y'|<\delta} e^{-2a\tau l_j(y')}dy' - C\tau e^{-2a\delta\tau}$$
$$-C\tau^{-1} - C\tau^{-2N+3},$$

where E_t and H_t are oscillating-decaying solutions for the penetrable case defined in $\Omega_t(\omega)$.

Proof. The proof is via the representation of the oscillating-decaying solutions of (E_t, H_t). For $\tau \gg 1(\tau \ll \tau^2)$, we have

$$\begin{aligned}
\int_D |H_t|^q dx &\leq C\tau^{2q}\int_D e^{-qa\tau(x\cdot\omega-t_0)}dx + C_q\int_D |\Gamma^{A,2}_{\chi_t,b,t,N,\omega}|^q dx \\
&\quad + C_q\int_D |r^{A,2}_{\chi_t,b,t,N,\omega}|^q dx \\
&\leq C\tau^{2q}\int_{D_\delta} e^{-qa\tau(x\cdot\omega-t_0)}dx + C\tau^{2q}\int_{D\setminus D_\delta} e^{-qa\tau(x\cdot\omega-t_0)}dx \\
&\quad + C_q\int_D |\Gamma^1_{A,B,\gamma,\mu}|^q dx + C_q\int_D |r^1_{A,B,\gamma,\mu}|^q dx \\
&\leq C\tau^{2q}\sum_{j=1}^{m}\iint_{|y'|<\delta}dy'\int_{l_j(y')}^{\delta} e^{-qa\tau y_3}dy_3 + C\tau^{2q}e^{-qa\tau} \\
&\quad + C\|\Gamma^{A,2}_{\chi_t,b,t,N,\omega}\|^2_{L^2(D)} + C\|r^{A,2}_{\chi_t,b,t,N,\omega}\|^2_{L^2(D)} \\
&\leq C\tau^{2q-1}\sum_{j=1}^{m}\iint_{|y'|<\delta} e^{-aq\tau l_j(y')}dy' - \frac{C}{q}\tau^{2q-1}e^{-qa\delta\tau} \\
&\quad + C\tau^{2q}e^{-qa\tau} + C\tau e^{-ca\tau} + C\tau^{-2N+5},
\end{aligned}$$

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where c is a positive constant and a depending only on a_A, a_B. For the lower bound of $\int_D |H_t|^2 dx$, we have

$$
\begin{aligned}
\int_D |H_t|^2 dx &\geq C\tau^4 \int_D e^{-2a\tau(x\cdot\omega - t_0)} dx - C\|\Gamma^{A,2}_{\chi_t,b,t,N,\omega}\|^2_{L^2(\Omega_{t_0}(\omega))} \\
&\quad -C\|r^{A,2}_{\chi_t,b,t,N,\omega}\|^2_{L^2(\Omega_{t_0}(\omega))} \\
&\geq C\tau^4 \int_{D_\delta} e^{-2a\tau(x\cdot\omega - t_0)} dx - C\tau e^{-c\tau} - C\tau^{-2N+5}. \\
&\geq C\tau^3 \sum_{j=1}^{m} \iint_{|y'|<\delta} e^{-2a\tau l_j(y')} dy' - C\tau^3 e^{-2a\delta\tau} \\
&\quad -C\tau e^{-ca\tau} - C\tau^{-2N+5}.
\end{aligned}
$$

It is similar to prove the remaining case, so we omit the proof. □

Lemma 3.31. *We have the following estimate*

$$
\frac{\|H_t\|^2_{L^2(D)}}{\|E_t\|^2_{L^2(D)}} \geq O(\tau^2),\ \tau \gg 1.
$$

Proof. Since ∂D is Lipschitz, we have $l_j(y') \leq C|y'|$. Therefore we have the following estimate

$$
\begin{aligned}
C\tau^3 \sum_{j=1}^{m} \iint_{|y'|<\delta} e^{-2a\tau l_j(y')} dy' &\geq C\tau^3 \sum_{j=1}^{m} \iint_{|y'|<\delta} e^{-2a\tau|y'|} \\
&\geq C\tau \sum_{j=1}^{m} \iint_{|y'|<\tau\delta} e^{-2a|y'|} dy' \\
&= O(\tau).
\end{aligned}
$$

Then we use Lemma 3.30 to get

$$
\begin{aligned}
\frac{\|H_t\|^2_{L^2(D)}}{\|E_t\|^2_{L^2(D)}} &\geq C\tau^2 \frac{1 - \frac{Ce^{-2a\delta\tau} + C\tau^{-2}e^{-2c\tau} + C\tau^{-2N+2}}{\sum_{j=1}^{m}\iint_{|y'|<\delta} e^{-2a\tau l_j(y')} dy'}}{1 - \frac{O(e^{-2\delta a\tau}) + O(\tau e^{-ca\tau}) + O(\tau^{-2N+2})}{\sum_{j=1}^{m}\iint_{|y'|<\delta} e^{-2a\tau l_j(y')} dy'}} \\
&= O(\tau^2) \text{ (if } \tau \gg 1).
\end{aligned}
$$

□

Lemma 3.32. *If $t = h_D(\rho)$, then for some positive constant C, we have*

$$
\liminf_{\tau\to\infty} \int_D \tau |\nabla \times H_t|^2 dx \geq C.
$$

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Proof. Since $l_j(y') \le C|y'|$, we have

$$\begin{aligned}\int_D |\nabla \times H_t(x)|^2 dx &\ge C\int_D |E_t(x)|^2 dx \\ &\ge C\tau \sum_{j=1}^{m} \iint_{|y'|<\delta} e^{-2a\tau l_j(y')} dy' - C\tau e^{-2a\delta\tau} \\ &\quad -C\tau^{-1} - C\tau^{-2N+3} \\ &\ge C\tau \sum_{j=1}^{m} \iint_{|y'|<\delta} e^{-2a\tau|y'|} dy' - C\tau e^{-2a\delta\tau} \\ &\quad -C\tau^{-1} - C\tau^{-2N+3} \\ &\ge C\tau[\tau^{-2}\sum_{j=1}^{m} \iint_{|y'|<\tau\delta} e^{-2a|y'|} dy'] - C\tau e^{-2a\delta\tau} \\ &\quad -C\tau^{-1} - C\tau^{-2N+3} \text{ (as } \tau \gg 1).\end{aligned}$$

Therefore, we have

$$\liminf_{\tau\to\infty} \int_D \tau |\nabla \times H_t|^2 dx \ge C.$$

□

Lemma 3.33. *For $p \in (\max\{\frac{4}{3}, \frac{2+\delta}{1+\delta}\}, 2]$. we have the following*

$$\lim_{\eta\to 0}\limsup_{\ell\to\infty} \frac{\|\widetilde{H_{\eta,\ell}}\|^2_{L^2(\Omega)}}{\|\nabla \times H_t\|^2_{L^2(D)}} \le C\tau^{1-\frac{2}{p}} \ (\tau \gg 1).$$

Proof. From the Proposition 3.28, we have

$$\lim_{\eta\to 0}\limsup_{\ell\to\infty} \|\widetilde{H_{\eta,\ell}}\|_{L^2(\Omega)} \le C\|\nabla \times H_t\|_{L^p(D)}.$$

Then it is easy to see the conclusion. □

Remark 3.34. Recall that the sequence $\{H_{\eta,\ell}\}$ converges to $H_{t+\eta}$ in $H(curl, K)$ as $\ell \to \infty$ for all compact subset $D \Subset K \Subset \Omega$ and $H_{t+\eta} \to H_t$ in $H^2(\Omega_t(\omega))$ as $\eta \to 0$, so we have

$$\|\nabla \times H_{\eta,\ell}\|_{L^p(D)} \to \|\nabla \times H_t\|_{L^p(D)} \text{ and } \|H_{\eta,\ell}\|_{L^2(D)} \to \|H_t\|_{L^2(D)}$$

as $\ell \to \infty$, $\eta \to 0$.

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3.8.2.2 End of the proof of Theorem 1.1 for the penetrable case

First, we prove the case $t < h_D(\rho)$. From (3.8.7), we have

$$
\begin{aligned}
-\tau^{-1} I_\rho^{\eta,\ell}(\tau,t) &= \int_\Omega \big((\epsilon^{-1}-\epsilon_0^{-1})\nabla\times H_{\eta,\ell}\big)\cdot(\nabla\times\overline{H_{\eta,\ell}})dx \\
&\quad -\int_\Omega (\epsilon^{-1}\nabla\times\widetilde{H_{\eta,\ell}})\cdot(\nabla\times\overline{\widetilde{H_{\eta,\ell}}})dx - k^2\int_\Omega \mu|\widetilde{H_{\eta,\ell}}|^2 dx. \qquad (3.8.32)
\end{aligned}
$$

Note that $(\widetilde{E_{\epsilon,\ell}}, \widetilde{H_{\epsilon.\ell}})$ satisfies

$$
\begin{cases}
\nabla\times\widetilde{E_{\eta,\ell}} - ik\mu\widetilde{H_{\eta,\ell}} = 0 & \text{in } \Omega, \\
\nabla\times\widetilde{H_{\eta,\ell}} + ik\gamma\widetilde{E_{\eta,\ell}} = ik(\epsilon_0-\epsilon)E_{\eta,\ell} & \text{in } \Omega,
\end{cases}
$$

and rewrite it as

$$
\nabla\times(\epsilon^{-1}\nabla\times\widetilde{E_{\eta,\ell}}) - k^2\gamma\widetilde{E_{\eta,\ell}} = k^2(\epsilon-\epsilon_0)E_{\eta,\ell}. \qquad (3.8.33)
$$

Thus, we can use the same argument from the Remark 5.4 again to (3.8.33), it is easy to see

$$
\|\widetilde{E_{\eta,\ell}}\|_{L^2(\Omega)} \le C\|E_{\eta,\ell}\|_{L^2(D)}.
$$

In addition, we use the Maxwell's equation and $\epsilon-\epsilon_0 = -\epsilon_D\chi_D$, then we have

$$
\begin{aligned}
\int_\Omega (\epsilon^{-1}\nabla\times\widetilde{H_{\eta,\ell}})\cdot(\nabla\times\overline{\widetilde{H_{\eta,\ell}}})dx &= \int_\Omega (-ik\epsilon\widetilde{E_{\eta,\ell}} + ik(\epsilon_0-\epsilon)E_{\eta,\ell}))\cdot(\nabla\times\overline{\widetilde{H_{\eta,\ell}}})dx \\
&\le C\int_\Omega |\widetilde{E_{\eta,\ell}}|^2dx + C\int_D |E_{\eta,\ell}|^2 dx \qquad (3.8.34) \\
&\le C\int_D |E_{\eta,\ell}|^2 dx.
\end{aligned}
$$

Thus, from (3.8.32), Proposition 3.28, Lemma 3.30 and (3.8.34), we can obtain

$$
|\frac{1}{\tau} I_\rho^{\eta,\ell}(\tau,t)| \le \|E_{\eta,\ell}\|^2_{H(curl,D)} + \|H_{\eta,\ell}\|^2_{H(curl,D)}.
$$

From taking $\ell\to\infty$ and $\eta\to 0$, we have

$$
\begin{aligned}
|\frac{1}{\tau} I_\rho(\tau,t)| &\le |\tau\sum_{j=1}^m \iint_{|y'|<\delta} e^{-2a\tau l_j(y')}dy' + O(\tau^2 e^{-2a\delta\tau}) \\
&\quad + O(\tau^2 e^{-2a\tau}) + O(\tau^{-3}) + O(\tau^{-2N+3}) \\
&\le O(\tau^{-1}) + O(\tau^2 e^{-2a\delta\tau}) \\
&\quad + O(\tau^2 e^{-2a\tau}) + O(\tau^{-3}) + O(\tau^{-2N+3}).
\end{aligned}
$$

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In particular, we get

$$\limsup_{\tau\to\infty} |\frac{1}{\tau} I_\rho(\tau,t)| = 0.$$

Second, we prove the case $t = h_D(\rho)$.

Case 1. $\xi \cdot (\epsilon^{-1} - \epsilon_0^{-1})\xi \geq \Lambda|\xi|^2$ for all $\xi \in \mathbb{R}^3$ for some $\Lambda > 0$.

From the inequality in Lemma 3.26, we have

$$\begin{aligned} -\tau^{-1} I_\rho^{\eta,\ell} &\geq \int_D [\epsilon(\epsilon-\epsilon_0^{-1})^{-1}\epsilon_0^{-1}\nabla\times H_{\eta,\ell}]\cdot(\nabla\times\overline{H_{\eta,\ell}})dx - k^2\int_\Omega \mu|\widetilde{H_{\eta,\ell}}|^2dx \\ &\quad -k^2\int_\Omega \mu|\widetilde{H_{\eta,\ell}}|^2dx \\ &\geq C\int_D |\nabla\times H_{\eta,\ell}|^2dx - c\|\widetilde{H_{\eta,\ell}}\|^2_{L^2(\Omega)}. \end{aligned}$$

By using the definition $I_\rho(\tau,t) := \lim_{\eta\to0}\lim_{\ell\to\infty} I_\rho^{\epsilon,\ell}(\tau,t)$, $\{H_{\eta,\ell}\}$ converges to H_t in $H(curl,K)$ for all compact subset $D \Subset K \Subset \Omega$ as $\ell\to\infty$, $\eta\to0$, we have

$$\begin{aligned} \frac{-I_\rho(\tau,t)}{\|\nabla\times H_t\|^2_{L^2(D)}} &\geq C\tau\left[1 - C\lim_{\epsilon\to0}\limsup_{\ell\to\infty}\frac{\|\widetilde{H}_\ell\|^2_{L^2(\Omega)}}{\|\nabla\times H_t\|^2_{L^2(D)}}\right] \\ &\geq C\tau(1-C\tau^{1-\frac{2}{p}}). \end{aligned}$$

Hence, using Lemma 3.32 we deduce that for $\tau \gg 1$,

$$|I_\rho(\tau,h_D(\rho))| \geq C > 0$$

which finishes the proof.

Case 2. $\xi\cdot(\gamma_0^{-1}-\gamma^{-1})\xi \geq \lambda|\xi|^2$ for all $\xi\in\mathbb{R}^3$ for some $\lambda > 0$.

Similarly, using the inequality in Lemma 3.26, we have

$$\tau^{-1}I_\rho^{\eta,\ell}(\tau,t) \geq \int_D((\epsilon_0^{-1}-\epsilon^{-1})\nabla\times H_{\eta,\ell})\cdot(\nabla\times\overline{H_{\eta,\ell}})dx - k^2\int_\Omega\mu|H_{\eta,\ell}|^2dx.$$

Then use the same argument as in **Case 1** we can finish the proof.

3.8.3 Impenetrable Case

We give the proof of the second part of Theorem 1.1, since it is the hardest part. The other cases are easy since we have proved it in the penetrable case. In addition, the upper bound is easy because of the well-posedness and the L^p estimate for the indicator function, but the lower bound is not easy to see. In the following proof, we will use the layer potential properties for the exterior isotropic Maxwell's equation (with the Silver-Müller radiation condition) and the perturbation argument from the anisotropic Maxwell's equation compared with the isotropic case. In the impenetrable

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case, we have chosen the oscillating-decaying solution as the following form

$$\begin{cases} E_t = G_B^2(x) e^{i\tau x\cdot\xi} e^{-\tau(x\cdot\omega - t)A_t^B(x')} b + \Gamma^{B,2}_{\chi_t,b,t,N,\omega}(x,\tau) + r^{,B,2}_{\chi_t,b,t,N,\omega}(x,\tau), \\ H_t = G_B^1(x) e^{i\tau x\cdot\xi} e^{-\tau(x\cdot\omega - t)A_t^B(x')} b + \Gamma^{B,1}_{\chi_t,b,t,N,\omega}(x,\tau) + r^{B,1}_{\chi_t,b,t,N,\omega}(x,\tau), \end{cases}$$

where $G_B^1(x) = O(\tau)$ and $G_B^2(x) = O(\tau^2)$ and $\Gamma^{B,j}_{\chi_t,b,t,N,\omega}$ satisfies (3.7.11) for $|\alpha| = j$ and $r^{B,j}_{\chi_t,b,t,N,\omega}$ satisfies (3.7.11) for $k = j$.

We start by the following lemma.

Lemma 3.35. *Assume that μ is a smooth scalar function and γ is a matrix-valued function. Let $(E,H) \in H(curl;\Omega\backslash\bar{D}) \times H(curl;\Omega\backslash\bar{D})$ be a solution of the problem*

$$\begin{cases} \nabla\times E - ik\mu H = 0 & \text{in } \Omega\backslash\bar{D}, \\ \nabla\times H + i\epsilon E = 0 & \text{in } \Omega\backslash\bar{D}, \\ \nu\times E = f & \text{on } \partial\Omega, \\ \nu\times H = 0 & \text{on } \partial D, \end{cases} \tag{3.8.35}$$

with $f \in TH^{-1/2}(\partial\Omega)$. If we put $f_{\eta,\ell} = \nu\times E_{\eta,\ell}$ with $\{E_{\eta,\ell}\}$ is obtained by the Runge approximation property. Then we have the identity

$$\begin{aligned} -\frac{1}{\tau} I_\rho^{\eta,\ell}(\tau,t) &= -\int_D \{|\nabla\times E_{\eta,\ell}(x)|^2 - k^2|E_{\eta,\ell}(x)|^2\}dx \\ &\quad - \int_{\Omega\backslash\bar{D}} \{|\nabla\times \widetilde{E_{\eta,\ell}}(x)|^2 - k^2|\widetilde{E_{\eta,\ell}}(x)|^2\}dx \\ &= \int_D \{|\nabla\times H_{\eta,\ell}(x)|^2 - k^2|H_{\eta,\ell}(x)|^2\}dx \\ &\quad + \int_{\Omega\backslash\bar{D}} \{|\nabla\times \widetilde{H_{\eta,\ell}}(x)|^2 - k^2|\widetilde{H_{\eta,\ell}}(x)|^2\}dx \end{aligned}$$

and the inequality

$$-\frac{1}{\tau} I_\rho^{\eta,\ell}(\tau,t) \geq \int_D \{|\nabla\times H_{\eta,\ell}(x)|^2 - k^2|H_{\eta,\ell}(x)|^2\}dx - k^2\int_{\Omega\backslash\bar{D}} |\widetilde{H_{\eta,\ell}}(x)|^2\}dx,$$

where $\widetilde{E_{\eta,\ell}} = E - E_{\eta,\ell}$ and $\widetilde{H_{\eta,\ell}} = H - H_{\eta,\ell}$ are described in section 5.

Proof. Use the integration by parts and the boundary condition, we have

$$\int_{\Omega\backslash\bar{D}} \epsilon^{-1}(\nabla\times E)\cdot\overline{(\nabla\times\widetilde{E_{\eta,\ell}})} - k^2\epsilon E\cdot\overline{\widetilde{E_{\eta,\ell}}}dx = -(\int_{\partial\Omega} - \int_{\partial D}) ik(\nu\times H)\cdot\overline{\widetilde{E_{\eta,\ell}}}dS = 0.$$

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Adding this to

$$\begin{aligned} I_\rho^{\eta,\ell} &= \int_{\partial\Omega} (\nu \times E_{\eta,\ell}) \cdot \overline{(-ikH + ikH_{\eta,\ell})} dS \\ &= \int_{\Omega\setminus\bar{D}} -(\mu^{-1}\nabla \times E_{\eta,\ell}) \cdot (\overline{\nabla \times E}) + k^2(\mu E_{\eta,\ell}) \cdot \bar{E} dx \\ &\quad + \int_{\Omega} \mu^{-1}|\nabla \times E_{\eta,\ell}|^2 - k^2(\mu E_{\eta,\ell}) \cdot \overline{E_{\eta,\ell}} dx + \int_{\partial D} (\nu \times E_{\eta,\ell}) \cdot (\overline{-ikH}) dS \end{aligned}$$

due to the zero boundary condition on ∂D we have the last term is vanishing. □

From the above estimate, it only need to control the lower order term $\int_{\Omega\setminus\bar{D}} |\widetilde{H_{\eta,\ell}}(x)|^2 dx$.

3.8.3.1 Estimate of the lower order term $\widetilde{H_{\eta,\ell}}$

Proposition 3.36. *Let Ω be a C^1 domain, $D \Subset \Omega$ be Lipschitz. Then there exists a positive constant C independent of $(\widetilde{E_{\eta,\ell}}, \widetilde{H_{\eta,\ell}})$ and $(E_{\eta,\ell}, H_{\eta,\ell})$ such that*

$$\int_{\Omega\setminus\bar{D}} |\widetilde{H_{\eta,\ell}}(x)|^2 dx \leq C\{\|\nabla \times H_{\eta,\ell}\|^2_{L^p(D)} + \|H_{\eta,\ell}\|^2_{H^{s+1/2}(D)}\},$$

for all p and s such that $\max\{2-\delta, 4/3\} < p \leq 2$ *and* $0 < s \leq 1$ *with* $\delta > 0$.

Proof. **Step 1.** Before proving the Proposition 3.36, we consider the anisotropic Maxwell's equation in Ω as follows:

$$\begin{cases} \nabla \times E_{\eta,\ell} - ik\mu H_{\eta,\ell} = 0 & \text{in } \Omega, \\ \nabla \times H_{\eta,\ell} + ik\epsilon E_{\eta,\ell} = 0 & \text{in } \Omega, \\ \nu \times E_{\eta,\ell} := f_{\eta,\ell} \in TH^{-1/2}(\partial\Omega) & \text{on } \partial\Omega, \end{cases} \tag{3.8.36}$$

where $E_{\eta,\ell}$ and $H_{\eta,\ell}$ are solutions of the anisotropic Maxwell's equation. Since $\widetilde{E_{\eta,\ell}} = E - E_{\eta,\ell}$, $\widetilde{H_{\eta,\ell}} = H - H_{\eta,\ell}$, we have

$$\begin{cases} \nabla \times \widetilde{E_{\eta,\ell}} - ik\mu\widetilde{H_{\eta,\ell}} = 0 & \text{in } \Omega\setminus\bar{D}, \\ \nabla \times \widetilde{H_{\eta,\ell}} + ik\gamma\widetilde{E_{\eta,\ell}} = 0 & \text{in } \Omega\setminus\bar{D}, \\ \nu \times \widetilde{E_{\eta,\ell}} = 0 & \text{on } \partial\Omega, \\ \nu \times \widetilde{H_{\eta,\ell}} = -\nu \times H_{\eta,\ell} & \text{on } \partial D. \end{cases} \tag{3.8.37}$$

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Step 2. Let $(E^{ex}_{\eta,\ell}, H^{ex}_{\eta,\ell})$ be the solution of the following well posed exterior Maxwell's problem

$$\begin{cases} \nabla\times E^{ex}_{\eta,\ell} - ikH^{ex}_{\eta,\ell} = 0 & \text{in } \mathbb{R}^3\backslash\bar{D}, \\ \nabla\times H^{ex}_{\eta,\ell} + ikE^{ex}_{\eta,\ell} = 0 & \text{in } \mathbb{R}^3\backslash\bar{D}, \\ \nu\times H^{ex}_{\eta,\ell} = -\nu\times H_{\eta,\ell} & \text{on } \partial D, \\ E^{ex}_{\eta,\ell}, H^{ex}_{\eta,\ell} \text{ satisfiy the Silver-Müller radiation condition.} \end{cases} \tag{3.8.38}$$

We can represent these solutions $E^{ex}_{\eta,\ell}$ and $H^{ex}_{\eta,\ell}$ by the following layer potentials

$$\begin{aligned} H^{ex}_{\eta,\ell}(x) &:= \nabla\times\int_{\partial D}\Phi_k(x,y)f(y)ds(y), \\ E^{ex}_{\eta,\ell}(x) &:= -\frac{1}{ik}\nabla\times H^{ex}_{\eta,\ell}(x),\ x\in\mathbb{R}^3\backslash\partial D, \end{aligned}$$

where $\Phi_k(x,y) = -\dfrac{e^{ik|x-y|}}{4\pi|x-y|}$, $x,y\in\mathbb{R}^3$, $x\neq y$, is the fundamental solution of the Helmholtz equation and f is the density. Now, we follow the arguments in section 2.1 of [27] and use the same argument for the isotropic Maxwell's equation (3.8.38), then we have

$$\begin{cases} \|E^{ex}_{\eta,\ell}\|_{L^p(\Omega\backslash\bar{D})} \le C\{\|\nu\times H_{\eta,\ell}\|_{L^p(\partial D)} + \|\nabla\times H_{\eta,\ell}\|_{L^p(D)}\}, \\ \|H^{ex}_{\eta,\ell}\|_{L^2(\Omega\backslash\bar{D})} \le C\{\|\nu\times H_{\eta,\ell}\|_{L^p(\partial D)} + \|\nabla\times H_{\eta,\ell}\|_{L^p(D)}\}, \end{cases} \tag{3.8.39}$$

for $p\in(\frac{4}{3},2]$. Moreover, if we define $\mathcal{E}_{\eta,\ell} = \widetilde{E_{\eta,\ell}} - E^{ex}_{\eta,\ell}$, $\mathcal{H}_{\eta,\ell} = \widetilde{H_{\eta,\ell}} - H^{ex}_{\eta,\ell}$, then $\mathcal{E}_{\eta,\ell}$ and $\mathcal{H}_{\eta,\ell}$ satisfy the following Maxwell's equation

$$\begin{cases} \nabla\times\mathcal{E}_{\eta,\ell} - ik\mu\mathcal{H}_{\eta,\ell} = ik(1-\mu)H^{ex}_{\eta,\ell} & \text{in } \Omega\backslash\bar{D}, \\ \nabla\times\mathcal{H}_{\eta,\ell} + ik\epsilon\mathcal{E}_{\eta,\ell} = ik(\gamma-I_3)E^{ex}_{\eta,\ell} & \text{in } \Omega\backslash\bar{D}, \\ \nu\times\mathcal{H}_{\eta,\ell} = 0 & \text{on } \partial\Omega, \\ \nu\times\mathcal{E}_{\eta,\ell} = -\nu\times E^{ex}_{\eta,\ell} & \text{on } \partial D. \end{cases} \tag{3.8.40}$$

Step 3. Now we decompose $\mathcal{E}_{\eta,\ell} = \mathcal{E}^1_{\eta,\ell} + \mathcal{E}^2_{\eta,\ell}$ and $\mathcal{H}_{\eta,\ell} = \mathcal{H}^1_{\eta,\ell} + \mathcal{H}^2_{\eta,\ell}$, where $(\mathcal{E}^1_{\eta,\ell}, \mathcal{H}^1_{\eta,\ell})$ satisfies the following zero boundary Maxwell's equation

$$\begin{cases} \nabla\times\mathcal{E}^1_{\eta,\ell} - ik\mu\mathcal{H}^1_{\eta,\ell} = ik(1-\mu)H^{ex}_{\eta,\ell} & \text{in } \Omega\backslash\bar{D}, \\ \nabla\times\mathcal{H}^1_{\eta,\ell} + ik\epsilon\mathcal{E}^1_{\eta,\ell} = ik(\epsilon-I_3)E^{ex}_{\eta,\ell} & \text{in } \Omega\backslash\bar{D}, \\ \nu\times\mathcal{E}^1_{\eta,\ell} = \nu\times\mathcal{H}^1_{\eta,\ell} = 0 & \text{on } \partial(\Omega\backslash\bar{D}), \end{cases} \tag{3.8.41}$$

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and $(\mathcal{E}^2_{\eta,\ell}, \mathcal{H}^2_{\eta,\ell})$ satisfies

$$\begin{cases} \nabla\times\mathcal{E}^2_{\eta,\ell} - ik\mu\mathcal{H}^2_{\eta,\ell} = 0 & \text{in } \Omega\backslash\bar{D}, \\ \nabla\times\mathcal{H}^2_{\eta,\ell} + ik\gamma\mathcal{E}^2_{\eta,\ell} = 0 & \text{in } \Omega\backslash\bar{D}, \\ \nu\times\mathcal{H}^2_{\eta,\ell} = 0 & \text{on } \partial\Omega, \\ \nu\times\mathcal{E}^2_{\eta,\ell} = -\nu\times E^{ex}_{\eta,\ell} & \text{on } \partial D. \end{cases} \tag{3.8.42}$$

First, we deal with the equation (3.8.41) by using the L^p estimate in $\Omega\backslash\bar{D}$. Note that $(\mathcal{E}^1_{\eta,\ell}, \mathcal{H}^1_{\eta,\ell})$ satisfies (3.8.41), then we have

$$\begin{cases} \nabla\times(\epsilon^{-1}\nabla\times\mathcal{E}^1_{\eta,\ell}) - k^2\gamma\mathcal{E}^1_{\eta,\ell} = ik\nabla\times[(\mu^{-1}-1)H^{ex}_{\eta,\ell}] + ik(\gamma - I_3)E^{ex}_{\eta,\ell} & \text{in } \Omega\backslash\bar{D}, \\ \nu\times\mathcal{E}^1_{\eta,\ell} = 0 & \text{on } \partial(\Omega\backslash\bar{D}), \end{cases}$$

and

$$\begin{cases} \nabla\times(\epsilon^{-1}\nabla\times\mathcal{H}^1_{\eta,\ell}) - k^2\mu\mathcal{H}^1_{\eta,\ell} = ik\nabla\times[(I_3-\epsilon^{-1})E^{ex}_{\eta,\ell}] + ik(1-\mu)H^{ex}_{\eta,\ell} & \text{in } \Omega\backslash\bar{D}, \\ \nu\times\mathcal{H}^1_{\eta,\ell} = 0 & \text{on } \partial(\Omega\backslash\bar{D}). \end{cases}$$

Now, if we use the same method in the proof of the Proposition 3.28, we will obtain

$$\begin{cases} \|\mathcal{E}^1_{\eta,\ell}\|_{L^p(\Omega\backslash\bar{D})} + \|\nabla\times\mathcal{E}^1_{\eta,\ell}\|_{L^p(\Omega\backslash\bar{D})} \le C\{\|H^{ex}_{\eta,\ell}\|_{L^p(\Omega\backslash\bar{D})} + \|E^{ex}_{\epsilon,\ell}\|_{L^2(\Omega\backslash\bar{D})}\}, \\ \|\mathcal{H}^1_{\eta,\ell}\|_{L^p(\Omega\backslash\bar{D})} + \|\nabla\times\mathcal{H}^1_{\eta,\ell}\|_{L^P(\Omega\backslash\bar{D})} \le C\{\|E^{ex}_{\eta,\ell}\|_{L^p(\Omega\backslash\bar{D})} + \|H^{ex}_{\eta,\ell}\|_{L^2(\Omega\backslash\bar{D})}, \end{cases} \tag{3.8.43}$$

for any $\frac{4}{3} < p \le 2$. If we combine (3.8.39) and (3.8.43) together, we have

$$\|H^1_{\eta,\ell}\|_{L^p(\Omega\backslash\bar{D})} \le C\{\|\nu\times H_{\eta,\ell}\|_{L^p(\partial D)} + \|\nabla\times H_{\eta,\ell}\|_{L^p(D)}\}. \tag{3.8.44}$$

For $(\mathcal{E}^2_{\eta,\ell}, \mathcal{H}^2_{\eta,\ell})$, we apply the L^2-theory for the anisotropic Maxwell's equation, we get

$$\|\mathcal{H}^2_{\eta,\ell}\|_{L^2(\Omega\backslash\bar{D})} \le \|\mathcal{E}^2_{\eta,\ell}\|_{H(curl,\Omega\backslash\bar{D})} \le C\|\nu\times\mathcal{E}^2_{\eta,\ell}\|_{H^{-1/2}(\partial\Omega)} \le C\|\nu\times E^{ex}_{\eta,\ell}\|_{H^{-1/2}(\partial\Omega)}.$$

Moreover, following the proof in the Lemma 2.3 of [27], we have

$$\|\nu\times E^{ex}_{\eta,\ell}\|_{H^{-1/2}(\partial\Omega)} \le C\|f\|_{L^p(\partial D)},\ \forall p \ge 1,$$

and

$$\|\mathcal{H}^2_{\eta,\ell}\|_{L^2(\Omega\backslash\bar{D})} \le C\{\|\nu\times H_{\eta,\ell}\|^2_{L^p(\partial D)} + \|\nabla\times H_{\eta,\ell}\|^2_{L^p(D)}\}, \tag{3.8.45}$$

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for all $p \in (\frac{4}{3}, 2]$. Recall that $\mathcal{H}_{\eta,\ell} = \mathcal{H}^1_{\eta,\ell} + \mathcal{H}^2_{\eta,\ell}$, by using (3.8.44) and (3.8.45), then we have

$$\|\mathcal{H}_{\eta,\ell}\|_{L^2(\Omega\setminus\bar{D})} \leq C\{\|\nu \times H_{\eta,\ell}\|_{L^p(\partial D)} + \|\nabla \times H_{\eta,\ell}\|_{L^p(D)}\} \tag{3.8.46}$$

for all $p \in (\frac{4}{3}, 2]$. Combining (3.8.39), (3.8.46) and $\widetilde{H_{\eta,\ell}} = \mathcal{H}_{\eta,\ell} + H^{ex}_{\eta,\ell}$, we get

$$\begin{aligned}\int_{\Omega\setminus\bar{D}} |\widetilde{H_{\eta,\ell}}(x)|^2 dx &\leq \|\mathcal{H}_{\eta,\ell}\|_{L^2(\Omega\setminus\bar{D})} + \|H^{ex}_{\eta,\ell}\|_{L^2(\Omega\setminus\bar{D})} \\ &\leq C\{\|\nu \times H_{\eta,\ell}\|^2_{L^p(\partial D)} + \|\nabla \times H_{\eta,\ell}\|^2_{L^p(D)}\}\end{aligned} \tag{3.8.47}$$

for all $p \in (\frac{4}{3}, 2]$. Finally, for $s > 0$ and $p \leq 2$ we have $H^s(\partial D) \subset L^2(\partial D) \subset L^p(\partial D)$, then we reduce that

$$\|\nu \times H_{\eta,\ell}\|_{L^p(\partial D)} \leq C\|H_{\eta,\ell}\|_{L^p(\partial D)} \leq C\|H_{\eta,\ell}\|_{H^s(\partial D)}.$$

Note that the trace map from $H^{s+1/2}(D) \to H^s(\partial D)$ is bounded for all $0 < s \leq 1$. So the estimate (3.8.47) will become

$$\int_{\Omega\setminus\bar{D}} |\widetilde{H_{\eta,\ell}}(x)|^2 dx \leq C\{\|H_{\eta,\ell}\|^2_{H^{s+1/2}(D)} + \|\nabla \times H_{\eta,\ell}\|^2_{L^p(D)}\},$$

for all $p \in (\frac{4}{3}, 2]$ and $0 < s \leq 1$. □

Remark 3.37. Now, if we take $\ell \to \infty$ and $\epsilon \to 0$, we will get

$$\lim_{\eta\to 0} \limsup_{\ell\to\infty} \int_{\Omega\setminus\bar{D}} |\widetilde{H_{\eta,\ell}}(x)|^2 dx \leq C\{\|H_t\|^2_{H^{s+1/2}(D)} + \|\nabla \times H_t\|^2_{L^p(D)}\},$$

where H_t is the oscillating-decaying solution defined on $\Omega_t(\omega)$.

We have the following lemmas for the oscillating-decaying solutions in the same way as we did in section 5, so we omit the proofs.

Lemma 3.38. *For $1 \leq q < \infty$, $\tau \gg 1$, we have the following estimates.*

1.

$$\begin{aligned}\int_D |H_t(x)|^q dx &\leq \tau^{q-1} \sum_{j=1}^{m} \iint_{|y'|<\delta} e^{-aq\tau l_j(y')} dy' + O(\tau^{q-1} e^{-qa\delta\tau}) \\ &\quad + O(\tau^q e^{-qa\tau}) + O(\tau^{-1}) + O(\tau^{-2N+3})\end{aligned}$$

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2.

$$\int_D |H_t|^2 dx \quad \geq \quad C\tau \sum_{j=1}^{m} \iint_{|y'|<\delta} e^{-2a\tau l_j(y')} dy' - C\tau e^{-2a\delta\tau} - C\tau^{-1} - C\tau^{-2N+3}$$

3.

$$\int_D |\nabla \times H_t(x)|^q dx \quad \leq \quad \tau^{2q-1} \sum_{j=1}^{m} \iint_{|y'|<\delta} e^{-aq\tau l_j(y')} dy' + O(\tau^{2q-1} e^{-qa\delta\tau}) + O(\tau^{2q} e^{-qa\tau}) + O(\tau e^{-c\tau}) + O(\tau^{-2N+5})$$

4.

$$\int_D |\nabla \times H_t(x)|^2 dx \quad \geq \quad C\tau^3 \sum_{j=1}^{m} \iint_{|y'|<\delta} e^{-2a\tau l_j(y')} dy' - C\tau^3 e^{-2a\delta\tau} - C\tau e^{-c\tau} - C\tau^{-2N+5}$$

Lemma 3.39. *We have the following estimate*

$$\frac{\|H_t\|^2_{L^2(D)}}{\|\nabla \times H_t\|^2_{L^2(D)}} \leq O(\tau^{-2}), \ \tau \gg 1.$$

For $p < 2$, we have the following estimate

$$\frac{\|\nabla \times H_t\|^2_{L^p(D)}}{\|\nabla \times H_t\|^2_{L^p(D)}} \leq C\tau^{1-\frac{2}{p}}, \ \tau \gg 1.$$

Lemma 3.40. *If $t = h_D(\rho)$,then for some positive constant C, we have*

$$\liminf_{\tau \to \infty} \int_D \tau |\nabla \times H_t|^2 dx \geq C.$$

3.8.3.2 End of the proof of Theorem 1.1 for the impenetrable case

By using the same argument in the penetrable case, it is easy to see that

$$\limsup_{\tau \to \infty} |\frac{1}{\tau} I_\rho(\tau, t)| = 0$$

for $t > h_D(\rho)$. Recall that from Lemma 3.35, we have

$$-\frac{1}{\tau} I_\rho^{\eta,\ell}(\tau, t) \geq \int_D \{|\nabla \times H_{\eta,\ell}(x)|^2 - k^2 |H_{\eta,\ell}(x)|^2\} dx - k^2 \int_{\Omega \setminus \bar{D}} |\widetilde{H_{\eta,\ell}}(x)|^2\} dx. \qquad (3.8.48)$$

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By using Proposition 3.32, we deduce

$$-\frac{1}{\tau}I_{\rho}^{\eta,\ell}(\tau,t) \geq \int_D \{|\nabla\times H_{\eta,\ell}(x)|^2 - k^2|H_{\eta,\ell}(x)|^2\}dx - C\{\|H_t\|^2_{H^{s+1/2}(D)} + \|\nabla\times H_t\|^2_{L^p(D)}\},$$

where $0 < s \leq 1$ and $\frac{4}{3} < p \leq 2$. We want to estimate $\frac{\|H_t\|^2_{H^{s+1/2}(D)}}{\|\nabla\times H_t\|^2_{L^2(D)}}$, for $0 < s \leq 1$. Set $r = s + 1/2$, then we need to estimate

$$\frac{\|H_t\|^2_{H^r(D)}}{\|\nabla\times H_t\|^2_{L^2(D)}}$$

for $r \in (\frac{1}{2}, \frac{3}{2}]$. Using the interpolation inequality, we have

$$\|H_t\|_{H^r(D)} \leq C\|H_t\|^{1-r}_{L^2(D)}\|H_t\|^r_{H^1(D)},\ 0 \leq r \leq 1.$$

By the Young's inequality $ab \leq \delta^{-\alpha}\frac{a^\alpha}{\alpha} + \delta^{\beta}\frac{b^\beta}{\beta}$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, we obtain

$$\begin{aligned}\|H_t\|^2_{H^r(D)} &\leq C\left[\frac{\delta^{-\alpha}}{\alpha}\|H_t\|^2_{L^2(D)} + \frac{\delta^\beta}{\beta}\|H_t\|^2_{H^1(D)}\right]\\ &\leq C\left[\{(1-r)\delta^{-(1-r)^{-1}} + r\delta^{r^{-1}}\}\|H_0\|^2_{L^2(D)} + r\delta^{r^{-1}}\|\nabla H_t\|^2_{L^2(D)}\right]. \quad (3.8.49)\end{aligned}$$

Recall that $H_t = G_B^1(x)e^{i\tau x\cdot\xi}e^{-\tau(x\cdot\omega-t)A_t^B(x')}b + \Gamma^{B,1}_{\chi_t,b,t,N,\omega}(x,\tau) + r^{B,1}_{\chi_t,b,t,N,\omega}(x,\tau)$ is a smooth function with $G_B^1(x) = O(\tau)$ and $\Gamma^{B,1}_{\chi_t,b,t,N,\omega}$ satisfies (3.7.11) for $|\alpha| = 1$ and $r^{B,1}_{\chi_t,b,t,N,\omega}$ satisfies (3.7.11) for $k = 1$. If we can differentiate H_t componentwisely, we will get $\frac{\partial H_t}{\partial x_j} = \frac{\partial G_B^1 e^{i\tau x\cdot\xi}e^{-\tau(x\cdot\omega-t)A_t^B}b}{\partial x_j} + \frac{\partial \Gamma^{B,1}_{\chi_t,b,t,N,\omega}}{\partial x_j} + \frac{\partial r^{B,1}_{\chi_t,b,t,N,\omega}}{\partial x_j}$ and

$$\begin{cases}\|\frac{\partial N^t_{A,B,\gamma,\mu}}{\partial x_j}\|^2_{L^2(D)} \leq C\tau^4\int_D e^{-2a\tau(x\cdot\rho-t)}dx,\\ \|\frac{\partial \Gamma^{2,t}_{A,B,\gamma,\mu}}{\partial x_j}\|_{L^2(D)} \leq c\tau^{-1/2}e^{-c\tau}.\\ \|\frac{\partial r^{2,t}_{A,B,\gamma,\mu}}{\partial x_j}\|_{L^2(D)} \leq c\tau^{-N+3/2}.\end{cases}$$

Then by using the same method as before, it is easy to see that

$$\begin{aligned}\|\nabla H_t\|^2_{L^2(D)} &= \sum_{j=1}^3 \|\frac{\partial H_t}{\partial x_j}\|^2_{L^2(D)}\\ &\leq C\tau^4\int_D e^{-2(x\cdot\rho-t)}dx + c\tau^{-1}e^{-2c\tau} + c\tau^{-2N+3}.\end{aligned}$$

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For $t = h_D(\rho)$, we have

$$\begin{aligned}
\|\nabla H_t\|^2_{L^2(D)} &\le C\tau^4 \int_D e^{-2a(x\cdot\rho - h_D(\rho))}dx + c\tau^{-1}e^{-2c\tau} + c\tau^{-2N+3} \\
&\le C\tau^4 (\int_{D_\delta} + \int_{D\setminus D_\delta}) e^{-2a(x\cdot\rho - h_D(\rho))}dx + c\tau^{-1}e^{-2\tau(s-t)a} \\
&\quad + c\tau^{-2N+3} \\
&\le C\tau^4 \sum_{j=1}^m \iint_{|y'|<\delta} dy' \int_{l_j(y')}^\delta e^{-2a\tau y_3}dy_3 + C\tau^4 e^{-2ac\tau} \\
&\quad + c\tau^{-1}e^{-2c\tau} + c\tau^{-2N+3} \\
&\le C\tau^3 \sum_{j=1}^m \iint_{|y'|<\delta} e^{-2a\tau l_j(y')}dy' - C\tau^3 e^{-2a\delta\tau} \\
&\quad + C\tau^3 e^{-2ac\tau} + c\tau^{-1}e^{-2c\tau} + c\tau^{-2N+3}. \qquad (3.8.50)
\end{aligned}$$

From Lemma 3.38 and (3.8.50), we have

$$\frac{\|\nabla H_t\|^2_{L^2(D)}}{\|\nabla \times H_t\|^2_{L^2(D)}} \le C. \qquad (3.8.51)$$

Combining Lemma 3.38, (3.8.49) and (3.8.51) we obtain

$$\begin{aligned}
\frac{\|H_t\|^2_{H^r(D)}}{\|\nabla \times H_t\|^2_{L^2(D)}} &\le C\{(1-r)\delta^{-(1-r)^{-1}} + r\delta^{r^{-1}}\} \frac{\|H_t\|^2_{L^2(D)}}{\|\nabla \times H_t\|^2_{L^2(D)}} \\
&\quad + Cr\delta^{r^{-1}} \frac{\|\nabla H_t\|^2_{L^2(D)}}{\|\nabla \times H_t\|^2_{L^2(D)}} \\
&\le C\{(1-r)\delta^{-(1-r)^{-1}} + r\delta^{r^{-1}}\}O(\tau^{-2}) + Cr\delta^{r^{-1}}.
\end{aligned}$$

We now choose $p \in (\frac{4}{3}, 2)$, combining (3.8.48), (3.8.49) and (3.8.51) we have

$$\begin{aligned}
\frac{-\frac{1}{\tau} I_\rho^{\epsilon,\ell}(\tau,t)}{\|\nabla \times H_t\|^2_{L^2(D)}} &\ge C - c_1 \frac{\|H_t\|^2_{L^2(D)}}{\|\nabla \times H_t\|^2_{L^2(D)}} - c_2 \frac{\|H_t\|^2_{H^r(D)}}{\|\nabla \times H_t\|^2_{L^2(D)}} - c_3 \frac{\|\nabla \times H_t\|^2_{L^p(D)}}{\|\nabla \times H_t\|^2_{L^2(D)}} \\
&\ge C - c_1\{(1-r)\delta^{-(1-r)^{-1}} + r\delta^{r^{-1}}\}O(\tau^{-2}) - Cr\delta^{r^{-1}} - c_3\tau^{1-\frac{2}{p}} \\
&\ge C - c_2 r\delta^{r^{-1}},\ \frac{1}{2} < r < 1,\ \tau \gg 1.
\end{aligned}$$

Hence from Lemma 3.40, we have

$$\liminf_{\tau\to\infty} |I_\rho(\tau, h_D(\rho))| \ge c > 0.$$

3.8.4 Reconstruction algorithm

1. Give $\omega \in S^2$ and choose $\eta, \zeta, \xi \in S^2$ so that $\{\eta, \zeta, \xi\}$ forms a basis of $\mathbb{R}^3$ and ξ lies in the span of η and ζ;

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2. Choose a starting t such that $\Omega \subset \{x \cdot \omega \geq t\}$;

3. Construct the oscillating-decaying solutions for the anisotropic Maxwell system from the reduction strongly elliptic system;

4. Define a suitable indicator function $I_\rho(\tau, t)$ and the support function $h_D(\rho) = \inf_{x \in D} x \cdot \rho$;

5. If $I_\rho(\tau, t) \to 0$ as $\tau \to \infty$, then choose $t' > t$ and repeat (iv), (v), (vi);

6. If $I_\rho(\tau, t') \nrightarrow 0$, then $t' = t_0 = h_D(\rho)$;

7. Varying $\rho \in S^2$ and repeat previous steps again, we can determine the convex hull of D.

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Chapter 4

Strong unique continuation for a residual stress system with Gevrey coefficients

We consider the problem of the strong unique continuation for an elasticity system with general residual stress. Due to the known counterexamples, we assume the coefficients of the elasticity system are in the Gevrey class of appropriate indices. The main tools are Carleman estimates for product of two second order elliptic operators.

First, we give some properties of the *strong unique continuation property* (SUCP) for the second order elliptic operators with Gevrey coefficients.

4.1 SUCP for the elliptic equations

Let $A_{ij}(x)\partial_{x_i x_j}$ be a second order uniformly elliptic differential operator and $A_{ij}(x)$ satisfies $\forall \xi \in \mathbb{R}^n$,

$$\lambda|\xi|^2 \leq A_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2, \tag{4.1.1}$$

for some $0 < \lambda \leq \Lambda$. Recall that we have the following scalar second order elliptic inequalities

$$|\sum_{i,j=1}^{n} A^{ij}(x)\partial^2_{x_i x_j} u| \leq C\{|u| + |\nabla u|\} \tag{4.1.2}$$

have the *SUCP* if the coefficients $A^{ij}(x)$ are real, Lipschitz continuous and satisfy (4.1.1). If u satisfies (4.1.2) and u satisfies

$$\sup_{r\leq\delta} r^{-N}\|u\|_{L^2(B_r(0))} < \infty$$

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for all N, then u vanishes near 0.

For higher order elliptic differential equation, we consider the following product form $A(x) := A_N \cdots A_1$ (using the Einstein convention), where all A_ℓ's satisfy (4.1.1), which are uniformly elliptic operators such that

$$A_\ell u := A^\ell_{ij}(x)\partial^2_{x_i x_j} u.$$

We assume that all coefficients are in the Gevrey class G^s (we will define in the later section) and $\exists \alpha > 0$ such that the eigenvalues $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$ of $A^\ell_{ij}(x)$ satisfy

$$\frac{\mu_n - \mu_1}{\mu_1} < \alpha \tag{4.1.3}$$

uniformly in x and ℓ. The following is the main result about the SUCP for the higher order product elliptic equation.

Proposition 4.1. *Let G^s be the Gevrey class of order s. Let $A_\ell(x) \in G^s$ satisfy (4.1.1) for all $\ell = 1, 2, \cdots, N$ and $\alpha > 0$ satisfying (4.1.3) at $x = x_0$ for some $x_0 \in \Omega$. Moreover, if $s > 0$ satisfies*

$$s < 1 + \frac{1}{\alpha},$$

then SUCP holds at $x = x_0$ for the following differential equation

$$Au = A_N \cdots A_1 u = \sum_{|\beta| \leq \lfloor \frac{3N}{2} \rfloor} a_\beta \partial^\beta u$$

provided that all $a_\beta \in G^s$.

In order to prove the Proposition, we use the Carleman estimates for the second order elliptic equation. Recall that for the second order elliptic equation

$$Av := \sum_{i,j} \partial_{x_i}(A_{ij}(x)\partial_j v),$$

with $A_{ij}(x) \in G^s$, we have the following Carleman estimates holds

$$\sum_{j=0}^{2} \tau^{3/2-j} \||x|^{\alpha/2}|x|^{(j-2)(1+\alpha)} D^j e^{\tau|x|^{-\alpha}} v\|_{L^2} \leq c\|e^{\tau|x|^{-\alpha}} Av\|_{L^2}, \tag{4.1.4}$$

where c is independent of v. For detailed proof, we refer readers to [9]. Then we can iterate the above estimates to higher order elliptic equation, which has the product form $A = A_N \cdots A_1$ with each A_ℓ's are second order elliptic operators. In Proposition 4.1, we can observe that the derivative order of the right hand side do not exceed $\left\lfloor \frac{3N}{2} \right\rfloor$. The reason is that we have used the

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iteration argument from the original Carleman estimates for the second order elliptic equations. Since the highest power of τ in left hand side of (4.1.4) is $\tau^{3/2}$, after iteration N times, the number of derivative cannot be bigger than $\frac{3N}{2}$.

4.2 Basic properties for the Gevrey class

First, we give the definition of the Gevrey class.

Definition 4.2. We say that $f \in C^\infty(\Omega)$ belongs to the Gevrey class of order s, denote it as $G^s(\Omega)$, if there exist constants c, A and multiindices β such that

$$|\partial^\beta f| \le cA^{|\beta|}|\beta|!^s \text{ in } \Omega.$$

To simplify the notation, from now on, we use G^s to denote $G^s(\Omega)$.

Note that the Gevrey class contains the following properties:

1. Gevrey regularity: If u is a solution of an elliptic equation with Gevrey coefficients G^s, then u also lies in the Gevrey class G^s.

2. $|u| \le ce^{-|x|^{-s}}$ near $x = x_0$ for some $x_0 \in \Omega$ and $u \in G^s$ will imply u vanishes near $x = x_0$.

In the following, we list basic properties for the Gevrey class which will be used in the following sections.

Lemma 4.3. *Let U be a bounded open set and suppose that $0 \in U$, $s \ge 1$ and $f \in G^s(U)$ satisfies*

$$\partial^\beta f(0) = 0$$

for all multiindices β. Let $s - 1 < \rho$, then

$$|f(x)| \le e^{-|x|^{-1/\rho}}$$

near $x = 0$.

Lemma 4.4. *We have*

$$e^{-|x|^{-1/\rho}} \in G^s(\mathbb{R}^3)$$

provided $1 + \rho = s$.

Lemma 4.5. *Let*

$$P(x, D)u = f \text{ in } U$$

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be an elliptic differential system with coefficients and right had side in the Gevrey class $G^s(U)$. Then $u \in G^s(V)$ for all bounded $V \Subset U$.

Proof. See [5], Proposition 2.13, we know that the Gevrey class are good classes of elliptic regularity. □

4.3 SUCP for the residual stress system with Gevrey coefficients

Our main result is to prove the strong unique continuation property (SUCP) for the isotropic elasticity system with residual stress under appropriate conditions. We formulate the mathematical problem in the following.

Let Ω be a connected open domain in $\mathbb{R}^3$ and consider the time-harmonic elasticity system

$$\nabla \cdot \sigma + \kappa^2 \rho u = 0 \text{ in } \Omega, \tag{4.3.1}$$

where $\sigma = (\sigma_{ij})_{i,j=1}^3$ is the stress tensor field, $\kappa \in \mathbb{C}$ is the frequency and $\rho = \rho(x) > 0$ denotes the density of the medium. The vector field $u(x) = (u_i(x))_{i=1}^3$ is the displacement vector. Suppose that the stress tensor is given by

$$\sigma(x) = T(x) + (\nabla u)T(x) + \lambda(x)(\text{tr}E)I + 2\mu(x)E,$$

where $E(x) = \dfrac{\nabla u + \nabla u^t}{2}$ is the infinitesimal strain and $\lambda(x), \mu(x)$ are the Lamé parameters. The second-rank tensor $T(x) = (t_{ij}(x))_{i,j=1}^3$ is the residual stress and satisfies

$$t_{ij}(x) = t_{ji}(x),\ \forall i, j = 1, 2, 3 \text{ and } x \in \Omega$$

and

$$\nabla \cdot T = \sum_j \partial_j t_{ij} = 0 \text{ in } \Omega,\ \forall i = 1, 2, 3.$$

If we define the elastic tensor $C = (C_{ijkl})_{i,j,k,l=1}^3$ with

$$C_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu(\delta_{jk}\delta_{jl} + \delta_{jk}\delta_{il}) + t_{jl}\delta_{ik},$$

then (4.3.1) is equivalent to

$$\nabla \cdot (C\nabla u) + \kappa^2 \rho u = 0 \text{ in } \Omega.$$

We concern the SUCP for (4.3.1), i.e., if $u \in H^2_{loc}(\Omega)$ satisfies (4.3.1) and $u(x)$ vanishes to

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infinite order at a point $x_0 \in \Omega$, then u must vanish identically in Ω. Without loss of generality, we assume $x_0 = 0$.

Historical notes

A brief history of the results on the (strong) unique continuation for (4.3.1) is in the following. In [50], Nakamura and Wang proved the unique continuation property for (4.3.1) under the condition $\max_{i,j} \|t_{ij}\|_\infty$ is small and $T(x), \lambda(x),\ \mu(x) \in W^{2,\infty}$ and $\rho(x) \in W^{1,\infty}$. In [36], Lin proved the SUCP for (4.3.1) under the assumptions that $T(0) = 0$, $\max_{i,j} \|t_{ij}\|_\infty$ is small, $\lambda(x), \mu(x)$ and $\rho(x)$ are in C^2. In addition, in [63], Uhlmann and Wang proved unique continuation principle for (4.3.1) under the conditions $T(x), \lambda(x),\ \mu(x) \in W^{2,\infty}$, $\rho(x) \in W^{1,\infty}$ and general residual stress.

Motivated by [63], we want to prove the SUCP for (4.3.1) with arbitrary residual stress. In this paper, we will give a reduction algorithm to transform (4.3.1) into a special fourth order elliptic system. The main difficulty is that when $T(0) \neq 0$, the leading terms of (4.3.1) will not be the Laplacian at zero, so we cannot use a perturbation argument to derive suitable Carleman estimates in order to obtain the SUCP. In [4], Alinhac and Baouendi proved the SUCP for any fourth order operator with smooth coefficients verifying $P = Q_2Q_1 + R$, where Q_i's are second order elliptic operators with $Q_i(0, D) = -\Delta$ for $i = 1, 2$. Moreover, in [32], Le Borgne proved the SUCP for fourth order differential inequality with Q_i's are Lipschitz continuous and $Q_i(0, D) = -\Delta$ for $i = 1, 2$. In [36], Lin introduced $v = \nabla \cdot u$ and $w = \nabla \times u$ to transform (4.3.1) into a second order differential system, but the system is weakly-coupled, i.e., the principal part of the second order derivatives are not diagonal. Moreover, Lin also introduced a fourth order elliptic system $P = \Delta Q_i$ with Q_i's are second order elliptic operators with $Q_i(0, D) = \Delta$ for $i = 1, 2$ and give another approach to derive the SUCP. For more details, we refer readers to [36].

In this note, our transformation will reduce (4.3.1) into a fourth order principally diagonal elliptic system with the same leading coefficients. The key observation is that the leading terms of the fourth order elliptic system are the same. Notice that principally diagonal strongly elliptic systems allow the application of Carleman estimates for scalar operators since these estimates are flexible with respect to perturbations by lower order terms. Therefore, it is possible to derive suitable Carleman estimates for the fourth order elliptic system.

In general, the SUCP doe not hold even the coefficients are smooth, Alinhac gave a counterexample in [3]. Thus, we consider all the coefficients in the Gevrey class and we will use the Carleman estimates proved in [9] for the scalar higher order elliptic equations in order to prove the SUCP for the new fourth order strongly elliptic system.

We assume all the coefficients $T(x), \lambda(x), \mu(x)$ and $\rho(x)$ lie in the Gevrey class G^s. We are interested in the SUCP for (4.3.1) with Gevery coefficients, which means if u satisfies (4.3.1) and

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u is flat at the origin in the sense that

$$\sup_{r\leq\delta} r^{-N}\|u\|_{L^2(B(0,r))} < \infty \tag{4.3.2}$$

for all N, then u vanishes near the origin. If u is smooth, the condition (4.3.2) is equivalent to all partial derivatives of u vanishing at 0.

The SUCP for the second order elliptic equations in the Gevrey class were studied in many literature [7, 8, 9, 34]. In 1981, Lerner [34] considered a second order elliptic operator L in $\mathbb{R}^2$ with simple characteristics and the coefficients in the Gevery class of order s. Lerner proved that if s is smaller than a quantity depending on the principal symbol of $l_2(0,\mathbb{R}^2)$, then L has the SUCP near 0. In [8], the authors extended Lerner's result to $\mathbb{R}^N$, which means the SUCP holds for a second order elliptic operator L in $\mathbb{R}^N$ with the Gevrey order s smaller than a quantity depending on the principal symbol of $l_2(0,\mathbb{R}^N)$.

Recall that the strongly elliptic condition is given as: there exists $c_0 > 0$ such that for all vectors $\xi = (\xi_i)_{i=1}^3$,

$$\sum_{ij} a_{ij}(x)\xi_i\xi_j \geq c_0|\xi|^2,\ \forall x\in\Omega.$$

In this paper, we assume P_1 and P_2 are two strongly elliptic operators, where

$$P_1(x,D) \ := \ \sum_{jk} a^1_{jk}(x)\partial^2_{x_jx_k} := \sum_{jk}(\mu\delta_{jk}+t_{jk})\partial^2_{x_jx_k}, \tag{4.3.3}$$

$$P_2(x,D) \ := \ \sum_{jk} a^2_{jk}(x)\partial^2_{x_jx_k} := \sum_{jk}((\lambda+2\mu)\delta_{jk}+t_{jk})\partial^2_{x_jx_k} \tag{4.3.4}$$

with $a^1_{jk}(x) = \mu(x)\delta_{jk}+t_{jk}(x)$ and $a^2_{jk}(x) = (\lambda(x)+2\mu(x))\delta_{jk}+t_{jk}(x)$. Further, there exists $c_0 > 0$ such that for any $\xi = (\xi_i)_{i=1}^3 \in \mathbb{R}^3$

$$\sum_{jk} a^1_{jk}(x)\xi_j\xi_k \ = \ \sum_{jk} t_{jk}\xi_j\xi_k + \mu|\xi|^2 \geq c_0|\xi|^2 \tag{4.3.5}$$

$$\sum_{jk} a^2_{jk}(x)\xi_j\xi_k \ = \ \sum_{jk} t_{jk}\xi_j\xi_k + (\lambda+2\mu)|\xi|^2 \geq c_0|\xi|^2 \tag{4.3.6}$$

for all $x\in\Omega$, note that $(a^\ell_{jk}(x))^3_{j,k=1}$ is a symmetric matrix for $\ell = 1,2$.

We also assume that there exists a constant $\alpha > 0$ such that the eigenvalues $\lambda^\ell_1 \leq \lambda^\ell_2 \leq \lambda^\ell_3$ of $(a^\ell_{jk}(0))$ satisfying

$$\alpha > \frac{\lambda^\ell_3-\lambda^\ell_1}{\lambda^\ell_1} \tag{4.3.7}$$

and

$$s < 1+\frac{1}{\alpha} \tag{4.3.8}$$

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for $\ell = 1, 2$.

The following theorem derives the SUCP for (4.3.1) when all the coefficients lie in the Gevrey class G^s.

Theorem 4.6. *Let the residual stress $(t_{ij}(x))_{i,j=1}^3$, the Lamé parameters $\lambda(x)$, $\mu(x)$ and the density of the medium $\rho(x)$ be in the Gevrey class $G^s(\Omega)$ with s satisfying (4.3.8). Then for all $u \in H^2_{loc}(\Omega; \mathbb{R}^3)$ solving (4.3.1) and for all $N > 0$*

$$\int_{R \le |x| \le 2R} |u|^2 dx = O(R^N) \text{ as } R \to 0,$$

then u is identically zero in Ω.

This paper is organized as follows. In section 2, we will reduce (4.3.1) into a fourth order principally diagonal elliptic system. We use the ideas in [36] and give more detailed transformations. In section 3, we will use the property of the strongly elliptic system in the Gevrey class, then we can get the asymptotic behavior of u near 0. In section 4, we state the SUCP for the fourth order elliptic system and prove the theorem by using the Carleman estimates.

4.4 Reduction to a fourth order strongly elliptic system

In this section, we want to transform (4.3.1) into a principally diagonal fourth order strongly elliptic system. As the calculation in [36]. Let

$$Ru = \nabla \cdot (\nabla u T) \tag{4.4.1}$$

with $Ru = ((Ru)_1, (Ru)_2, (Ru)_3)$, where $(Ru)_i = \sum_{jk} t_{jk} \partial^2_{jk} u_i$, $i = 1, 2, 3$.

As in Section 2, we set $U = (u, v, w)^t$, where $v = \nabla \cdot u$, $w = \nabla \times u$ and u satisfies (4.3.1). From (4.3.1), (4.4.1), let P_1 and P_2 be two elliptic operators

$$\begin{aligned} P_1(x, D) &= R + \mu\Delta, \\ P_2(x, D) &= R + (\lambda + 2\mu)\Delta, \end{aligned}$$

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then (u,v,w) satisfies

$$
\begin{aligned}
P_1(x,D)u &= A_{1,1}(u,v)+A_{1,0}(u,v), && (4.4.2)\\
P_2(x,D)v &= -\sum_{jk}\nabla(t_{jk})\cdot\partial^2_{jk}u && (4.4.3)\\
&\quad +A_{2,1}(u,v,w)+A_{2,0}(u,v,w),\\
P_1(x,D)w &= -\sum_{jk}\nabla(t_{jk})\times\partial^2_{jk}u && (4.4.4)\\
&\quad +A_{3,1}(u,v,w)+A_{3,2}(u,v,w),
\end{aligned}
$$

where $A_{\ell,m}$ are m-th order differential operators. For more details, we refer reader to [36].

Notice that $u\in H^2_{loc}(\Omega;\mathbb{R}^3)$ satisfies (4.4.2) and $v=\nabla\cdot u\in H^1_{loc}(\Omega)$ and $\nabla v\in L^2_{loc}(\Omega)$, then the right hand side of (4.4.2) lies in $L^2_{loc}(\Omega)$. Therefore, we use the standard elliptic higher order regularity theory for (4.4.2) (see Theorem 2.2 in [13]) and the strongly elliptic property, then we have $u\in H^3_{loc}(\Omega;\mathbb{R}^3)$. Iterate the procedures, we obtain $u\in H^k_{loc}(\Omega;\mathbb{R}^3)$ $\forall k\in\mathbb{N}$ (which implies $v,w\in H^k_{loc}(\Omega)$ $\forall k\in\mathbb{N}$).

Let $P(x,D)$ be the principal part of the system to get

$$P(x,D)U=(P_1(x,D)u,P_2(x,D)v,P_1(x,D)w)^t,$$

where $U:=(u,v,w)^t:\Omega\to\mathbb{R}^7$. Component-wise, we have

$$
\begin{aligned}
(P(x,D)U)_i &= \mu\Delta u_i+\sum_{jk}t_{jk}\partial^2_{jk}u_i,\ i=1,2,3\\
(P(x,D)U)_i &= (\lambda+2\mu)\Delta v+\sum_{jk}t_{jk}\partial^2_{jk}v,\ i=4\\
(P(x,D)U)_i &= \mu\Delta w_{i-4}+\sum_{jk}t_{jk}\partial^2_{jk}w_{i-4}\ i=5,6,7.
\end{aligned}
$$

Now, let us take the second order elliptic operator $P_2(x,D)$ on (4.4.2), we get

$$
\begin{aligned}
P_2P_1(x,D)u &= P_2(x,D)[A_{1,1}(u,v)+A_{1,0}(u,v)] && (4.4.5)\\
&:= \sum_{m=0}^{3}B_{1,m}(u,v),
\end{aligned}
$$

where $B_{1,m}$ is an m-th order differential operator. Similarly, we can take $P_1(x,D)$ on (4.4.3) and

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$P_2(x,D)$ on (4.4.4), then we obtain

$$\begin{aligned} & P_1P_2(x,D)v \quad (4.4.6) \\ & = P_1(x,D)(-\sum_{jk}\nabla(t_{jk})\cdot\partial^2_{jk}u) + P_1(x,D)(A_{2,1}(u,v,w)+A_{2,0}(u,v,w)) \\ & = -P_1(x,D)(\sum_{jk}\nabla(t_{jk})\cdot\partial^2_{jk}u) + \sum_{m=0}^{3} B_{2,m}(u,v,w), \end{aligned}$$

and

$$\begin{aligned} & P_2P_1(x,D)w \quad (4.4.7) \\ & = P_2(x,D)(-\sum_{jk}\nabla(t_{jk})\times\partial^2_{jk}u) + P_2(x,D)(A(u,v,w)+A_{3,2}(u,v,w)) \\ & = -P_2(x,D)(\sum_{jk}\nabla(t_{jk})\times\partial^2_{jk}u) + \sum_{m=0}^{3} B_{3,m}(u,v,w). \end{aligned}$$

Now, if we interchange P_1, P_2 on (4.4.6), and use

$$P_2P_1 = P_1P_2 - [P_1,P_2],$$

where $[P_1,P_2]$ is the commutator of two second order elliptic operators, then $[P_1,P_2]$ is a third order differential operator. Thus, (4.4.6) becomes

$$\begin{aligned} P_2P_1(x,D)v \;&=\; -P_1(x,D)[\sum_{jk}\nabla(t_{jk})\cdot\partial^2_{jk}u] \quad (4.4.8) \\ &\quad + \sum_{m=0}^{3}\widetilde{B_{2,m}}(u,v,w), \end{aligned}$$

where $\widetilde{B_{2,m}}$ is an m-th order differential operator and

$$\sum_{m=0}^{3}\widetilde{B_{2,m}}(u,v,w) = \sum_{m=0}^{3} B_{2,m}(u,v,w) - [P_1,P_2](x,D)v.$$

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Now, combine (4.4.5), (4.4.7) and (4.4.8) together, we have

$$P_2P_1(x,D)\begin{pmatrix} u \\ v \\ w \end{pmatrix} = -\begin{pmatrix} 0 \\ P_1(x,D)[\sum_{jk}\nabla(t_{jk})\cdot\partial_{jk}^2 u] \\ P_2(x,D)[\sum_{jk}\nabla(t_{jk})\times\partial_{jk}^2 u] \end{pmatrix} + \sum_{m=0}^{3}\begin{pmatrix} B_{1,m}(u,v) \\ \widetilde{B_{2,m}}(u,v,w) \\ B_{3,m}(u,v,w) \end{pmatrix}. \tag{4.4.9}$$

Now, for $P_1\left(\sum_{jk}\nabla(t_{jk})\cdot\partial_{jk}^2 u\right)$ in (4.4.9), recall that $P_1(x,D)=R+\mu\Delta$ and $Ru=\nabla\cdot(\nabla uT)$, then we have

$$P_1(x,D)[\sum_{jk}\nabla(t_{jk})\cdot\partial_{jk}^2 u] = R(\sum_{jk}\nabla(t_{jk})\cdot\partial_{jk}^2 u)+\mu\Delta(\sum_{jk}\nabla(t_{jk})\cdot\partial_{jk}^2 u). \tag{4.4.10}$$

For the second term of (4.4.10), by using the vector identity $\Delta u=\nabla(\nabla\cdot u)-\nabla\times\nabla\times u=\nabla v-\nabla\times w$, it is easy to see

$$\begin{aligned} &\Delta(\sum_{jk}\nabla(t_{jk})\cdot\partial_{jk}^2 u) \\ &=\sum_{jk}\nabla(t_{jk})\cdot\partial_{jk}^2(\Delta u)+\widetilde{A_{2,3}}(u)+\widetilde{A_{2,2}}(u) \\ &=\sum_{jk}\nabla(t_{jk})\cdot\partial_{jk}^2(\nabla v-\nabla\times w)+\widetilde{A_{2,3}}(u)+\widetilde{A_{2,2}}(u) \\ &=\widetilde{B_{2,3}}(u,v,w)+\widetilde{A_{2,2}}(u), \end{aligned}$$

where $\widetilde{A_{2,m}}$ and $\widetilde{B_{2,m}}$ are m-th order differential operators and

$$\widetilde{B_{2,3}}(u,v,w)=\sum_{jk}\nabla(t_{jk})\cdot\partial_{jk}^2(\nabla v-\nabla\times w)+\widetilde{A_{2,3}}(u).$$

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For the first term of (4.4.10), we have

$$
\begin{aligned}
&R(\sum_{jk} \nabla(t_{jk}) \cdot \partial_{jk}^2 u) \\
&= \sum_{\ell m} t_{\ell m} \partial_{\ell m}^2 (\sum_{jk} \nabla(t_{jk}) \cdot \partial_{jk}^2 u)) \\
&= \sum_{jk} \nabla(t_{jk}) \cdot [\sum_{\ell m} t_{\ell m} \partial_{\ell m}^2 \partial_{jk}^2 u] + \widetilde{C_{2,3}}(u) + \widetilde{C_{2,2}}(u) \\
&= \sum_{jk} \nabla(t_{jk}) \cdot \partial_{jk}^2 (\sum_{\ell m} t_{\ell m} \partial_{\ell m}^2 u) + \widetilde{D_{2,3}}(u) + \widetilde{D_{2,2}}(u) \\
&= \sum_{jk} \nabla(t_{jk}) \cdot Ru + \widetilde{D_{2,3}}(u) + \widetilde{D_{2,2}}(u),
\end{aligned}
$$

and use (4.4.2), we have $Ru = -\mu \Delta u + A_{1,1,}(u,v) + A_{1,0}(u,v)$, we have

$$
\begin{aligned}
&R(\sum_{jk} \nabla(t_{jk}) \cdot \partial_{jk}^2 u) \\
&= \sum_{jk} \nabla(t_{jk}) \cdot \partial_{jk}^2 (-\mu \Delta u + A_{1,1}(u,v) + A_{1,0}(u,v)) + \widetilde{D_{2,3}}(u) + \widetilde{D_{2,2}}(u) \\
&= \sum_{jk} \nabla(t_{jk}) \cdot \partial_{jk}^2 (-\mu(\nabla v - \nabla \times w)) + \sum_{m=0}^{3} \widetilde{E_{2,3}}(u,v) \\
&= \sum_{m=0}^{3} \widetilde{F_{2,m}}(u,v,w)
\end{aligned}
$$

where $\widetilde{C_{2,m}}, \widetilde{D_{2,m}}, \widetilde{E_{2,m}}$ and $\widetilde{F_{2,m}}$ are m-th order differential operators. From the above calculation and (4.4.9), we have

$$
P_2 P_1(x,D)v = \sum_{m=0}^{3} \widehat{E_{2,m}}(u,v,w), \tag{4.4.11}
$$

where $\widehat{E_{2,m}}$ are m-th order differential operators. Similarly, for $P_2 \left(\sum_{jk} \nabla(t_{jk}) \times \partial_{jk}^2 u \right)$, it is easy too see that

$$
\Delta(\sum_{jk} \nabla(t_{jk}) \times \partial_{jk}^2 u) = \widetilde{A_{3,3}}(u,v,w) + \widetilde{A_{3,2}}(u),
$$

where $\widetilde{A_{3,m}}$ is an m-th order differential operator. Similarly, for $R(\sum_{jk} \nabla(t_{jk}) \times \partial_{jk}^2 u)$, component-

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wise, we have

$$
\begin{aligned}
&\left[R(\sum_{jk}\nabla(t_{jk})\times\partial^2_{jk}u)\right]_i\\
&=\sum_{\ell m}t_{\ell m}\partial^2_{\ell m}(\sum_{jk}\nabla(t_{jk})\times\partial^2_{jk}u)_i\\
&=\sum_{\ell m}\sum_{jk}(\nabla(t_{jk})\times t_{\ell m}\partial^2_{\ell m}\partial^2_{jk}u)_i+\widetilde{B_{3,3}}(u)+\widetilde{B_{3,2}}(u)\\
&=\sum_{jk}(\nabla(t_{jk})\times\partial^2_{jk}(\sum_{\ell m}t_{\ell m}\partial^2_{\ell m}u)_i)+\widetilde{C_{3,3}}(u)+\widetilde{C_{3,2}}(u)\\
&=\sum_{jk}(\nabla(t_{jk})\times\partial^2_{jk}Ru)_i+\widetilde{D_{3,3}}(u)+\widetilde{D_{3,2}}(u)
\end{aligned}
$$

and use (4.4.2) again, we obtain

$$
\begin{aligned}
&\left[R(\sum_{jk}\nabla(t_{jk})\times\partial^2_{jk}u)\right]_i\\
&=\sum_{jk}\big(\nabla(t_{jk})\times\partial^2_{jk}[-\mu(\nabla v-\nabla\times w)]\big)_i+\sum_{m=0}^{3}\widetilde{E_{3,m}}(u,v)\\
&=\sum_{m=0}^{3}\widetilde{F_{3,m}}(u,v,w),
\end{aligned}
$$

where $\widetilde{B_{3,m}},\widetilde{C_{3,m}},\widetilde{D_{3,m}},\widetilde{E_{3,m}}$ and $\widetilde{F_{3,m}}$ are m-th order differential operators.

Therefore, we transform the equation (4.4.7) into

$$
P_2P_1(x,D)w=\sum_{m=0}^{3}\widehat{E_{3,m}}(u,v,w), \tag{4.4.12}
$$

where $\widehat{E_{3,m}}$ are m-th order differential operators. From (4.4.11), (4.4.12) and (4.4.9), we can obtain

$$
P_2P_1(x,D)\begin{pmatrix}u\\ v\\ w\end{pmatrix}=\sum_{m=0}^{3}\begin{pmatrix}\widehat{E_{1,m}}(u,v,w)\\ \widehat{E_{2,m}}(u,v,w)\\ \widehat{E_{3,m}}(u,v,w)\end{pmatrix},
$$

with $\widehat{E_{\ell,m}}$ are m-th order differential operators, or equivalently,

$$
P_2P_1U=\sum_{m=0}^{3}\widehat{E_m}(U), \tag{4.4.13}
$$

with $\widehat{E_m}=(\widehat{E_{3,m}},\widehat{E_{3,m}},\widehat{E_{3,m}})^t$ is an m-th order differential operator and $U=(u,v,w)^t$, which means this fourth-order differential equation has the same leading term P_2P_1 and all coefficients of

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(4.4.13) lie in G^s. Moreover, use the elliptic regularity for (4.4.13) with Gevrey coefficients, then $U \in G^s$ by Proposition 2.13 in [5].

4.5 The asymptotic behavior of u near 0

As in Section 2, we set $U = (u, v, w)^t$, where $v = \nabla \cdot u$ and $w = \nabla \times u$. If we can prove that U solves (4.4.13) and satisfies the SUCP, then u solves (4.3.1) and fulfills the SUCP. In the following lemma, we describe the asymptotic behavior of u near 0. Recall that if $u \in H^2_{loc}(\Omega; \mathbb{R}^3)$, then $u \in C^\infty(\Omega)$ by the standard elliptic regularity. Thus, $\forall k \in \mathbb{N}$, we can consider $u \in H^k_{loc}(\Omega; \mathbb{R}^3)$ for arbitrary $k \in \mathbb{N}$ in the following results.

Theorem 4.7. *[36] Let u be a solution to (4.3.1) and for all $N > 0$*

$$\int_{R \le |x| \le 2R} |u|^2 dx = O(R^N) \text{ as } R \to 0.$$

Then for $|\beta| \le 2$, we have

$$\int_{R \le |x| \le 2R} |R^{|\beta|} D^\beta u|^2 dx = O(R^N) \text{ as } R \to 0.$$

Proof. The lemma was proved by the Corollary 17.1.4 in Hörmander [15]. By using the Theorem 4.7, we will get the following Corollary. □

Corollary 4.8. *Let $U = (u, v, w)^t$ with $v = \nabla \cdot u$ and $w = \nabla \times u$. Then for $|\beta| \le 1$, $\forall N > 0$, we have*

$$\int_{R \le |x| \le 2R} |D^\beta U|^2 dx = O(R^N) \text{ as } R \to 0. \tag{4.5.1}$$

In fact, we can get higher derivatives for $|\beta| \ge 2$ in the Corollary 3.2.

Lemma 4.9. *[15] If U satisfies a fourth order strongly elliptic system (4.4.13)*

$$PU = \sum_{m=0}^{3} \widehat{E_m}(U),$$

and U satisfies $\forall N > 0$,

$$\int_{R \le |x| \le 2R} |U|^2 dx = O(R^N) \text{ as } R \to 0.$$

Then it follows that if $|\beta| \le 4$ that

$$\int_{R \le |x| \le 2R} |R^{|\beta|} D^\beta U|^2 dx = O(R^N) \text{ as } R \to 0. \tag{4.5.2}$$

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Proof. Since U satisfies (4.4.13), a fourth order strongly elliptic system, by using the Corollary 17.1.4 in [15], we can obtain (4.5.2). $\square$

Remark 4.10. In the section 3 of [36], the author proved (4.5.2) holding for $|\beta| \leq 2$. From Lemma 4.9 and the coefficients of P are in the Gevrey class G^s, we have $U \in G^s$ and

$$\int_{|x|\leq R} |D^\beta U|^2 dx = O(R^N) \text{ as } R \to 0,$$

for $|\beta| \leq 4$ and $\forall N > 0$.

4.6 Proof of the main theorem

In this section, we want to prove Theorem 1.1. If $U = (u, v, w)^t$ satisfies (4.4.13) and the SUCP, then the SUCP holds for u, where u fulfills (4.3.1).

4.6.1 SUCP for U

In the following theorem, we will prove the SUCP for U.

Theorem 4.11. *Suppose that the second order elliptic operators P_ℓ satisfies (4.3.3), (4.3.4), (4.3.5) and (4.3.6) for $\ell = 1, 2$. $\alpha > 0$ satisfies (4.3.7) at $x = 0$ and s satisfies (4.3.8). Let $P = P_2 P_1$ be a fourth order elliptic operator. Then the SUCP holds for the elliptic system*

$$PU = \sum_{|\beta|\leq 3} a_\beta \partial^\beta U \tag{4.6.1}$$

provided the coefficients of P_ℓ are in the Gevery class G^s.

Proof. The proof follows from [9] and section 1. To prove Theorem 4.1, there are two steps. First, Gevrey regularity of the elliptic system implies the solution U of (4.6.1) is in the Gevrey class G^s (see Proposition 2.13 in [5]). Use the vanishing order assumption and $U \in G^s$, we have

$$|U| \lesssim e^{-|x|^{-\gamma}}, \tag{4.6.2}$$

near $x = 0$ and for some constant $\gamma > 0$. Second, we can show that (4.6.2) implies U vanishes near 0 by using appropriate Carleman estimates. In addition, since U vanishes near 0, by the results in [63], we have $U \equiv 0$ in Ω. $\square$

Remark 4.12. We give the Carleman estimates which were used in the proof of Theorem 4.1 in the following section.

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4.6.2 Carleman Estimates

We are going to derive the Carleman estimates with the weight $e^{\tau|x|^{-\alpha}}$ for the fourth order elliptic operator $P = P_2P_1$ in this section. The following Carleman estimates for the scalar case has been proven in [8] and [9]. Similar to the scalar elliptic equation, we can derive the following Carleman estimate for the special elliptic system.

Proposition 4.13. *Let $P_\ell(x, D) = \sum_{jk} a^\ell_{jk}(x)\partial^2_{jk}$ be a principally diagonal second order elliptic operator where $a^\ell_{jk}(x) \in G^s$ satisfies(4.3.3), (4.3.4), (4.3.5) and (4.3.6) for $\ell = 1, 2$. $\alpha > 0$ satisfies (4.3.7) at $x = 0$ and s satisfies (4.3.8). Then there exist $\tau_0 > 0$ and $r_0 > 0$ such that for $\tau > \tau_0$ and for all $V \in C^\infty((B_{r_0}\backslash\{0\}); \mathbb{R}^7)$, $\ell = 1, 2$, the following inequality holds:*

$$\begin{aligned}&\tau\int |D^2(|x|^{\alpha/2}e^{\tau|x|^{-\alpha}}V|^2dx + \tau^3\int |x|^{-4-3\alpha}e^{2\tau|x|^{-\alpha}}V|^2dx\\ &\lesssim \int |e^{2\tau|x|^{-\alpha}}(P_\ell V)|^2dx.\end{aligned}$$

Proof. Since P_ℓ is the principally diagonal second order elliptic operator for $\ell = 1, 2$, we can directly follow the consequences in [9] and use the proof in [8]. For more details and classical results, we refer readers to [14, 59]. □

By using the integration by parts, we can get a stronger inequality in the following. For more details, we refer readers to [9] and section 3, then we have

$$\begin{aligned}&\sum_{j=0}^{2}\tau^{3-2j}\int e^{2\tau|x|^{-\alpha}}|x|^\alpha|x|^{2(j-2)(1+\alpha)}|D^jV|^2dx\\ &\lesssim \int |e^{2\tau|x|^{-\alpha}}|P_\ell V|^2dx,\end{aligned}$$

with P_ℓ satisfying all the assumptions in Proposition 4.2 for $\ell = 1, 2$. Note that the right hand side of (4.4.3) and (4.4.4) involve second order derivatives of u, we cannot apply the Carleman estimates for the second order differential systems directly to get the SUCP for U. Since we have transformed (4.3.1) into a special fourth order elliptic system with the same leading operator, see (4.4.13), then we can derive the Carleman estimates for the operator $P = P_2P_1$.

Corollary 4.14. *[9] Let*

$$A = \sum_{jk} a_{jk}(x)\partial^2_{x_jx_k}$$

be a second order strongly elliptic operator with a_{ij} in the Gevrey class G^s. Suppose $\alpha > 0$ satisfying

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(4.3.7) at $x = 0$. Then there exists τ_0 such that for all $|s|$, $k \le \nu$ and $\tau \ge \tau_0$

$$\sum_{j=0}^{k+2} \tau^{3-2j} \int |x|^{\alpha+2j(1+\alpha)}|x|^{2s} e^{2\tau|x|^{-\alpha}} |D^j V|^2 dx \tag{4.6.3}$$
$$\lesssim \sum_{j=0}^{k} \tau^{-2j} \int |x|^{2(2+j)(1+\alpha)}|x|^{2s} e^{2\tau|x|^{-\alpha}} |D^j (AV)|^2 dx.$$

Proof. See [9] and section 3. We can use the induction hypothesis to prove the Corollary 4.3. □

For the fourth order elliptic operator $PU = P_2P_1U$ is the product of two second order elliptic operators which satisfies (4.4.13), where $U = (u, v, w)^t$ and $P_\ell(x, D)U = \sum_{jk} a^\ell_{jk}(x)\partial^2_{jk}U$. Recall that $a^\ell_{jk} \in G^s$ and $\alpha > 0$ satisfying (4.3.7) uniformly in x and for $\ell = 1, 2$, then we have the following key estimates.

Proposition 4.15. *We have the following Carleman estimates*

$$\sum_{j=0}^{4} \tau^{6-2j} \int |x|^{-8-6\alpha}|x|^{2j(1+\alpha)} e^{2\tau|x|^{-\alpha}} |D^j V|^2 dx \le C \int e^{2\tau|x|^{-\alpha}} |PV|^2 dx.$$

Proof. Apply the Corollary 4.2 iteratively, then we have

$$\sum_{j=0}^{4} \tau^{6-2j} \int |x|^{-8-6\alpha}|x|^{2j(1+\alpha)} e^{2\tau|x|^{-\alpha}} |D^j V|^2 dx \tag{4.6.4}$$
$$\lesssim \sum_{j=9}^{2} \tau^{3-2j} \int |x|^{-4-3\alpha}|x|^{j(1+\alpha)} e^{2\tau|x|^{-\alpha}} |D^j (P_1V)|^2 dx$$
$$\lesssim \int e^{2\tau|x|^{-\alpha}} |(P_2P_1V)|^2 dx = \int e^{2\tau|x|^{-\alpha}} |PV|^2 dx,$$

where the first inequality is obtained by (4.6.3) with $k = 2$, $s = -4 - \frac{7}{2}\alpha$ and the second inequality is obtained by (4.6.3) with $k = 0$, $s = -2(1 + \alpha)$. For more details, we refer reader to see [9]. □

Now, we want to prove the SUCP for (4.3.1). Here we prove the theorem 2.2.

Proof of Theorem 4.1: The operator $P = P_2P_1$ is strongly elliptic in the Gevrey class G^s, then U is also in the Gevrey class G^s. Therefore, we have the vanishing of infinite order implies that

$$|u| \lesssim e^{-|x|^{-\gamma}}$$

for some $\gamma > \alpha$. Let $\chi \in C_0^\infty(\mathbb{R}^3)$ be such that $\chi \equiv 1$ for $|x| \le R$ and $\chi \equiv 0$ for $|x| \ge 2R$ ($R > 0$ is

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small enough). Then we can apply (4.6.4) to the function χU, which means

$$
\begin{aligned}
& C\sum_{|\beta|=0}^{4}\tau^{6-2|\beta|}\int_{|x|<R}|x|^{(2|\beta|-6)(1+\alpha)-2}e^{2\tau|x|^{-\alpha}}|D^{\beta}U|^{2}dx \\
\leq & \int e^{2\tau|x|^{-\alpha}}|PU|^{2}dx \\
\leq & \int_{|x|<R} e^{2\tau|x|^{-\alpha}}|PU|^{2}dx + \int_{|x|>R} e^{2\tau|x|^{-\alpha}}|P(\chi U)|^{2} \\
\leq & \int_{|x|<R} e^{2\tau|x|^{-\alpha}}|\sum_{m=0}^{3}\widehat{E_m}(U)|^{2}dx + \int_{|x|>R} e^{2\tau|x|^{-\alpha}}|P(\chi U)|^{2},
\end{aligned}
\tag{4.6.5}
$$

by using the reduction elliptic system (4.4.13).

If τ is large and R is sufficiently small, then (4.6.5) implies

$$
\begin{aligned}
& C\sum_{|\beta|=0}^{4}\tau^{6-2|\beta|}\int_{|x|<R}|x|^{(2|\beta|-6)(1+\alpha)-2}e^{2\tau|x|^{-\alpha}}|D^{\beta}U|^{2}dx \\
\leq & \int_{|x|>R} e^{2\tau|x|^{-\alpha}}|P(\chi U)|^{2},
\end{aligned}
\tag{4.6.6}
$$

for some constant $C > 0$. Notice that $e^{\tau|x|^{-\alpha}} \geq e^{\tau R^{-\alpha}}$ for $|x| < R$ and $e^{\tau|x|^{-\alpha}} \leq e^{\tau R^{-\alpha}}$ for $|x| > R$. Therefore, we can use (4.6.6) to obtain

$$
\begin{aligned}
& C\sum_{|\beta|=0}^{4}\tau^{6-2|\beta|}\int_{|x|<R}|x|^{(2|\beta|-6)(1+\alpha)-2}|D^{\beta}U|^{2}dx \\
\leq & \int_{|x|>R}|P(\chi U)|^{2}.
\end{aligned}
$$

Let $\tau \to \infty$, we get $U = 0$ in $\{|x| < R\}$ for R small, which implies $u = 0$ in $\{|x| < R\}$. Furthermore, by using the unique continuation principal in [63], we can obtain $u \equiv 0$ in Ω, then we are done.

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Chapter 5

Future work

5.1 Fundamental solutions for the anisotropic Maxwell system

The last chapter of this thesis is going to list out related opening problems. We list some future works which related to this thesis in the following. In the above chapters, we have already mentioned the enclosure-type method for the anisotropic medium. We gave reconstruction algorithms for both anisotropic elliptic equation (Chapter 2) and anisotropic Maxwell system (Chapter 3). Recall that for the enclosure-type method, we have two tools: One is to define a suitable indicator function and the other is to construct appropriate special solutions for the mathematical model. In Chapter 2 and 3, we have constructed oscillating-decaying (OD) solutions for both anisotropic elliptic equation and anisotropic Maxwell system. The drawback of this special type solutions is that we need to use the *Runge approximation property* to find a sequence of solutions defined on the whole domain and to satisfy the same equation which approximates to OD solutions. It looks like the Runge approximation property used in the thesis is not constructive. If we can make the proof in a constructive way, then this may be useful if one tries to implement the method numerically.

The Runge approximation has a constructive version. Indeed, we can use the density property of the single layer operator between appropriate Sobolev spaces (as L^2-spaces) and the well-posedness of the forward problem. This argument can be used as soon as we have the corresponding fundamental solution and the unique continuation property of the Maxwell model. If ϵ and μ are isotropic, this is of course possible. In the anisotropic cases, we need the construction of the fundamental solution (and justify its type of singularities) in addition to the unique continuation property. We could not find these properties in the literature, so we need to do more constructive work for the fundamental solutions. One of our future work is to construct the fundamental solutions for the anistropic Maxwell system.

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5.2 More L^p estimates for the anisotropic Maxwell system

For the anisotropic Maxwell system in Chapter 3, we only consider the electric permittivity to be allowed to have the jump and the anisotropy and the magnetic permeability μ is a scalar function. It is known that the same method would work when the role of the two parameters exchange. Recall that we have defined the impedance map as $\Lambda_D : \nu \times H|_{\partial\Omega} \to \nu \times E|_{\partial\Omega}$ and we can allow ϵ to be anisotropic and to have jump discontinuity. If we exchange ϵ and μ, we need to use the other impedance map $\widetilde{\Lambda_D} : \nu \times E|_{\partial\Omega} \to \nu \times H|_{\partial\Omega}$. Indeed, we needed to assume that the other coefficients are smooth. The technique is due to the type of L^p estimates we are using. Then, we are able to remove these assumptions by using the Layer potential techniques. and we could test this idea for the scalar divergence form PDE model and it works. Hence, we do hope that this idea can go smoothly to Maxwell as well. We are working on it.

5.3 Strong unique continuation for the general second order elliptic system

In many literature, we know the strong unique continuation property (SUCP) holds for the scalar elliptic equation case. However, for more general elliptic system, we do not know much about the result. In Chapter 4, we gave a very special method to derive the SUCP for the residual stress system with Gevrey coefficients. We use the "product" of two elliptic operators to derive the SUCP for this system. Our future work is try to use similar method to derive more SUCP for more general elliptic system with Gevrey coefficients. In [8], the authors derived the SUCP for the second order elliptic operator $P(x, D)$ with complex coefficients. We want to generalize their ideas to second order elliptic system without any assumptions.

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