國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

MATH107A / GEAI1216A

數學導論（一） **Introduction to Mathematics I**

第一次期中考 October 17, 2025 **Midterm 1**

姓名 Name : solution

學號 Student ID # :

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page,
	5 pages of questions,
	score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**20 points** + 2 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- Please answer the problems in English.

1. [1pt] State the definition of $\lim_{x\to c} f(x) = L$.

We say $\lim_{x\to c} f(x) = L$ if
for any $\varepsilon > 0$, there exists $\delta > 0$ such that
$0 < |x-c| < \delta$ implies $|f(x) - L| < \varepsilon$.

2. [1pt] State the negation of the definition of $\lim_{x\to c} f(x) = L$.

There exists some $\varepsilon > 0$ such that
for any $\delta > 0$, there is x that
satisfies $0 < |x-c| < \delta$ but not $|f(x) - L| < \varepsilon$.

3. [1pt] Give a positive example and a negative example for the definition of $\lim_{x\to c} f(x) = L$. (You don't need to prove it.)

Positive.
$\lim_{x\to 3} x^2 = 9$
is True

Negative
$\lim_{x\to 3} x^2 = 20$
is False.

4. [1pt] State the definition of a linear combination.

Let S be a set of vectors, say $S = \{\vec{u}_1, \dots, \vec{u}_n\}$.
A vector $\vec{p}$ is a linear combination of S
if $\vec{p} = c_1\vec{u}_1 + \dots + c_n\vec{u}_n$ for some $c_1, \dots, c_n \in \mathbb{R}$.

5. [1pt] Give a positive example and a negative example for the definition of a linear combination. (You don't need to prove it.)

Let $S = \{(1,1,1)\}$.

Then $(3,3,3) = 3 \cdot (1,1,1)$ is a lin comb of S,

but $(1,2,3)$ is not a lin comb of S.

 Every sentence should be complete.

6. [3pt] Prove that $\lim_{x\to 2} 3x^2 + 5 = 17$ using the ϵ-δ definition..

Proof: Let $c=2$ and $L=17$.

Let $\varepsilon > 0$ be given.

Choose $\delta = \min\{1, \frac{1}{15}\varepsilon\}$.

Suppose $0<|x-c|<\delta$.

Then $|x-2|<1$ and $|x-2|<\frac{\varepsilon}{15}$.

Note that $|x-2|<1$ implies $1<x<3$,

so $|x+2|<5$.

Thus, $|f(x)-L| = |3x^2+5-17|$

$= |(3x^2+5)-(3\cdot 2^2+5)|$

$= 3|x+2||x-2|$

$< 3\cdot 5\cdot \frac{\varepsilon}{15} = \varepsilon$. #

Think:

$|f(x)-L| = |3x^2+5-17|$

$= |(3x^2+5)-(3\cdot 2^2+5)|$

$= 3|x^2-2^2|$

$= 3|x+2||x-2|$.

If we choose $\delta<1$ and $\delta<\frac{\varepsilon}{15}$, then $|x+2|<5$ and $|x-2|<\frac{\varepsilon}{15}$.

7. [2pt] Disprove that $\lim_{x\to 2} 3x^2 + 5 = 10$ using the ϵ-δ definition.

Proof:

Let $c=2$. and $L=10$.

Choose $\varepsilon = 1$.

We will show that for any $\delta>0$ there is x satisfying $0<|x-c|<\delta$ but $|f(x)-L|>\varepsilon$.

Suppose $\delta>1$. Choose $x=2.1$ so that $0<|x-c|<\delta$.

Then $|f(x)-L| = |3\cdot 4.41+5-10| = 8.23 > \varepsilon$.
#

Suppose $\delta\le 1$. Choose $x=2+\frac{1}{2}\delta$, so that $0<|x-c|<\delta$.

Then $|f(x)-L| = |3(2+\frac{1}{2}\delta)^2+5-10|$

$= |12+6\delta+\frac{3}{4}\delta^2-5|$

$= |7+6\delta+\frac{3}{4}\delta^2| > 7 > \varepsilon$ since $\delta>0$.

Think:

When $0<\delta<k$,

we choose $x=2+\frac{1}{2}\delta$.

Then $|f(x)-L|$

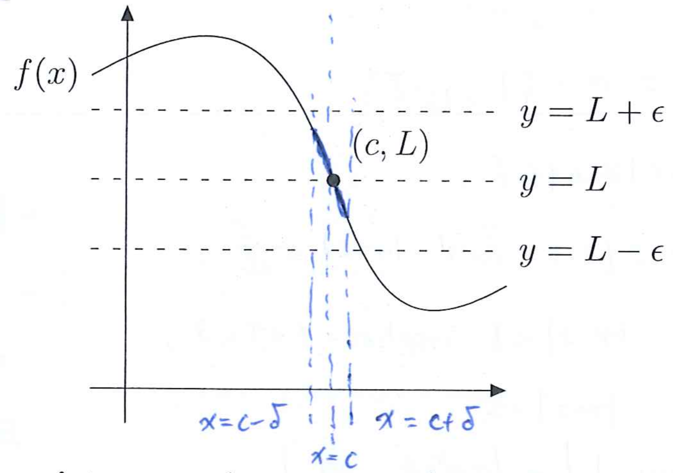
$= |3(2+\frac{1}{2}\delta)^2+5-10|$

$= |12+6\delta+\frac{3}{4}\delta^2-5|$

$= |7+6\delta+\frac{3}{4}\delta^2| > 7$, since $0<\delta<k$.

 Every sentence should be complete.

8. [5pt] Consider a function $f(x)$, a point (c, L), and $\epsilon > 0$ as shown in the picture.



Draw on the picture to demonstrate that you can find some $\delta > 0$ such that

$$0 < |x - c| < \delta \text{ implies } |f(x) - L| < \epsilon. \tag{1}$$

Then **explain carefully by words** why your idea is correct.

We need to find a small $\delta > 0$ such that the part of f on $0 < |x-c| < \delta$ is within the band between $y = L - \varepsilon$ and $y = L + \varepsilon$.

 Every sentence should be complete.

9. Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, where

$$\mathbf{u}_1 = (1, -2, 1, 0, 0),$$
$$\mathbf{u}_2 = (0, 1, -2, 1, 0),$$
$$\mathbf{u}_3 = (0, 0, 1, -2, 1).$$

(a) [2pt] Prove that $\mathbf{p}_1 = (3, -2, 0, -6, 5)$ is a linear combination of S.

Suppose $\vec{p_1} = a\vec{u_1} + b\vec{u_2} + c\vec{u_3}$.

This is equivalent to

$$\begin{cases} a = 3, \\ -2a + b = -2, \\ a - 2b + c = 0, \\ b - 2c = -6, \\ c = 5. \end{cases}$$

By solving the system of equations, we have $a=3$, $b=4$, and $c=5$.

Therefore, $\vec{p_1} = 3\vec{u_1} + 4\vec{u_2} + 5\vec{u_3}$ is a linear combination of S.

(b) [3pt] Prove that $\mathbf{p}_2 = (1, 2, 3, 4, 5)$ is not a linear combination of S.

Suppose $\vec{p_2} = a\vec{u_1} + b\vec{u_2} + c\vec{u_3}$.

This is equivalent to

$$\begin{cases} a = 1, \\ -2a + b = 2, \\ a - 2b + c = 3, \\ b - 2c = 4, \\ c = 5. \end{cases}$$

By the first two equations, we have $a=1$ and $b=4$.

By the last two equations, we have $c=5$ and $b=14$.

Since $b=4$ and $b=14$ contradict to each other, the system of equations have no solution.

Therefore, $\vec{p_2}$ is not a linear combination of S.

10. [extra 2pt] Show that every point in $\mathbb{R}^3$ can be written as a linear combination of $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, where

$$\mathbf{u}_1 = (1, 1, 1),$$
$$\mathbf{u}_2 = (1, 2, 4),$$
$$\mathbf{u}_3 = (1, 3, 9).$$

Let $\vec{p} = (x, y, z)$ be any point in R^3.

We claim that there are a, b, c such that $\vec{p} = a\vec{u_1} + b\vec{u_2} + c\vec{u_3}$.

To see this, we may solve

$$\begin{cases} a + b + c = x, \\ a + 2b + 3c = y, \\ a + 4b + 9c = z. \end{cases}$$

By row operations,

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & x \\ 1 & 2 & 3 & y \\ 1 & 4 & 9 & z \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & x \\ 0 & 1 & 2 & y-x \\ 0 & 3 & 8 & z-x \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & x \\ 0 & 1 & 2 & y-x \\ 0 & 0 & 2 & 2x-3y+z \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & x \\ 0 & 1 & 0 & -3x+4y-z \\ 0 & 0 & 1 & x-\frac{3}{2}y+\frac{1}{2}z \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & & & 3x-\frac{5}{2}y+\frac{1}{2}z \\ & 1 & & -3x+4y-z \\ & & 1 & x-\frac{3}{2}y+\frac{1}{2}z \end{array}\right].$$

By choosing $\begin{cases} a = 3x - \frac{5}{2}y + \frac{1}{2}z, \\ b = -3x + 4y - z, \\ c = x - \frac{3}{2}y + \frac{1}{2}z, \end{cases}$ we know any $\vec{p} = a\vec{u_1} + b\vec{u_2} + c\vec{u_3}$

in $\mathbb{R}^3$ is a linear combination of S.

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	2	
Total	20 (+2)	