Linear preservers of matrix pairs with some extremal norm properties

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Based on joint work with several colleagues.

Part I: Preservers and Isometries

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if and only if there are $M, N \in M_n$ with $\det(MN) = 1$ such that T has the form

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(b) There are $Z \in M_n$ and invertible matrices $M, N \in M_n$ such that T has the form

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- By the Mazur-Ulam theorem (1932), ϕ is a distance preserving map if and only if it is a real affine map, a real linear map followed by a translation.

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$$A \mapsto UAV + Z, \quad A \mapsto UA^tV + Z, \quad A \mapsto U\bar{A}V + Z, \quad A \mapsto UA^*V + Z.$$

Part II: Matrix Product with Extreme Property

Matrix pairs with extremal norm properties

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- Problem: Characterize linear maps $T : M_n \to M_n$ such that:

$$\|T(A)\|\|T(B)\| = \|T(A)T(B)\| \quad \text{whenever} \quad \|A\|\|B\| = \|AB\|,$$
$$\|T(A)\|\|T(B)\| = \|T(A)T(B)^*\| \quad \text{whenever} \quad \|A\|\|B\| = \|AB^*\|, \text{ or}$$
$$\|T(A)\|\|T(B)\| = \|T(A)^*T(B)\| \quad \text{whenever} \quad \|A\|\|B\| = \|A^*B\|.$$

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- The real linear map $A \mapsto \overline{A}$ on complex matrices would work.

Result

We let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and consider real bijective linear maps $T : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ inspired by the results on distance preserving maps.

Theorem [Kuzma,Li,Poon,2026]

Let $n \geq 3$ and $T : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ be a bijective linear map. The following conditions are equivalent.

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Result

We let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and consider real bijective linear maps $T : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ inspired by the results on distance preserving maps.

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(b) $\mathbb{F} = \mathbb{C}$, there exist $U, V \in \mathcal{U}_2(\mathbb{C})$, $\gamma > 0$ and $\mu \in \mathbb{C} \setminus \mathbb{R}$ such that the real-linear map $L\colon X \mapsto \gamma V T(UXU^*) V^*$ has the form

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If T is complex linear, then $\mu = i$.

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- Extension of our results to $B(H)$ and general UI norms are of interest.

Part III: Matrix Sum with Extreme Property

Linear preservers of TEA pairs

- In a normed vector space $(\mathbf{V}, |\cdot|)$ over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, we have

$$|\mathbf{x}+\mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}| \qquad \text{for any } \mathbf{x}, \mathbf{y} \in \mathbf{V}.$$

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- A map on $T : \mathbf{V} \to \mathbf{V}$ preserving triangular equality attaining (TEA) pairs if

$$|T(\mathbf{x}) + T(\mathbf{y})| = |T(\mathbf{x})| + |T(\mathbf{y})| \quad \text{whenever} \quad |\mathbf{x}+\mathbf{y}| = |\mathbf{x}| + |by|.$$

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Theorem [Li,Tsai,Wang,Wong, 2016]

Let $n \geq 3$. A linear map $T : M_n \to M_n$ preserves TEA pair with respect to the operator norm if and only if T has the form

$$A \mapsto \gamma UAV, \quad A \mapsto \gamma UA^t V \quad \text{or} \quad A \mapsto f(A)Z,$$

for some $\gamma \in \mathbb{F}$, $Z, U, V \in M_n$ with $U^*U = V^*V = I$, and a linear functional $f : M_n \to \mathbb{F}$.

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The case for $n = 2$ was also treated; the results and proofs are more intricate.

The Ky-Fan norms

Let $1 \le k \le n$. The Ky-Fan k-norm of $A \in M_n$ is defined by

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  That is: (i) $A \mapsto \gamma UAV$, $A \mapsto \gamma UA^tV$, or

  (ii) $(\mathbb{F}, n, k) = (\mathbb{R}, 4, 2)$, $A \mapsto \phi(UAV)$ or $A \mapsto \phi(UA^tV)$, where $\Phi(A_1 + A_2) = A_1 - A_2$ when $A_1 \in \mathbf{V}$, $A_2 \in \mathbf{V}^\perp$ for a certain 4 dimensional subspace $\mathbf{V}$ in $M_4(\mathbb{R})$.

The higher numerical radius

Let $1 \le k \le n$. The k-numerical radius of A is defined by

$$w_k(A) = \max\{|\mathrm{tr}\,(PA)| : P^* = P = P^2, \mathrm{tr}\,A = k\}.$$

Note that and $w_n(A) = \mathrm{tr}\,(A)$, and is (classical) numerical radius:

$$\begin{aligned} w_1(A) &= \max\{|x^*Ax| : x \in \mathbb{C}^n, x^*x = 1\} \\ &= \max\{|\mathrm{tr}\,(Axx^*)| : x \in \mathbb{C}^n, x^*x = 1\}. \end{aligned}$$

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Let $1 \leq k < n$ and $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a linear map. The following conditions are equivalent.

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(ii) $n = 2k > 2$ and γT has the form

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Results on Hermitian matrices are obtained; the results and proofs are more intricate.

Additional topics

- One may general maps without the linearity assumption.

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- What are your favorite preserver problems?

Thank you for your attention!