國 立 陽 明 交 通 大 學
NATIONAL YANG MING CHIAO TUNG UNIVERSITY

Python 與機器學習演算法 **SCMA30084**
Python and Machine Learning Algorithms

第一次段考 March 30, 2026 **Exam 1**

姓名 Name : Solution

學號 Student ID # : ____________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page,
	5 pages of questions,
	score page at the end
To be answered:	on the test paper
Duration:	**180 minutes**
Total points:	**20 points** + 2 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- If the answer is too long, round it to **two decimal places**. Numerical answers with a **tolerance** $\pm$**0.05** are accepted.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- Please answer the problems in **English**.

1. Define the NumPy arrays as follows.

```
arr1 = np.array([1,2,3])
arr2 = np.array([[1],[2],[3]])
arr3 = np.array([[1,2,3]])
arr4 = np.array([[1,2],[3,4]])
```

(a) [1pt] For each array above, write down its shape.

arr1 $\to$ (3,)

arr2 $\to$ (3,1)

arr3 $\to$ (1,3)

arr4 $\to$ (2,2)

(b) [2pt] What is the output of `(arr2 - 1) * arr3`? Describe the broadcasting rule.

$$\text{arr2} - 1 = \cancel{[1\ 2\ 3] - 1} = \cancel{[1\ 2\ 3] - [1\ 1\ 1]}$$

$$\begin{bmatrix}1\\2\\3\end{bmatrix} - 1 \xrightarrow{\text{broadcasting}} \begin{bmatrix}1\\2\\3\end{bmatrix} - \begin{bmatrix}1\\1\\1\end{bmatrix} = \cancel{[0} \begin{bmatrix}0\\1\\2\end{bmatrix}$$

$$(\text{arr2}-1)\ \cancel{\text{arr2}} * \text{arr3} = \begin{bmatrix}0\\1\\2\end{bmatrix} * [1\ 2\ 3] \underset{\text{broadcasting}}{=} \begin{bmatrix}0&0&0\\1&1&1\\2&2&2\end{bmatrix} * \begin{bmatrix}1&2&3\\1&2&3\\1&2&3\end{bmatrix}$$

$$= \begin{bmatrix}0&0&0\\1&2&3\\2&4&6\end{bmatrix}$$

(c) [2pt] What is the output of `arr4 - arr4.mean(axis=`~~1~~ 0`)`? Describe the broadcasting rule.

arr4.mean(axis = 1) = average of each ~~col~~ column

$$\cancel{= \left[\frac{1+2}{2}\ \ 2\right]} \quad \left[\begin{bmatrix}1\\3\end{bmatrix}\begin{bmatrix}2\\4\end{bmatrix}\right]$$

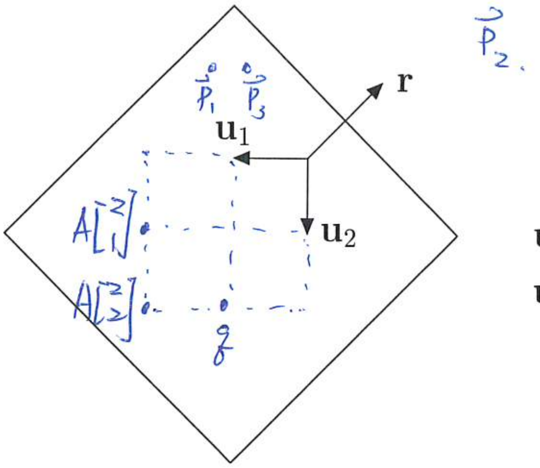
ave ave

$$= \left[\frac{1+3}{2}\ \ \frac{2+4}{2}\right] = [2, 3]$$

$$\text{arr4} - \text{arr.mean(axis} = 1) = \begin{bmatrix}1&2\\3&4\end{bmatrix} - [2\ 3]$$

$$\underset{\text{broadcasting}}{=} \begin{bmatrix}1&2\\3&4\end{bmatrix} - \begin{bmatrix}2&3\\2&3\end{bmatrix} = \begin{bmatrix}-1&-1\\1&1\end{bmatrix}.$$

2. Consider the hyperplane with its normal vector $\mathbf{r}$ as shown below. Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be vectors falling on the hyperplane.



$$\mathbf{r} = (1, -1, 1, -1)$$
$$\mathbf{u}_1 = (1, 1, 1, 1)$$
$$\mathbf{u}_2 = (1, -1, -1, 1)$$

(a) [2pt] For each of the following points, **draw** it on the picture above to demonstrate its relative position with the hyperplane (on the hyperplane, on the same side as $\mathbf{r}$, or on the opposite side) and **find** its projection onto the the direction of $\mathbf{r}$.

$\langle P_1, r\rangle = 0 \rightarrow$ on hyperplane $\quad \mathbf{p}_1 = (2, 2, 1, 1)$, $\quad$ proj of $\vec{P}_3 = \frac{\langle P_3, r\rangle}{\|r\|^2} = \underline{\underline{\vec{0}}}$

$\mathbf{p}_2 = (2, 1, 2, 1)$,

$\langle P_2, r\rangle = 2 \rightarrow \vec{r}$ side $\quad \mathbf{p}_3 = (2, 1, 1, 2)$.

$\langle P_3, r\rangle = 0 \rightarrow$ on hyperplane

proj of $\vec{P}_1 = \frac{\langle P_1, r\rangle}{\|r\|^2} \cdot r = \underline{\underline{\vec{0}}}$

proj of $\vec{P}_2 = \frac{\langle P_2, r\rangle}{\|r\|^2} \cdot r = \frac{2}{4} \cdot \begin{bmatrix}1\\-1\\1\\-1\end{bmatrix} = \underline{\underline{\frac{1}{2}\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}}}$

(b) [3pt] Let A be the matrix whose columns are $\mathbf{u}_1$ and $\mathbf{u}_2$ in order. Draw

$$A\begin{bmatrix}2\\1\end{bmatrix}, \quad A\begin{bmatrix}2\\2\end{bmatrix}, \text{ and } \mathbf{q} = (3, -1, -1, 3)$$

on the picture above.

$A = \begin{bmatrix} | & | \\ u_1 & u_2 \\ | & | \end{bmatrix}$, so $A\begin{bmatrix}2\\1\end{bmatrix} = 2u_1 + 1u_2$

$A\begin{bmatrix}2\\2\end{bmatrix} = 2u_1 + 2u_2$

Solve $\vec{q} = 1u_1 + 2u_2$.

3. Let

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}.$$

(a) [1pt] Find the eigenvalues of A and an orthonormal eigenbasis of it.

eigenvalues $-1, 1, 1, 3$ eigenvectors $\begin{bmatrix} -0.5 \\ -0.5 \\ 0.5 \\ 0.5 \end{bmatrix} \begin{bmatrix} -0.71 \\ 0.71 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -0.71 \\ 0.71 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}$

(b) [1pt] Let $R_A(\mathbf{x}) = \frac{\mathbf{x}^\top A \mathbf{x}}{\mathbf{x}^\top \mathbf{x}}$. Solve

$$\min_{\mathbf{x} \neq \mathbf{0}} R_A(\mathbf{x})$$

and find a vector $\mathbf{x}$ that achieves the minimum.

By the Rayleig quotient thm,

$\min = \lambda_{min} = -1$, achieved by eigenvector $\tilde{x} = \begin{bmatrix} -0.5 \\ -0.5 \\ 0.5 \\ 0.5 \end{bmatrix}$.

(c) [$\cancel{2}$ 1pt] Find a vector $\mathbf{x}$ such that $R_A(\mathbf{x}) = 0$. Describe your reasoning.

~~Find a~~ Solve $c_1^2(-1) + c_2^2 1 + c_3^2 1 + c_4^2 3 = 0$ for an arbitrary solution.

For example, let $c_1 = 1$, $c_2 = 1$

Then $x = 1\begin{bmatrix} -0.5 \\ -0.5 \\ 0.5 \\ 0.5 \end{bmatrix} + 1\begin{bmatrix} -0.71 \\ 0.71 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1.21 \\ 0.21 \\ 0.5 \\ 0.5 \end{bmatrix}$ (answer not unique).

(d) [2pt] Apply the Power Method to A using the initial vector $\mathbf{x}_0 = (1, -1, 0, 0)$. Which vector does $\mathbf{x}_k$ converge to? Explain why $\mathbf{x}_k$ does not converge to the eigenvector corresponding to the largest eigenvalue, ~~and discuss how likely this situation is to occur.~~

By algorithm, $x_k \to (0.71, -0.71, 0, 0)$

This is because $A x_0 = 1 x_0$, so x_0 never goes to the direction of the desired eigenvector.

4. [5pt] Mathematical essay: Write a few paragraphs to introduce the *spectral decomposition.*

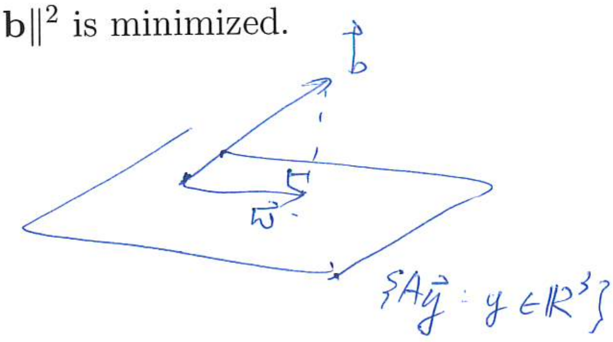
   Your score will be based on the following criteria.

   - The definition is clear.
   - Some sentences are added to explain the definition.
   - Examples or pictures are included to help understanding.
   - The sentences are complete.

5. [extra 2pt] Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 2 \end{bmatrix}.$$

Find the vector $\mathbf{x}$ such that $\|A\mathbf{x} - \mathbf{b}\|^2$ is minimized.



Solve $Ax = w$ instead,

$\vec{w}$ = projection of $\vec{b}$ onto Col (A)

$= A(A^TA)^{-1}A\vec{b}$

Solve $A\vec{x} = A(A^TA)^{-1}A^T\vec{b}$ and get $\vec{x} = (A^TA)^{-1}A^T\vec{b}$

$$= \begin{bmatrix} 4.75 \\ -3.65 \\ 0.75 \end{bmatrix}$$

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	2	
Total	20 (+2)	