

Engaging students with collaborative tasks in linear algebra

Jephian C.-H. Lin 林晉宏

Department of Applied Mathematics, National Yang Ming Chiao Tung University

Driver's training in Taiwan



Source: Driving-Coach

Driver's training in Taiwan



Only the **exam** matters.

Source: Driving-Coach

English training in Taiwan



Source: Facebook

English training in Taiwan



Only the **standard answer** matters.

Source: Facebook

Consequence in higher education in Taiwan

A function $f(x)$ is continuous at a point c means:

Textbook version

For any any $\epsilon > 0$, there is $\delta > 0$ such that $|x - c| < \delta$ implies $|f(x) - f(c)| < \epsilon$.

Student version

$\epsilon > 0, \delta > 0, |x - c| < \delta, |f(x) - f(c)| < \epsilon$

For students, mathematics is **more about calculation, but not too much about the logic.**

Potential solutions...

More exploratory exercises.

- Learn to find answers in different ways.
- Learn the beauty of open answers.

→ Finding the determinant in a tedious way

More intuition deliveries.

- Learn the meaning beyond the symbols.
- Learn the logic of mathematical settings.

→ Feeling the vector representation

Potential solutions...

More exploratory exercises.

- Learn to find answers in different ways.
- Learn the beauty of open answers.

→ Finding the determinant in a tedious way

More intuition deliveries.

- Learn the meaning beyond the symbols.
- Learn the logic of mathematical settings.

→ Feeling the vector representation

Finding the determinant in a tedious way

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

permutation: 4321

$$\det(A) = \cdots + \det \begin{bmatrix} 0 & 0 & 0 & 4 \\ 0 & 0 & 7 & 0 \\ 0 & 10 & 0 & 0 \\ 13 & 0 & 0 & 0 \end{bmatrix} + \cdots$$

= sum of 24 terms

= 0.

$\det(A) = \text{sum of 24 terms}$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

permutation	1234	1243	1324	1342	1423	1432
value	1056					
permutation	2134	2143	2314	2341	2413	2431
value						
permutation	3124	3142	3214	3241	3412	3421
value						
permutation	4123	4132	4213	4231	4312	4321
value						

$\det(A) = \text{sum of 24 terms}$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

permutation	1234	1243	1324	1342	1423	1432
value	1056	-1080				
permutation	2134	2143	2314	2341	2413	2431
value						
permutation	3124	3142	3214	3241	3412	3421
value						
permutation	4123	4132	4213	4231	4312	4321
value						

$\det(A) = \text{sum of 24 terms}$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

permutation	1234	1243	1324	1342	1423	1432
value	1056	-1080	-1120			
permutation	2134	2143	2314	2341	2413	2431
value						
permutation	3124	3142	3214	3241	3412	3421
value						
permutation	4123	4132	4213	4231	4312	4321
value						

$\det(A) = \text{sum of 24 terms}$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

permutation	1234	1243	1324	1342	1423	1432
value	1056	-1080	-1120	1176		
permutation	2134	2143	2314	2341	2413	2431
value						
permutation	3124	3142	3214	3241	3412	3421
value						
permutation	4123	4132	4213	4231	4312	4321
value						

$\det(A) = \text{sum of 24 terms}$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

permutation	1234	1243	1324	1342	1423	1432
value	1056	-1080	-1120	1176	1200	
permutation	2134	2143	2314	2341	2413	2431
value						
permutation	3124	3142	3214	3241	3412	3421
value						
permutation	4123	4132	4213	4231	4312	4321
value						

$\det(A) = \text{sum of 24 terms}$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

permutation	1234	1243	1324	1342	1423	1432
value	1056	-1080	-1120	1176	1200	-1232
permutation	2134	2143	2314	2341	2413	2431
value						
permutation	3124	3142	3214	3241	3412	3421
value						
permutation	4123	4132	4213	4231	4312	4321
value						

$\det(A) = \text{sum of 24 terms}$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

permutation	1234	1243	1324	1342	1423	1432
value	1056	-1080	-1120	1176	1200	-1232
permutation	2134	2143	2314	2341	2413	2431
value	-1760					
permutation	3124	3142	3214	3241	3412	3421
value						
permutation	4123	4132	4213	4231	4312	4321
value						

$\det(A) = \text{sum of 24 terms}$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

permutation	1234	1243	1324	1342	1423	1432
value	1056	-1080	-1120	1176	1200	-1232
permutation	2134	2143	2314	2341	2413	2431
value	-1760	1800				
permutation	3124	3142	3214	3241	3412	3421
value						
permutation	4123	4132	4213	4231	4312	4321
value						

$\det(A) = \text{sum of 24 terms}$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

permutation	1234	1243	1324	1342	1423	1432
value	1056	-1080	-1120	1176	1200	-1232
permutation	2134	2143	2314	2341	2413	2431
value	-1760	1800	2016			
permutation	3124	3142	3214	3241	3412	3421
value						
permutation	4123	4132	4213	4231	4312	4321
value						

$\det(A) = \text{sum of 24 terms}$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

permutation	1234	1243	1324	1342	1423	1432
value	1056	-1080	-1120	1176	1200	-1232
permutation	2134	2143	2314	2341	2413	2431
value	-1760	1800	2016	-2184		
permutation	3124	3142	3214	3241	3412	3421
value						
permutation	4123	4132	4213	4231	4312	4321
value						

$\det(A) = \text{sum of 24 terms}$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

permutation	1234	1243	1324	1342	1423	1432
value	1056	-1080	-1120	1176	1200	-1232
permutation	2134	2143	2314	2341	2413	2431
value	-1760	1800	2016	-2184	-2160	
permutation	3124	3142	3214	3241	3412	3421
value						
permutation	4123	4132	4213	4231	4312	4321
value						

$\det(A) = \text{sum of 24 terms}$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

permutation	1234	1243	1324	1342	1423	1432
value	1056	-1080	-1120	1176	1200	-1232
permutation	2134	2143	2314	2341	2413	2431
value	-1760	1800	2016	-2184	-2160	2288
permutation	3124	3142	3214	3241	3412	3421
value						
permutation	4123	4132	4213	4231	4312	4321
value						

$\det(A) = \text{sum of 24 terms}$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

permutation	1234	1243	1324	1342	1423	1432
value	1056	-1080	-1120	1176	1200	-1232
permutation	2134	2143	2314	2341	2413	2431
value	-1760	1800	2016	-2184	-2160	2288
permutation	3124	3142	3214	3241	3412	3421
value	2400					
permutation	4123	4132	4213	4231	4312	4321
value						

$\det(A) = \text{sum of 24 terms}$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

permutation	1234	1243	1324	1342	1423	1432
value	1056	-1080	-1120	1176	1200	-1232
permutation	2134	2143	2314	2341	2413	2431
value	-1760	1800	2016	-2184	-2160	2288
permutation	3124	3142	3214	3241	3412	3421
value	2400	-2520				
permutation	4123	4132	4213	4231	4312	4321
value						

$\det(A) = \text{sum of 24 terms}$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

permutation	1234	1243	1324	1342	1423	1432
value	1056	-1080	-1120	1176	1200	-1232
permutation	2134	2143	2314	2341	2413	2431
value	-1760	1800	2016	-2184	-2160	2288
permutation	3124	3142	3214	3241	3412	3421
value	2400	-2520	-2592			
permutation	4123	4132	4213	4231	4312	4321
value						

$\det(A) = \text{sum of 24 terms}$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

permutation	1234	1243	1324	1342	1423	1432
value	1056	-1080	-1120	1176	1200	-1232
permutation	2134	2143	2314	2341	2413	2431
value	-1760	1800	2016	-2184	-2160	2288
permutation	3124	3142	3214	3241	3412	3421
value	2400	-2520	-2592	2808		
permutation	4123	4132	4213	4231	4312	4321
value						

$\det(A) = \text{sum of 24 terms}$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

permutation	1234	1243	1324	1342	1423	1432
value	1056	-1080	-1120	1176	1200	-1232
permutation	2134	2143	2314	2341	2413	2431
value	-1760	1800	2016	-2184	-2160	2288
permutation	3124	3142	3214	3241	3412	3421
value	2400	-2520	-2592	2808	3024	
permutation	4123	4132	4213	4231	4312	4321
value						

$\det(A) = \text{sum of 24 terms}$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

permutation	1234	1243	1324	1342	1423	1432
value	1056	-1080	-1120	1176	1200	-1232
permutation	2134	2143	2314	2341	2413	2431
value	-1760	1800	2016	-2184	-2160	2288
permutation	3124	3142	3214	3241	3412	3421
value	2400	-2520	-2592	2808	3024	-3120
permutation	4123	4132	4213	4231	4312	4321
value						

$\det(A) = \text{sum of 24 terms}$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

permutation	1234	1243	1324	1342	1423	1432
value	1056	-1080	-1120	1176	1200	-1232
permutation	2134	2143	2314	2341	2413	2431
value	-1760	1800	2016	-2184	-2160	2288
permutation	3124	3142	3214	3241	3412	3421
value	2400	-2520	-2592	2808	3024	-3120
permutation	4123	4132	4213	4231	4312	4321
value	-3000					

$\det(A) = \text{sum of 24 terms}$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

permutation	1234	1243	1324	1342	1423	1432
value	1056	-1080	-1120	1176	1200	-1232
permutation	2134	2143	2314	2341	2413	2431
value	-1760	1800	2016	-2184	-2160	2288
permutation	3124	3142	3214	3241	3412	3421
value	2400	-2520	-2592	2808	3024	-3120
permutation	4123	4132	4213	4231	4312	4321
value	-3000	3080				

$\det(A) = \text{sum of 24 terms}$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

permutation	1234	1243	1324	1342	1423	1432
value	1056	-1080	-1120	1176	1200	-1232
permutation	2134	2143	2314	2341	2413	2431
value	-1760	1800	2016	-2184	-2160	2288
permutation	3124	3142	3214	3241	3412	3421
value	2400	-2520	-2592	2808	3024	-3120
permutation	4123	4132	4213	4231	4312	4321
value	-3000	3080	3240			

$\det(A) = \text{sum of 24 terms}$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

permutation	1234	1243	1324	1342	1423	1432
value	1056	-1080	-1120	1176	1200	-1232
permutation	2134	2143	2314	2341	2413	2431
value	-1760	1800	2016	-2184	-2160	2288
permutation	3124	3142	3214	3241	3412	3421
value	2400	-2520	-2592	2808	3024	-3120
permutation	4123	4132	4213	4231	4312	4321
value	-3000	3080	3240	-3432		

$\det(A) = \text{sum of 24 terms}$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

permutation	1234	1243	1324	1342	1423	1432
value	1056	-1080	-1120	1176	1200	-1232
permutation	2134	2143	2314	2341	2413	2431
value	-1760	1800	2016	-2184	-2160	2288
permutation	3124	3142	3214	3241	3412	3421
value	2400	-2520	-2592	2808	3024	-3120
permutation	4123	4132	4213	4231	4312	4321
value	-3000	3080	3240	-3432	-3528	

$\det(A) = \text{sum of 24 terms}$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

permutation	1234	1243	1324	1342	1423	1432
value	1056	-1080	-1120	1176	1200	-1232
permutation	2134	2143	2314	2341	2413	2431
value	-1760	1800	2016	-2184	-2160	2288
permutation	3124	3142	3214	3241	3412	3421
value	2400	-2520	-2592	2808	3024	-3120
permutation	4123	4132	4213	4231	4312	4321
value	-3000	3080	3240	-3432	-3528	3640

$\det(A) = \text{sum of 24 terms}$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

permutation	1234	1243	1324	1342	1423	1432
value	1056	-1080	-1120	1176	1200	-1232
permutation	2134	2143	2314	2341	2413	2431
value	-1760	1800	2016	-2184	-2160	2288
permutation	3124	3142	3214	3241	3412	3421
value	2400	-2520	-2592	2808	3024	-3120
permutation	4123	4132	4213	4231	4312	4321
value	-3000	3080	3240	-3432	-3528	3640

Do they sum up to be 0?

Guess

$$\det \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} =$$

11
11x11 matx



1234

1056

2134

-1760

3124

2400

4123

-3800

-1243

-1080

2143

1800

3142

-2520

4132

3080

1324

-1120

2314

2016

3214

-2592

4213

3240

1342

1176

2341

-2184

3241

2808

4231

3432

1423

1200

2413

-2160

3412

3024

4312

-3528

1432

1232

2431

2288 :0

3421

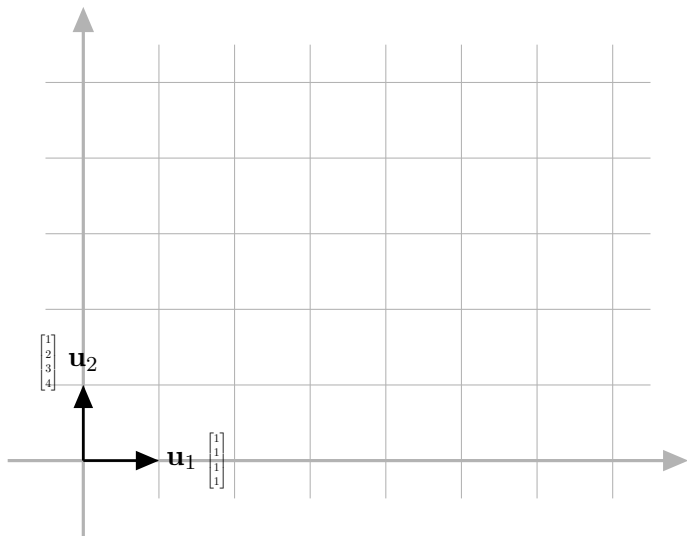
-3120

4321

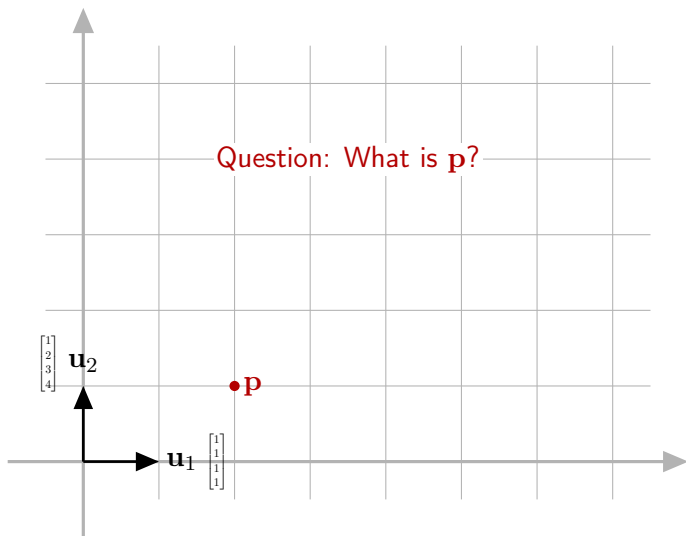
3640

some different ways

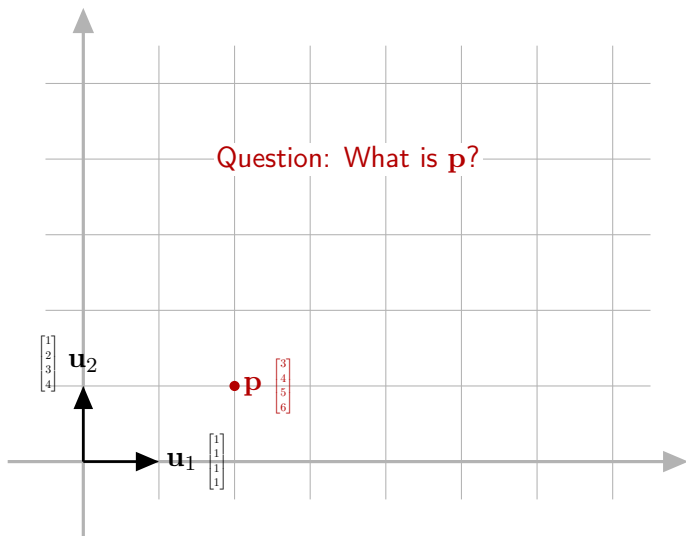
Feeling the vector representation



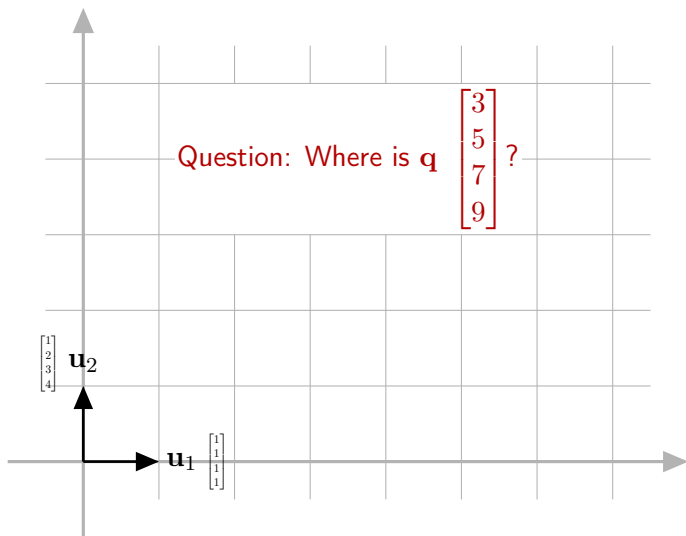
Feeling the vector representation



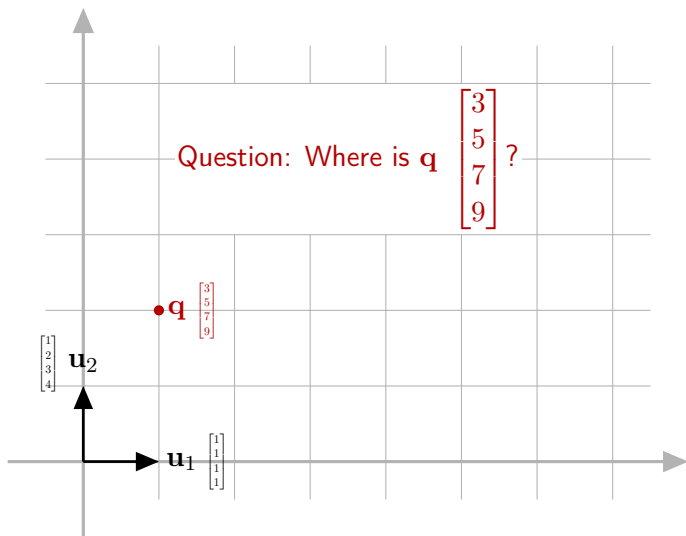
Feeling the vector representation



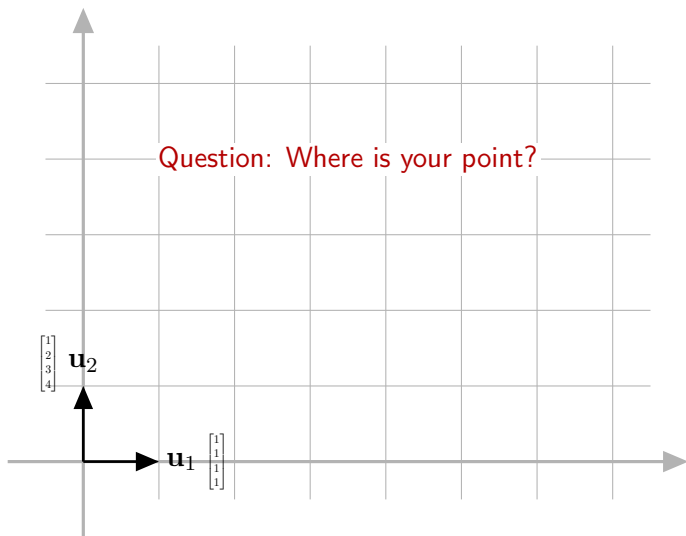
Feeling the vector representation



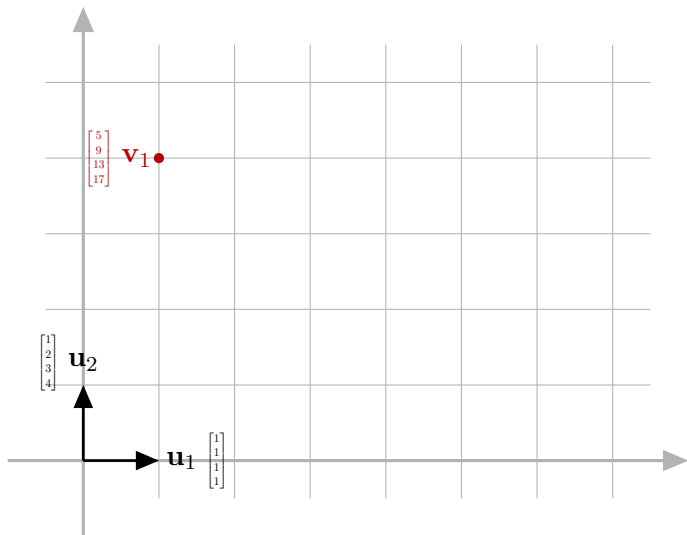
Feeling the vector representation



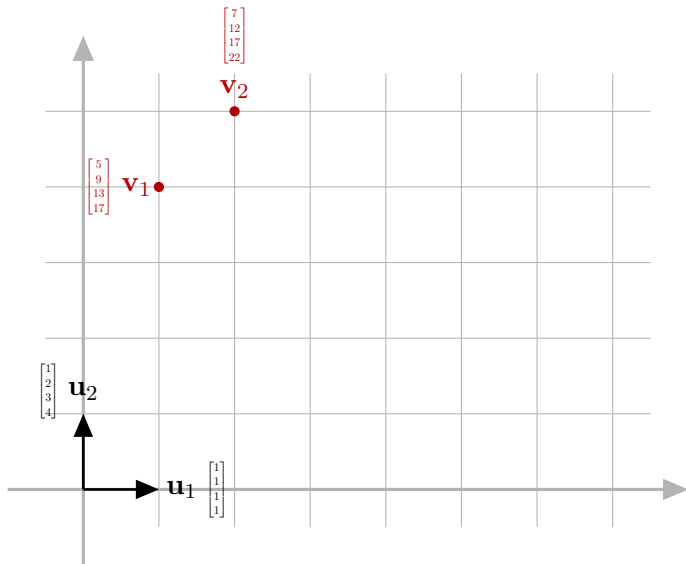
Feeling the vector representation



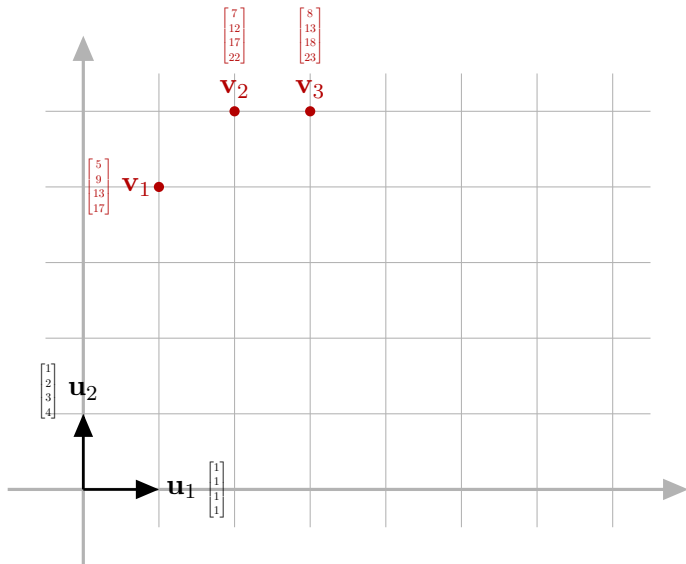
Feeling the vector representation



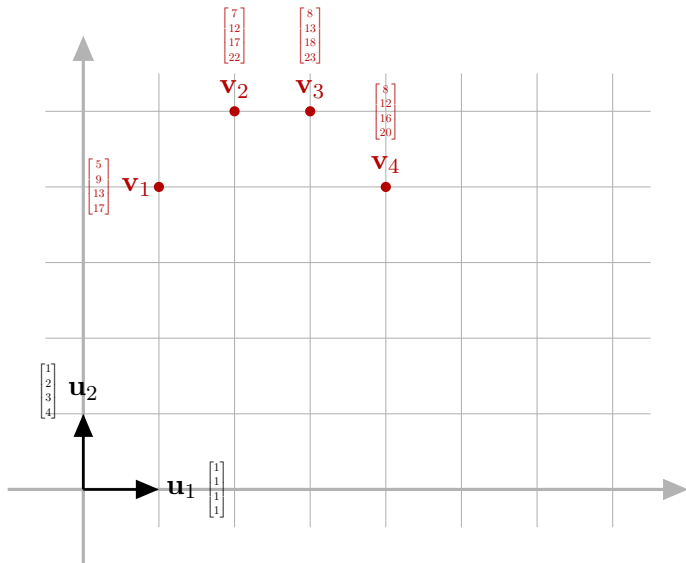
Feeling the vector representation



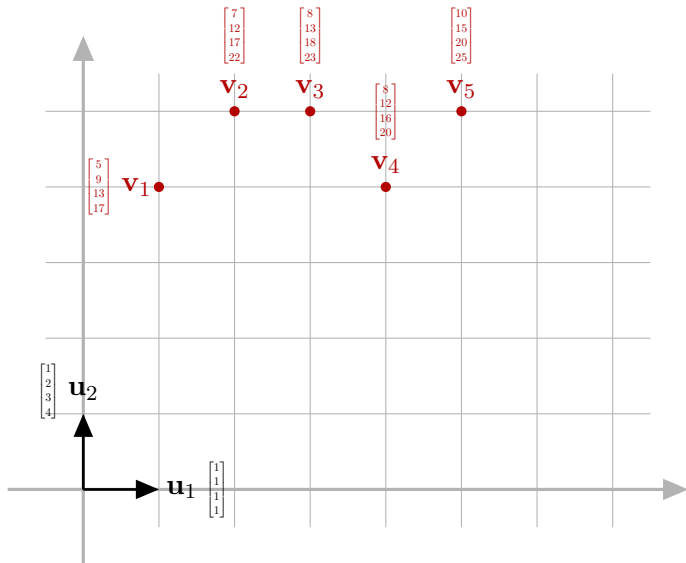
Feeling the vector representation



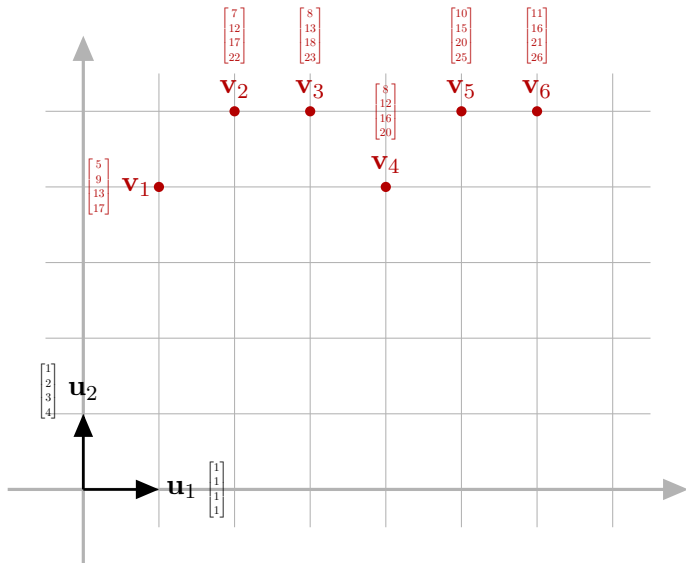
Feeling the vector representation



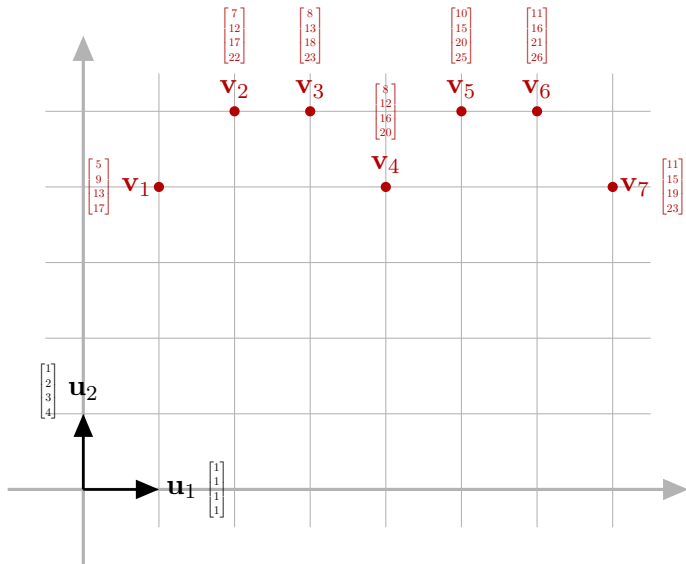
Feeling the vector representation



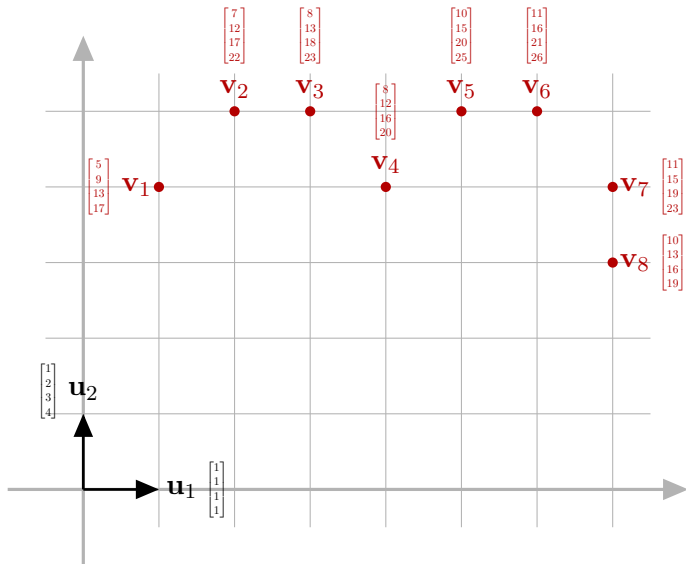
Feeling the vector representation



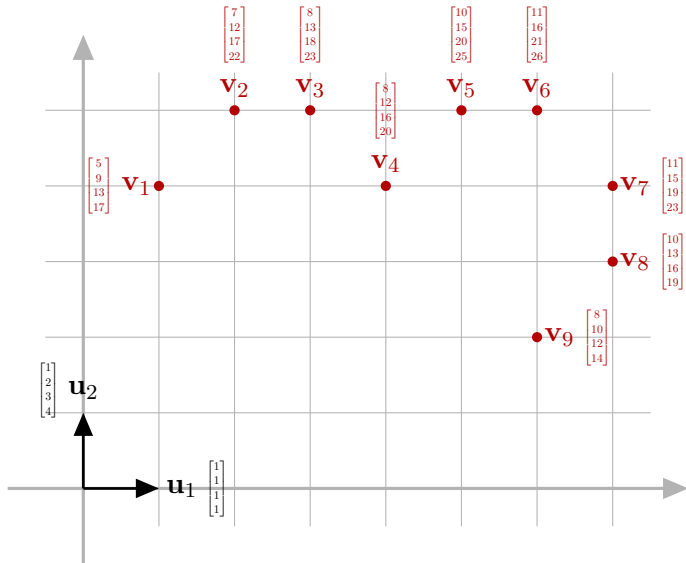
Feeling the vector representation



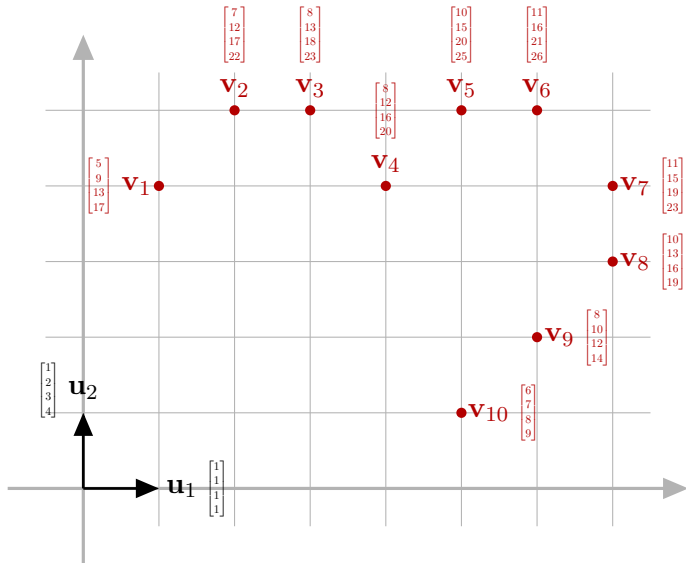
Feeling the vector representation



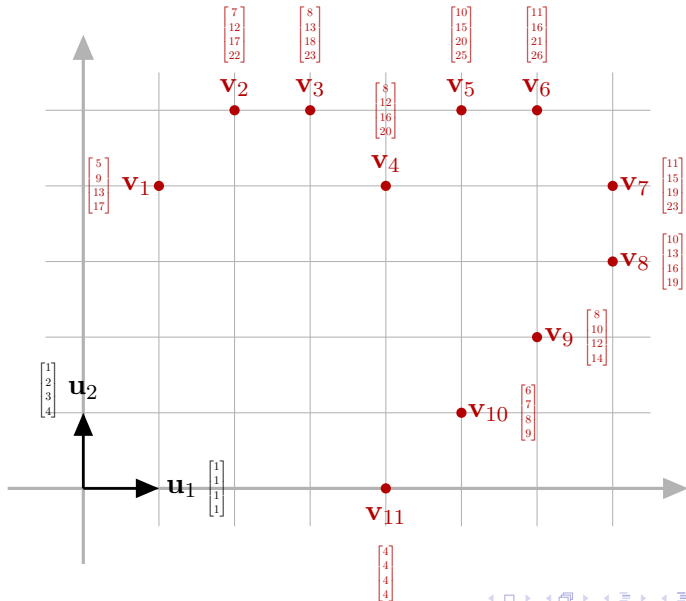
Feeling the vector representation



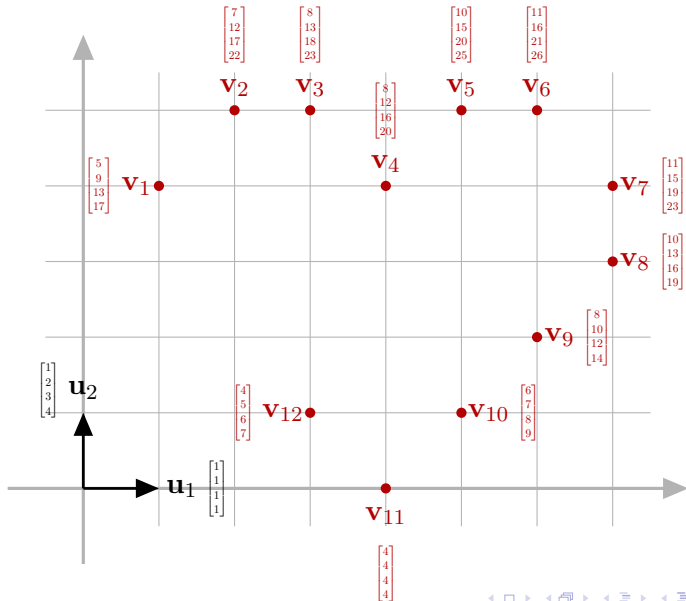
Feeling the vector representation



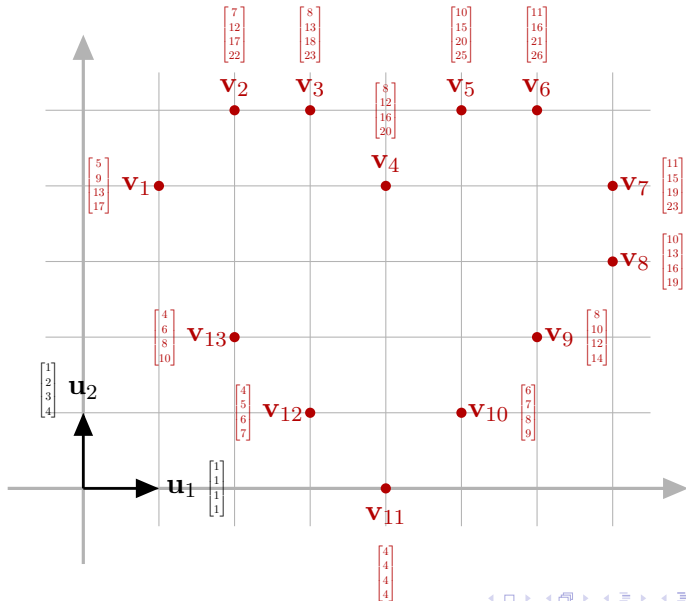
Feeling the vector representation



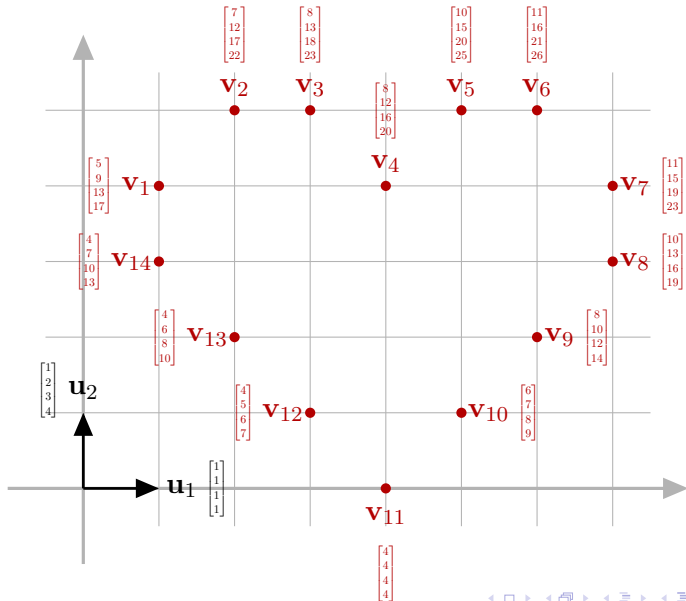
Feeling the vector representation



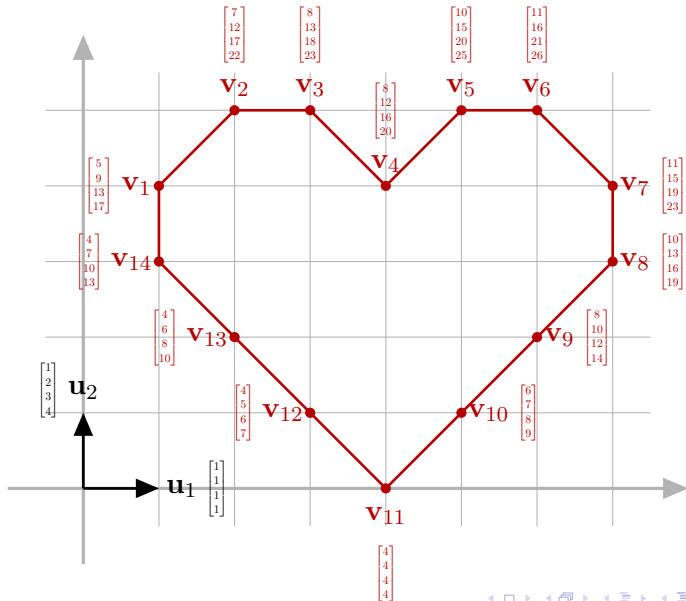
Feeling the vector representation

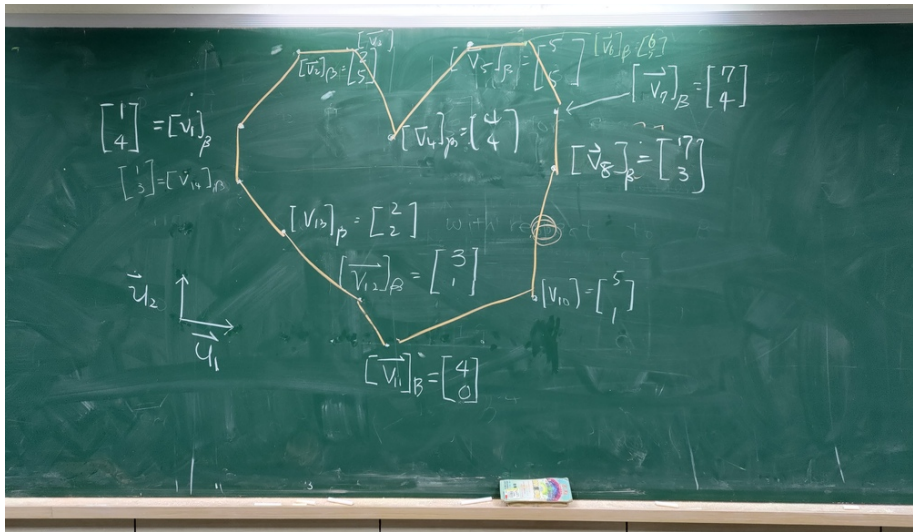


Feeling the vector representation

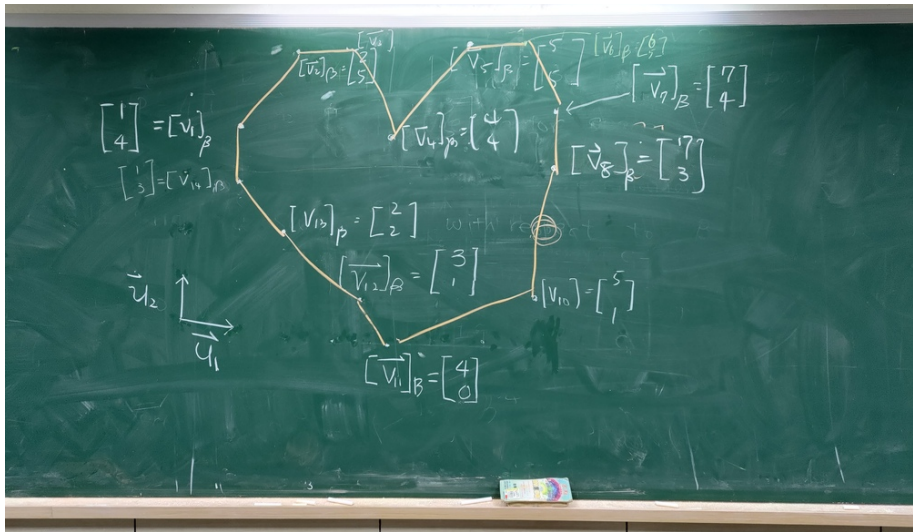


Feeling the vector representation





Thank you!



Thank you!