Variational Formulation of BVP

Consider functionals $f: V \to R$

here V is some set of continuous real function

define on a closed set $([a,b])$.

V: the set of admissible function of f

$S(u;\varepsilon) = \{v \in V \mid \|u - v\| < \varepsilon\}$, here $\|\cdot\|$ is the $L_2 - norm$

$V = \{\eta \mid \eta = \underset{(\text{u fixed})}{u - v},\ u, v \in V\}$ is called the set of test functions

Notice that (1) if V is a linear space then V is also linear.

(2) $S(u;\varepsilon)$ can be rewritten as

$$S(u;\varepsilon) = \{v \in V \mid v = u + \tau\eta,\ \eta \in \widetilde{V,\ \|}\eta\| = 1,\ \tau \in [0,\varepsilon)\}$$

Definition 1. Let f be defined on V. Then $u \in V$ is a (strong) local minimizer of f if $\exists \varepsilon > 0$

$$f(u) \underset{(<)}{\leq} f(u) \text{ for all } \underset{(u \neq u)}{u \in S(u;\varepsilon)}$$

Definition 2. Let f be defined on V. Then $u \in V$ is a (strong) global minimizer of f if

$$f(u) \underset{(<)}{\leq} f(u) \quad \text{for all } \underset{(u \neq u)}{u \in V}$$

Definition 3. Let f be defined on V. Let $u \in V$ and $\eta \in V$ be given where $\|\eta\| = 1$. The mth order directional derivative of f at u in the direction η is defined as

$$f^{(m)}(u;\eta) = \left. \frac{d^m f(u + \tau\eta)}{dt^m} \right|_{\tau = 0}$$

Definition 4. Let f be defined on V.

Suppose $f^{(1)}(u;\eta) = 0$ for all $\eta \in V$, $\|\eta\| = 1$

u is called a stationary point of f.

Theorem 1. Suppose $f^{(1)}\left(u;\eta\right)$ exists for all η at some $u \in V$.

If u is a local minimizer then u is stationary.

pf: $\dfrac{f\left(u+\tau\eta\right)-f\left(u\right)}{\tau} = f^{(1)}\left(u;\eta\right)+\mathrm{O}\left(\tau\right)$ for all $\eta \in V$, $\left\|\eta\right\|=1$

Suppose u is not stationary $\Rightarrow f^{(1)}\left(u;\eta\right)\neq 0$ for some $\eta \in \widehat{V}$

$\Rightarrow f\left(\widehat{u+\tau\eta}\right)-f\left(\widehat{u}\right)=\tau\left(f^{(1)}\left(\widehat{u;\eta}\right)+\mathrm{O}\left(\eta\right)\right)$

$\Rightarrow \exists\,\tau^{*}$ small enough such that

$sign\left(f^{(1)}\left(\widehat{u;\eta}\right)+\mathrm{O}\left(\tau\right)\right)=sign\left(f^{(1)}\left(\widehat{u;\eta}\right)\right)$ for $\left|\tau\right|<\left|\tau^{*}\right|$.

Let's choose τ with opposite sign of $f^{(1)}\left(\widehat{u;\eta}\right)$

Clearly, $f\left(\widehat{u+\tau\eta}\right)-f\left(\widehat{u}\right)<0$

$\Rightarrow f\left(\widehat{u+\tau\eta}\right)<f\left(\widehat{u}\right)$

$\Rightarrow f\left(\widehat{u}\right)$ is not a local minimizer $\rightarrow\leftarrow$.

Therefor, $\widehat{u}$ is stationary.

Theorem 2. Let $u \in V$ be a stationary point of f.

Suppose $f^{(2)}\left(u;\eta\right)$ exist for all direction η. If

u is a local minimizer, then $f^{(2)}\left(u;\eta\right)\geq 0$ for all

directions η.

pf: Consider the taylor formula of f.

$$f\left(u+\tau\eta\right)=f\left(u\right)+\tau f^{(1)}\left(u;\eta\right)+\frac{1}{2}\tau^{2}f^{(2)}\left(u;\eta\right)+\mathrm{O}\left(\tau^{3}\right)$$

$$=f\left(u\right)+\frac{1}{2}\tau^{2}f^{(2)}\left(u;\eta\right)$$

u is a local minimizer.

$$\Rightarrow f\left(u+\tau\eta\right)-f\left(u\right)=\frac{1}{2}\tau^{2}f^{(2)}\left(u;\eta\right)+\mathrm{O}\left(\tau^{3}\right)>0$$

$\Rightarrow$ For τ small enough, one has $\mathrm{O}\left(\tau^{3}\right)<\frac{1}{2}\left(f\left(u+\tau\eta\right)-f\left(u\right)\right)$

$\Rightarrow f^{(2)}\left(u;\eta\right)>0$

Theorem 3. Let f be a guadratic functional defined on V

$$\begin{pmatrix} f(u+\tau\eta) = f(u) + \tau f^{(1)}(u;\eta) + \frac{1}{2}\tau^2 f^{(2)}(u;\eta) \quad -(*) \\ \text{for all } \eta \in V, \ \|\eta\| = 1 \text{ and } \tau \in R \end{pmatrix}$$

Then $u \in V$ is the unigue strong local and strong global minimizer if

$$\begin{cases} f^{(1)}\left(\widehat{u;\eta}\right) = 0 \\ f^{(2)}\left(\widehat{u;\eta}\right) > 0 \end{cases} \quad \text{for all } \eta \in V \text{ and } \|\eta\| = 1.$$

(trival)

Remark: differentiate f with respect to τ, we have

$$\begin{cases} f^{(1)}(u;\eta) = \frac{d}{d\tau} f(u+\tau\eta)\Big|_{\tau=0} \\ f^{(2)}(u;\eta) = \frac{d^2}{d\tau^2} f(u+\tau\eta)\Big|_{\tau=0} \end{cases}$$

Euler-Lagrange Equation:

(1-D problem)

Consider $F(x,r,s)$ be a real function defined on $a \le x \le b, \ -\infty < r, s < \infty$

let $V = \left\{ v \in C^2[a,b] \middle\| v(a) = \alpha, \ v(b) = \beta \right\}$

$$f(u) = \int_a^b F\left(x, u(x), u^{'}(x)\right) dx, \ u \in V$$

One has

$$\widetilde{V = \{} v \in C^2[a,b] \| \eta(a) = \eta(b) = 0 \}$$

and

$$f(u+\tau\eta) = \int_a^b F\left(x, u+\tau\eta, u^{'} + \tau\eta^{'}\right) dx$$

Now differentiate $f\left(u+\tau\eta\right)$ with respect to τ

we have

$$f^{(1)}\left(u;\eta\right)=\frac{d}{d\tau}\int_a^b F\left(x,u+\tau\eta,u^{'}+\tau\eta^{'}\right)dx\Big|_{\tau=0}$$

$$=\int_a^b \frac{\partial F}{\partial u}\cdot\eta+\frac{\partial F}{\partial u^{'}}\cdot\eta^{'}dx$$

$\Rightarrow u$ is a stationary point of $f \Leftrightarrow \int_a^b \frac{\partial F}{\partial u}\cdot\eta+\frac{\partial F}{\partial u^{'}}\cdot\eta^{'}dx=0 \quad -\left(**\right)$

$$\underset{\text{by part}}{\overset{\text{integration}}{\Rightarrow}}\int_a^b \frac{\partial F}{\partial u}\cdot\eta-\frac{d}{dx}\left(\frac{\partial F}{\partial u^{'}}\right)\cdot\eta dx+\left(\frac{\partial F}{\partial u^{'}}\right)\eta\Big|_a^b=0$$

$$\overset{\eta(a)=\eta(b)=0}{\Rightarrow}\int_a^b\left(\frac{\partial F}{\partial u}-\frac{d}{dx}\left(\frac{\partial F}{\partial u^{'}}\right)\right)\eta dx=0 \quad \text{for all } \eta\in\widetilde{V}$$

$$\Rightarrow\frac{\partial F}{\partial u}-\frac{d}{dx}\left(\frac{\partial F}{\partial u^{'}}\right)=0 \quad a<x<b \quad -\left(***\right)$$

The above equation $\left(***\right)$ is called the Euler-Lagrange equation.

The formulation $\left(**\right)$ is called the variational formulation of $\left(***\right)$.

Ex1. Let $V=\left\{v\in C^2\left[0,1\right]\middle| v\left(0\right)=0,\ v\left(1\right)=1\right\}$

and $F\left(x,u,u^{'}\right)=\frac{1}{2}\left(u^{'}\right)^2-u,\ r\left(x\right)\in C\left[0,1\right]$

Find the Euler-Lagrange equation and the associated variational formulation.

Ans: Euler-Lagrange equation:

$$\frac{\partial F}{\partial u}-\frac{d}{dx}\left(\frac{\partial F}{\partial u^{'}}\right)=0 \Rightarrow -r\left(x\right)-\frac{d}{dx}\left(u^{'}\right)=0$$

$$\Rightarrow u^{''}\left(x\right)+r\left(x\right)=0$$

variational formulation:

$$\int_a^b\left(u^{''}\left(x\right)+r\left(x\right)\right)\eta\left(x\right)dx \underset{\substack{\text{integration by parts by}\\ \eta\in\widetilde{V}=\left\{v\in C^2\left[0,1\right]\middle| v\left(0\right)=v\left(1\right)=0\right\}}}{=} \int_a^b u^{'}\left(x\right)\eta^{'}\left(x\right)+r\left(x\right)\eta\left(x\right)dx=0$$

Remark 1.

Notice that $f\left(u+\tau\eta\right)=\int_0^1\frac{1}{2}\left(u^{'}+\tau\eta\right)^2-r\left(u+\tau\eta\right)dx$

$$=\int_0^1\left(\frac{1}{2}u^{'2}-ru\right)dx+\tau\int_0^1 u^{'}\eta^{'}-r\eta dx+\frac{1}{2}\tau^2\int_0^1\left(\eta^{'}\right)^2 dx$$

and
$$\begin{cases} f\left(u\right)=\int_a^b F\left(x,u,u^{'}\right)dx=\int_a^b\frac{1}{2}\left(u^{'}\left(x\right)\right)^2-r\left(x\right)u\left(x\right)dx \\ f^{(1)}\left(u;\eta\right)=\int_a^b u^{'}\left(x\right)\eta^{'}\left(x\right)+r\left(x\right)\eta\left(x\right)dx \\ f^{(2)}\left(u;\eta\right)=\frac{d^2}{d\tau^2}f\left(u+\tau\eta\right)=\int_a^b\frac{d^2}{d\tau^2}\left[\frac{1}{2}\left(\left(u+\tau\eta\right)^{'}\right)^2-r\left(x\right)\left(u+\tau\eta\right)\right]dx \\ \qquad =\int_a^b\frac{d}{d\tau}\left[\left(u+\tau\eta\right)^{'}\cdot\mu^{'}-r\left(x\right)\eta\right]dx=\int_a^b\left(\eta^{'}\right)^2 dx>0 \end{cases}$$

$\Rightarrow f$ is a quadratic functional.

By Theorem 3, the solution of $\begin{cases} u^{''}\left(x\right)+r\left(x\right)=0 \\ u\left(0\right)=0;\ u\left(1\right)=1 \end{cases}$

is the strong global minimizer of F.

Exercise: Find the Euler-Lagrange equation of the functional

$$f\left(u\right)=\int_a^b\frac{1}{2}\left(P\left(x\right)\left(u^{'}\right)^2+q\left(x\right)u^2\right)-g\left(x\right)udx$$

here $u\in V=\left\{v\in C^2\left[a,b\right]\middle| v\left(a\right)=v\left(b\right)=0\right\}$.

(2-D problems)

Exercise: $F\left(x,y,u^{'},u^{''}\right)=\frac{1}{2}\left(u^{''}\right)^2$. Find the Euler-Lagrange equation.

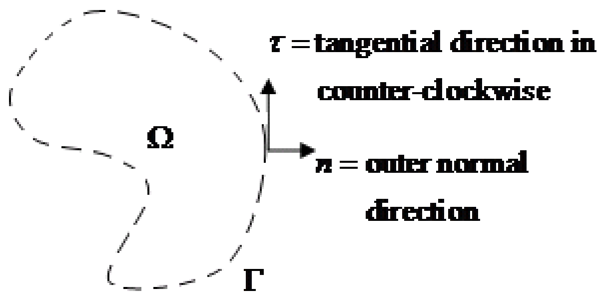
What boundary condition shall be imposed?

Consider Ω: open bounded simply connected

Γ: smooth except at finite number of corners (Lipschtz continuous)

$\vec{n}=\left(u_1,u_2\right);\ \vec{\tau}=\left(-u_2,u_1\right)$

$$\begin{aligned} u_n&=\vec{n}\cdot\nabla u \\ u_\tau&=\vec{\tau}\cdot\nabla u \end{aligned}\qquad \text{here } \nabla u=\left(u_x,u_y\right)^T=\begin{pmatrix}\frac{\partial u}{\partial x}\\ \frac{\partial u}{\partial y}\end{pmatrix},\ \nabla=\begin{pmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\end{pmatrix}$$



Some notations and identies

$$\mathrm{div}(u) = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} \quad \text{here } u = (u_1, u_2)^T, \ \mathrm{div} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$$

$$\Delta u = \mathrm{div}(\nabla u) = \mathrm{div}\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)^T = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$\nabla^2 u = \nabla^T (\nabla u) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \cdot \left[\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} \cdot u\right] = \frac{\partial^2}{\partial x^2} u + \frac{\partial^2}{\partial y^2} u$$

Gauss integral identies

$$\iint_\Omega u v_x \, dxdy = \int_\Gamma n_1 u v ds - \iint_\Omega u_x v dxdy$$

$$\iint_\Omega u v_y \, dxdy = \int_\Gamma n_2 u v ds - \iint_\Omega u_y v dxdy$$

$$\left(\begin{array}{l} \text{divergence theorem} \\ \iint_\Omega \mathrm{div}(u) \, dA = \int_\Gamma \vec{n} \cdot u ds \end{array}\right)$$

Integration by parts

$$\begin{aligned} \iint_\Omega \mathrm{div}(u) v dA &= \iint_\Omega \frac{\partial u_1}{\partial x} v dA + \iint_\Omega \frac{\partial u_2}{\partial y} v dA \\ &= \int_\Gamma n_1 u_1 v ds - \iint_\Omega u_1 \frac{\partial V}{\partial x} dA + \int_\Gamma n_2 u_2 v ds - \iint_\Omega u_2 \frac{\partial V}{\partial y} dA \\ &= \int_\Gamma (\vec{n} \cdot u) v ds - \iint_\Omega u^T \cdot \nabla v dA \end{aligned}$$

Let's consider $F(x,y,r,s,t)$ be a function defined on $(x,y)\in\Omega$ and $-\infty<r,s,t<\infty$.

Let $V=\left\{v\in C^2\left(\overline{\Omega}\right)\middle| v=\alpha \text{ on } \Gamma\right\}$ and

$$f(u)=\iint_\Omega F\left(x,y,u,u_x,u_y\right)dxdy,\ u\in V$$

Clearly, one has

$$V=\left\{v\in C^2\left(\overline{\Omega}\right)\middle| v=0 \text{ on } \Gamma\right\} \text{ and}$$

$$f(u+\tau\eta)=\iint_\Omega F\left(x,y,u+\tau\eta,u_x+\tau\eta_x,u_y+\tau\eta_y\right)dA$$

Now, differentiate $f(u+\tau\eta)$ with respect to τ.

We have $f^{(1)}(u;\eta)=\iint_\Omega \frac{d}{d\tau}F\left(x,y,u+\tau\eta,u_x+\tau\eta_x,u_y+\tau\eta_y\right)dA$

$$=\iint_\Omega \frac{\partial F}{\partial u}\cdot\eta+\frac{\partial F}{\partial u_x}\cdot\eta_x+\frac{\partial F}{\partial u_y}\cdot\eta_y dA$$

Hence, u is a stationary point

$$\Leftrightarrow \iint_\Omega \frac{\partial F}{\partial u}\eta+\frac{\partial F}{\partial u_x}\cdot\eta_x+\frac{\partial F}{\partial u_y}\cdot\eta_y=0 \quad -(++)$$

using integration by parts, $(++)$ implies

$$\iint_\Omega \frac{\partial F}{\partial u}\eta dA+\iint_\Omega\left(\frac{\partial F}{\partial u_x},\frac{\partial F}{\partial u_y}\right)^T\cdot\nabla\eta dA$$

$$=\iint_\Omega \frac{\partial F}{\partial u}\eta dA-\iint_\Omega \operatorname{div}\left(\left(\frac{\partial F}{\partial u_x},\frac{\partial F}{\partial u_y}\right)^T\right)\cdot\eta dA+\int_\Gamma \vec{n}\cdot\left(\frac{\partial F}{\partial u_x},\frac{\partial F}{\partial u_y}\right)^T\cdot\eta dA \quad -(+)$$

$$=\iint_\Omega\left[\frac{\partial F}{\partial u}-\operatorname{div}\left(\left(\frac{\partial F}{\partial u_x},\frac{\partial F}{\partial u_y}\right)^T\right)\right]\eta dA=0 \text{ for all } \eta\in \widetilde{V}$$

$\Rightarrow$ The Euler-Lagrange equation is

$$\frac{\partial F}{\partial u}-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)=0 \text{ in } \Omega \quad -(+++)$$

The formulation $(++)$ is called the variational formulation of $(+++)$

Ex2. Let $F\left(x,y,u,u_x,u_y\right)=\frac{1}{2}p\left(x,y\right)\left(u_x^2,u_y^2\right)+\frac{1}{2}q\left(x,y\right)u^2-r\left(x,y\right)u$

and $f\left(u\right)=\iint\limits_{\Omega}F\left(x,y,u,u_x,u_y\right)dA,$

here $u\in V=\left\{v\in C^2\left(\overline{\Omega}\right)\middle|\, v=\alpha \text{ on } \Gamma \text{ and } p>0\ q\geq 0\right\}$

Find the Euler-Lagrange equation and the associated variational formulation.

Remark 2.

If the admissible set is simply $C^2\left(\overline{\Omega}\right)$ without given boundary data $\left(v=\alpha \text{ on } \Gamma\right)$, the natural boundary condition for the Euler-Lagrange equation is $\vec{n}\cdot\left(\frac{\partial F}{\partial u_x},\frac{\partial F}{\partial u_y}\right)=0$ from $(+)$

In example 2, the natural boundary condition is

$$\vec{n}\cdot\left(p\left(x,y\right)u_x,\,p\left(x,u\right)u_y\right)=0$$

$\equiv p\left(x,y\right)\cdot\vec{n}\cdot\nabla u=0$ $\equiv$ the Neumann boundary condition.

Note: boundary of Cartilevel beam $\begin{cases} w\left(0\right)=\theta\left(0\right)=0 \\ w'\left(i\right)=\theta'\left(i\right)=0 \end{cases}$

Ans: By $(+++)$, the Euler-Lagrange equation is

$$\frac{\partial F}{\partial u}-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)=q\left(x,y\right)u-r\left(x,y\right)-\frac{\partial}{\partial x}p\left(x,y\right)u_x$$

$$-\frac{\partial}{\partial y}p\left(x,y\right)u_y=0$$

$$\Rightarrow\begin{cases} -\text{div}\left(p\left(x,y\right)\nabla u\right)+q\left(x,y\right)u-r\left(x,y\right)=0 \text{ in } \Omega \\ u\big|_{\Gamma}=\alpha \end{cases}$$

and the variational formulation is

$$\iint\limits_{\Omega}\left(-\text{div}\left(p\nabla u\right)+qu-r\right)\cdot\eta dA=0$$

$$\underset{\text{integration by parts}}{\Rightarrow}\iint\limits_{\Omega}p\nabla u\cdot\nabla\eta dA+\iint\limits_{\Omega}qu\cdot\eta dA+\iint\limits_{\Omega}r\eta dA=0$$

Remark 3.

In example 2, we have

$$f^{(1)}(u;\eta)=\frac{d}{d\tau}\left[\frac{1}{2}p\left((u+\tau\eta)_x^2+(u+\tau\eta)_y^2\right)+\frac{1}{2}q(u+\tau\eta)^2+r(u+\tau\eta)\right]\Bigg|_{\tau=0}$$

$$=p\left[(u+\tau\eta)_x\cdot\eta_x+(u+\tau\eta)_y\cdot\eta_y\right]+q(u+\tau\eta)\cdot\eta+r\eta\Big|_{\tau=0}$$

$$=p\left[u_x\eta_x+u_y\eta_y\right]+qu\eta+r\eta$$

$$f^{(2)}(u;\mu)=\frac{d}{d\tau}\left(p\left[(u+\tau\eta)_x\eta_x+(u+\tau\eta)_y\eta_y\right]+q(u+\tau\eta)\eta+r\mu\right)\Big|_{\tau=0}$$

$$=p\left(\eta_x\eta_x+\eta_y\eta_y\right)+q\eta^2>0$$

Clearly, we have

$$f(u+\tau\eta)=\frac{1}{2}p\left[(u+\tau\eta)_x^2+(u+\tau\eta)_y^2\right]+\frac{1}{2}q(u+\tau\eta)^2+r(u+\tau\eta)$$

$$=\frac{1}{2}p\left[u_x^2+u_y^2\right]+\frac{1}{2}qu+ru+\tau\left(p\left[u_x\eta_x+u_y\eta_y\right]+qu\eta+r\eta\right)+$$

$$\frac{1}{2}\tau^2\left(p\left[\eta_x^2+\eta_y^2\right]+q\eta^2\right)$$

$$=f(u)+\tau f^{(1)}(u;\eta)+\frac{1}{2}\tau^2 f^{(2)}(u;\eta)$$

Hence, f is a quadratic functional.

By theorem 3, the solution of the Euler-Lagrange equation is a strong global minimizer of the functional f.

Remark 4. The solution of the variational formulation is called the weak solution of the associated Euler-Lagrange equation.

Poincare'-Friedrich inequality: $\int w^2\le\int_0^1\left(\frac{dw}{dx}\right)^2$

$$w(x)=w(0)+\int_0^x\frac{dw}{dx}(\varsigma)\,d\varsigma\quad(\text{let } w(0)=0)$$

$$w^2(x)=\left|\int_0^x\frac{dw}{dx}\right|^2\le\int_0^x 1^2\cdot\int_0^x\left(\frac{dw}{dx}\right)^2\le\int_0^x\left(\frac{dw}{dx}\right)^2$$

$$\Rightarrow\int_0^1 w^2(x)\,dx\le\int_0^1\left(\int_0^x\left(\frac{dw}{dx}\right)^2dx\right)=\int_0^x\left(\frac{dw}{dx}\right)^2dx$$