In taking derivatives for the functional $f(u)$ on an admissible space V, there is a basic question to be asked.

(1) Given a sequence of functions $\{u_m\}$, does the sequence converge to a function u in V if given any $\varepsilon > 0$ $\|u_n - u_m\| < \varepsilon$ for all n, m large enough?

Is the space V complete under the given norm $\|\cdot\|_v$?

(2) Whar a good norm $\|\cdot\|_v$ is?

Let's consider the functional $f(u) = \frac{1}{2} a(u,u) - G(u)$

here $a(u,v) = \int_a^b p(x) u'v' + g(x) uv dx$

$$G(u) = \int_a^b g(x) u dx,\ p_1 > p > p_0 > 0,\ q_1 > q \geq 0$$

and $u \in V = \{v \in C^2[a,b] \| v(a) = v(b) = 0\}$, also let's consider the energy norm $\|u\|_a = a(u,u)^{\frac{1}{2}}$ in V.

We want to have the following properties

(i) Given any cauchy sequence $\{u_n\}$. $u_n \underset{\|\cdot\|_a}{\rightarrow} u \in V$

(ii) $u, u' \in L^2$

Assume $\{u_n\} \subset V$ and $\lim_{n,m\to\infty} \|u_n - u_m\|_a = 0$

By the Poincar'e-Friedrich inequality, $(v', v') \geq \frac{2}{(b-a)^2}(v,v)$

$$\|v\|_a^2 = (pv', v') + q(v,v) > p_0(v',v') \geq \frac{p_0}{2}(v',v') + \frac{p_0}{(b-a)^2}(v,v)$$

$$\geq p\left(\|v'\|^2 + \|v\|^2\right) \quad \forall u \in V$$

Let $v = u_n - u_m$. One has $\|u_n - u_m\|_a^2 \geq p\left(\left\|u_n' - u_m'\right\|^2 + \|u_n - u_m\|^2\right)$

Hence $\|u_n - u_m\|_a \to 0$ impies $\left\|u_n' - u_m'\right\| \to 0$ $\|u_n - u_m\| \to 0$

By the completeness of L^2 space, there exists u and $u \in L^2$

such that $u_n \underset{L^2}{\rightarrow} u$ and $u_n' \underset{L_2}{\rightarrow} u$

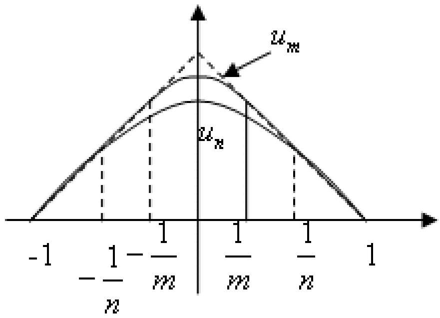
Now the questions here to be asked become

(1) does $u \in V$

(2) does $\widetilde{u = u'}$ or does u' not even exist?

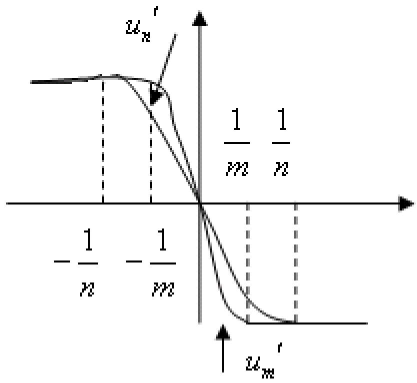
To answer the above questions, we give the following example.

consider $u_n(x) = \begin{cases} 1 - \dfrac{1}{n} + \dfrac{2}{n\pi}\cos\dfrac{n\pi x}{2} & \text{for } |x| < \dfrac{1}{n} \\ 1 - |x| & \text{for } \dfrac{1}{n} \le x \le 1 \end{cases}$



$$u_n'(x) = \begin{cases} 1 & -1 \le x \le -\dfrac{1}{n} \\ -\sin\dfrac{n\pi x}{2} & -\dfrac{1}{n} < x < \dfrac{1}{n} \\ -1 & \dfrac{1}{n} \le x \le 1 \end{cases}$$

$u_n \in C' = V$



we have

$$\|u_m - u_n\|_a^2 = \int_{-1}^{1} p\left(u_m{}' - u_n{}'\right)^2 dx + \int_{-1}^{1} q\left(u_m - u_n\right)^2 dx$$

$$\le p_1 \int_{-1}^{1}\left(u_m{}' - u_n{}'\right)^2 dx + q_1 \int_{-1}^{1}\left(u_m - u_n\right)^2 dx$$

where $\left|u_m{}' - u_n{}'\right| \le \begin{cases} 1 & \text{for } |x| < \dfrac{1}{n} \\ 0 & \text{otherwise} \end{cases}$; $\quad \left|u_m - u_n\right| \le \begin{cases} \dfrac{2}{n} & \text{for } |x| < \dfrac{1}{n} \\ 0 & \text{otherwise} \end{cases}$

$$\Rightarrow \|u_m - u_n\|_a^2 \le \frac{2p_1}{n} + \frac{4q_1}{n^3}$$

$$\Rightarrow \lim_{m,n\to\infty} \|u_m - u_n\|_a = 0$$

However, one has

$$\begin{cases} \lim\limits_{n\to\infty} u_n = 1 - |x| \\ \lim\limits_{n\to\infty} u_n{}' = \begin{cases} 1 & -1 \le x < 0 \\ 0 & x = 0 \\ -1 & 0 < x \le 1 \end{cases} \end{cases}$$

Clearly $\begin{cases} u = \lim\limits_{n\to\infty} u_n \\ u = \lim\limits_{n\to\infty} u_n{}' \end{cases} \notin \mathrm{C}'$

Therefore, the answer of (1) is "No"

(2) Since $u = 1 - |x|$, clearly u is not differentiable in the classical sense. However, u is "differentiable" in the weak sense. Indeed, one can show

$$\int_{-1}^{1} u'\phi dx = -\int_{-1}^{1} u\phi' dx = -\int_{-1}^{1} \left(1 - |x|\right)\phi' dx$$

$$= -\int_{-1}^{0} (1+x)\phi' dx - \int_{0}^{1} (1-x)\phi' dx$$

$$= \int_{-1}^{0} \phi dx + (1+x)\phi\Big|_{-1}^{0} - \int_{0}^{1} \phi dx - (1-x)\phi\Big|_{0}^{1}$$

$$= \int_{-1}^{0} \phi dx - \int_{0}^{1} \phi dx$$

$$= \int_{-1}^{1} \widetilde{u(x)}\phi(x) dx \qquad \text{for all } \phi \in V$$

Therefore, $u'(x) = \widetilde{u(x)}$ when the derivative is considered in the weak sense.

From the above observations, we need to enlarge the admissible space V' in order to find the minimizer of the functional $f(u)$. Moreover, the concept of generalized$($weak$)$ derivative is necessary in the new admissible space.

Sobolev Space:

Review of Lebesgue lntegration theorey:

Lebesgue space:

$$L^p(\Omega)=\left\{f \mid \|f\|_{L^p(\Omega)}<\infty\right\} \quad 1\le p\le\infty$$

here $\|f\|_{L^p(\Omega)}=\left(\int_\Omega f^p dx\right)^{\frac{1}{p}}$, $p<\infty$ $\left(\text{denote } \|f\|_{L^p}=\|f\|_p\right)$

Remark:

(1) In Lebesgue theory, dx denotes a "measure" and $f=g$ is $f(x)=g(x)$ for all x in Ω except on a measure zero set.

Example: $f(x)=\begin{cases}1 & x\ge 0\\ 0 & x<0\end{cases}$, $\quad g(x)=\begin{cases}1 & x>0\\ 0 & x\le 0\end{cases}$

$\Rightarrow$ $f(x)=g(x)$ except at $x=0$ since the length of a point $x=0$ is zero, we simply say $f(x)=g(x)$

f and g are also called "in equivalent class".

(1) For the case $p=\infty$,

$$\|f\|_{L^\infty(\Omega)}=\underset{(\text{essential sup})}{\text{ess sup}}\left\{|f(x)|:x\in\Omega\right\}$$

$$=\inf_{\text{minimum}}\left\{a\in R \,\middle|\, \underbrace{\mu\left(\underbrace{\{x \mid f(x)>a\}}_{\text{a is a upperbound of } f}\right)=0}_{\text{最小上界 except a measure zero set.}}\right\}$$

Example: $f(x)=\begin{cases}x^3 & x\in Q\\ \tan^{-1}x & x\in R\setminus Q\end{cases}$

In general sense, $f(x)$ is unbounded,

but $\|f\|_{L^\infty}=1 \Rightarrow f(x)\in L^\infty(\Omega)$.

(3) f is called locally integrable if $\int_K f \ dx < \infty$ for any compact set $K \subset \Omega$

we denote $f \in L_{loc}{}' (\Omega)$.

•Well known inequalities

Minkowski's inequality: $1 \le p \le \infty,\ f \cdot g \in L^p$

$\| f + g \|_{L^p} \le \| f \|_{L^p} + \| g \|_{L^p}$

Holder inequality: If $f \in L^p,\ g \in L^q \quad \dfrac{1}{p} + \dfrac{1}{q} = 1$

then $f \cdot g \in L'(\Omega)$ and

$$\| f \cdot g \|_{L'(\Omega)} \le \| f \|_{L^p(\Omega)} \| g \|_{L^q(\Omega)}$$

Schwarz's inequality: $\quad p = 2,\ q = 2$

$$\int_\Omega | f(x) g(x) | dx \le \| f \|_{L^2(\Omega)} \| g \|_{L^2(\Omega)}$$

Normed space: $(V, \| \cdot \|)$ here V is a linear space and $\| \cdot \|$ is a norm satisfies

(i) $\|v\| \ge 0 \quad \forall v \in V$ & $\|v\| = 0 \Leftrightarrow v = 0$ (in Lebesgue sense)

(ii) $\|c \cdot v\| = |c| \cdot \|v\|,\ c \in R,\ v \in V$

(iii) $\|v + w\| \le \|v\| + \|w\| \quad \forall v, w \in V$

A metric d is defined by $d(v, w) = \|v - w\|$ for any $v, w \in V$

A normed linear space V called complete if every cauchy sequence $\{v_j\}$ has a limit $v \in V$ here cauchy sequence is defined by $d(v_i, v_j) \to 0$ as $i, j \to \infty$.

(The complete normed linear space is called a Banach space.)

Theorem 4. $L^p(\Omega)$ is a Banach space, $1 \le p \le \infty$

$\left(L^1(\Omega) \supset L^2(\Omega) \supset \cdots\cdots \supset L^\infty(\Omega) \right)$

Def: A function $f \in L^1_{loc}(\Omega)$ has a weak derivative $D^\partial_w f$

if $\exists$ a function $g \in L^1_{loc}(\Omega)$ such that

$$\int_\Omega g(x) \phi(x) dx = (-1)^{|\partial|} \int_\Omega f(x) \phi^{(\partial)}(x) dx \quad \text{for all } \phi \in C_0^\infty(\Omega)$$

here $C_0^\infty(\Omega)$ denote the set of C^∞ function with compact support in Ω.

Notation: $v^{(\partial)} = D^{(\partial)}v$ and $v^{(\partial)} \in L^2(\Omega)$ meaning $g(x) \in L^2(\Omega)$

in the above theorem.

Let $\partial_1, \partial_2, \cdots\cdots, \partial_n$ be nonnegative integers

$$\partial = (\partial_1, \partial_2, \cdots\cdots, \partial_n),\ |\partial| = \sum_i \partial_i,\ D^{\partial} = \frac{\partial^{|\partial|}}{\partial x_1^{\partial_1} \partial x_2^{\partial_2} \cdots \partial x_n^{\partial^n}} \quad u^{(\partial)} = D^{\partial} u$$

and $\tilde{c}^k(\Omega) = \left\{ v \in C^k(\Omega) \middle| v^{(\partial)} \in L^2(\Omega),\ 0 \le |\partial| \le k \right\}$

The sobolev inner product on $\tilde{c}^k(\Omega)$ is

$$(u,v)_k = \sum_{0 \le |\partial| \le k} \left(u^{(\partial)}, v^{(\partial)}\right), \text{ where } (u,v) = \int_{\Omega} u \cdot v dx$$

and the sobolev norm

$$\|u\|_{k,2} = \left(\sum_{o \le |\partial| \le k} \left\| u^{(\partial)} \right\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}$$

In 2-D

$$\|u\|^2_{2,2} = \int_{\Omega} u^2 + u_x^2 + u_y^2 + u_{xx}^2 + u_{xy}^2 + u_{yy}^2 dxdy$$

The sobolev space $H_2^k(\Omega)$ is the completion of $\tilde{c}^k(\Omega)$ with respect to $\| \cdot \|_{k,2}$ or

$$H_2^k(\Omega) = \left\{ u \in L^2(\Omega) \middle| u^{(\partial)} \text{ exists for all } \partial,\ 0 \le |\partial| \le k \right\}$$

$$H_{2,0}^k(\Omega) = \left\{ u \in H^k(\Omega) \middle|\ u|_{\partial\Omega} = 0 \right\}$$

H_k^{∞} is the completion of c^k under H_{∞}^k norm $= \max_{|\partial| \le k} \left\| u^{(\partial)} \right\|_L^{\partial}$

Define H_1^k here, define $\| \cdot \|_{k,p} = \left(\sum_{0 \le |\partial| \le k} \left\| u^{(\partial)} \right\|^p_{L^p} \right)^{\frac{1}{p}}$

It can be shown that $H_p^2(\Omega)$ us a Hilbert space with respect to sobolev inner product $(\ \cdot\ ,\ \cdot\ )_k$ and sobolev norm $\| \cdot \|_k$

(Hilbert space is complete by definition)

Theorem 5. For $k \le m$, For $1 \le p \le q \le \infty$, $\begin{cases} H_p^m(\Omega) \subset H_p^k(\Omega) \\ H_q^k(\Omega) \subset H_p^k(\Omega) \end{cases}$

possible singularity of H^1 function

consider $u(x,y) = \log\left(\log \frac{2}{r}\right)$ $r = \sqrt{x^2+y^2}$ in $D = \left\{(x,y) \middle| x^2+y^2<1\right\}$

$$\int_0^{\frac{1}{2}} \frac{dr}{r\log^2 r} < \infty \quad \Rightarrow u \in H^1$$

$$\left(\begin{array}{l} \int_0^1 vD^{(1)} \log\left(\log \frac{2}{r}\right) dr,\ v \in C_0^\infty(D) \\ = v \log\log \frac{2}{r}\Big|_{\partial D} - \int_0^1 (v') \log\log \frac{2}{r} dr \\ \overset{\tilde{r}=\frac{r}{2}}{\Rightarrow} \text{ consider } \int_0^{\frac{1}{2}} w \log\log \frac{1}{r} d\tilde{r} = \int_0^{\frac{1}{2}} w \log\left(-\log \tilde{r}\right) \cdot \log^2 \tilde{r} \cdot \tilde{r} \cdot \frac{1}{\tilde{r}\log^2 \tilde{r}} d\tilde{r} \end{array} \right)$$