Singularity of H^1 function:

Consider $u(x,y)=\log\left(\log\frac{2}{r}\right)$, $r=\sqrt{x^2+y^2}$, in $D=\{(x,y)|x^2+y^2<1\}$

(1) It is easy to show $\int_0^{\frac{1}{2}}\frac{dr}{r\log^2 r}<\infty$

(2) To show $u\in H^1$:

consider $v\in C_0^\infty(D)$ and $\int_0^1 vD^{(1)}\log\left(\log\frac{2}{r}\right)dr$

$$=-\int_0^1(v')\log\left(\log\frac{2}{r}\right)dr \quad \left(\text{let } \tilde{r}=\frac{r}{2}\ \ w=v'\right)$$

$$=\int_0^{\frac{1}{2}} w\log\log\frac{1}{r}\,d\tilde{r}$$

$$=\int_0^{\frac{1}{2}} \underbrace{w\log\left(-\log\tilde{r}\right)\cdot\log\tilde{r}\cdot\tilde{r}^{\frac{1}{2}}}_{(I)}\cdot\underbrace{\frac{1}{\tilde{r}^{\frac{1}{2}}\log\tilde{r}}}_{(II)}\,d\tilde{r}$$

$$\le\underbrace{\left(\int_0^{\frac{1}{2}}(I)^2\right)^{\frac{1}{2}}}_{(A)}\underbrace{\left(\int_0^{\frac{1}{2}}(II)^2\right)^{\frac{1}{2}}}_{(B)}\quad(\text{By Holder inequality})$$

By (1), we have $(B)<\infty$. We only need to show $A<\infty$.

Since $\lim_{\tilde{r}\to 0}\log\left(-\log\tilde{r}\right)\cdot\log\tilde{r}\cdot\tilde{r}^{\frac{1}{2}}\overset{\text{let } u=-\log\tilde{r}}{=}\lim_{u\to\infty}\log u\cdot(u)\cdot e^{-\frac{u}{2}}=0$

Clearly, $w\cdot\log\left(-\log\tilde{r}\right)\cdot\log\tilde{r}\cdot\tilde{r}=w$ must be continuous and bounded.(suppose bounded by M)

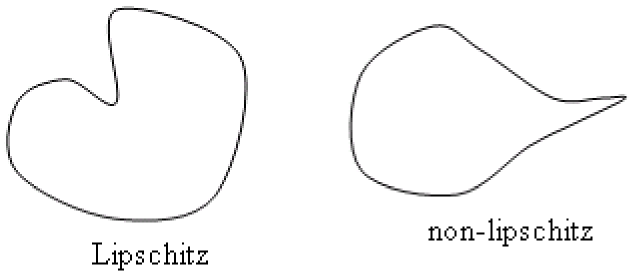
Therefore, (A) is bounded.

Hence, $u(x,y)\in H^1(D)$

Clearly, this example shows that $H^1(\Omega)\not\subset c(\Omega)$

Def: A domain Ω has a Lipschitz boundary if $\partial\Omega$ can be described by the graph of a Lipschitz function locally.

Example:



Theorem 6.

Sobolev's inequality (1)

Let Ω be a n-dimensional domain with Lipschitz boundary

Let k be a positive integer and $1 \le p < \infty$

such that $\begin{cases} k \ge n & \text{when } p = 1 \\ k > \dfrac{n}{p} & \end{cases}$

then there is a comstant c such that $u \in H_p^k$

$$\|u\|_\infty < c\|u\|_{k,p} \quad -(*)$$

$\Rightarrow$ There is a continuous function in $c^0(l)$ equivalent class of u

Something needs to be careful:

(1) it can be shown

if $\Omega = [a,b] \in R^1\ (n=1)$ and $u \in H_2^1$, then u has an equivalent in $c[a,b]$

$n=1,\ p=2$ and $k = 1 > \dfrac{n}{p} = \dfrac{1}{2}$ $\left(\text{from } (*),\text{ we can conclude}\right)$

(2) The previous example, $n=2,\ p=2,\ k=1,$ the assumption $k > \dfrac{n}{p}$ does not hold. Indeed, the example shows $\log\log\dfrac{2}{r} \in H^1$ but is unbounded

$\Rightarrow (*)$ does not hold! $\Rightarrow (*)$ is sharp in this sense.

Theorem 7.

Sobolev inequality (2)

Let Ω be a n-dimensional domain with Lipschtz boundary

Let k, m be positive integer $m < k$ and let $1 \le p < \infty$ such that

$$\begin{cases} k - m \ge n & \text{when } p = 1 \\ k - m > \dfrac{n}{p} & \text{when } p > 1 \end{cases}$$

then $\exists$ a constant c such that $\|u\|_{m,\infty} \le c\|u\|_{k,p} \quad -(**)$

$\Rightarrow$ There is a c^m function class of u in the equivalent.

Example: consider $(n = 2 \text{ or } 3)$, $k = 2, p = 2$ and $m = 0$ $\left(2 - 0 > \dfrac{2}{2}\right)$

The inequality shows $u \in H_2^2(\Omega) \Rightarrow u$ has a equivalent representer.

Compact Imbeddings:

A continuous linear mapping $L : U \to V$

U, V are normed linear space is called compact if the image of unit ball in U is relatively compact in V.

In particular, if $U \subset V$, L is called compact imbedding.

Rellich Selection Theorem:

Given $m > 0$, Ω a lipschitz domain. Then the imbedding $H_{(\Omega)}^{m+1} \to H_{(\Omega)}^{m}$ is compact.

More general case:

$$H_p^k \to H_q^l \quad \text{when} \quad \begin{matrix} k \ge 1 \\ k - \dfrac{n}{p} \ge l - \dfrac{n}{q} \end{matrix}$$

General problems:

(1) $f \in H_p^k$ $\Rightarrow$ how many continuous derivatives f has? (sobolev inequality)

(2) If Ω is sufficiently smooth, is it possible to determine the trace $\phi(x)$ of $f(x)$ at $x \in \partial\Omega = T$?

i.e. the limit value of $f(s)$ as $s \to x$, $s \in \Omega^{\circ}$

(3) What is the differentiability of $\phi(x)$?

Trace Theorem:

example: consider $u \in c'\left(\overline{\Omega}\right),\ \Omega=\left\{(x,y)\middle| x^2+y^2<1\right\}$

$$\Rightarrow \underset{(\text{boundary value})}{u(1,\theta)^2} = \int_0^1 \frac{\partial}{\partial r}\left(r^2 u(r,\theta)^2\right)dr$$

$$= \int_0^1 2\left(r^2 u u_r + r u^2\right)dr$$

$$= 2\int_0^1 r^2 u \nabla u \cdot \frac{(x,y)}{r} + r u^2 dr$$

$$\leq 2\int_0^1 r|u||\nabla u| + r u^2 dr$$

$$\leq 2\int_0^1 \left(|u||\nabla u| + u^2\right) r dr$$

$$\Rightarrow \int_{\partial\Omega} u^2 d\theta \leq 2\int_{\Omega} |u||\nabla u| + u^2 dxdy$$

$$\Rightarrow \|u\|^2_{L^2(\partial\Omega)} \leq 2\|u\|_{L^2(\Omega)}\left(\int_{\Omega}|\nabla u|^2\right)^{\frac{1}{2}} + 2\int_{\Omega} u^2 dxdy$$

$$= 2\|u\|_{L^2(\Omega)}\left(\left(\int(\nabla u)^2\right)^{\frac{1}{2}} + \left(\int u^2\right)^{\frac{1}{2}}\right)$$

using the inqguality $ab < \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2$ for any $\varepsilon > 0$

$$\Rightarrow a+b < \left((1+\varepsilon)a^2 + \left(1+\frac{1}{\varepsilon}\right)b^2\right)^{\frac{1}{2}}$$

$$\Rightarrow \left(\int(\nabla u)^2\right)^{\frac{1}{2}} + \left(\int u^2\right)^{\frac{1}{2}} < \left(2\left(\int|\nabla u|^2 + \int u^2\right)\right)^{\frac{1}{2}}$$

$$\Rightarrow \|u\|^2_{L^2(\partial\Omega)} < \sqrt{8}\left(\|u\|_{L^2(\Omega)}\|u\|_{1,2}\right)$$

$$\Rightarrow \|u\|_{L^2(\partial\Omega)} < \sqrt{8}\left(\|u\|^{\frac{1}{2}}_{L^2(\Omega)}\|u\|^{\frac{1}{2}}_{1,2}\right) \quad -(***)$$

The inequality sugguests

$\Rightarrow$ if the function is in $H_2^1(\Omega)$, it makes sense to restrict $u|_{\partial\Omega}$ and

$u \in L^2(\partial\Omega)$

Proposition:

Let Ω denote the unit disk in R^2. For all $u \in H_2^1(\Omega)$, the restriction $u|_{\partial\Omega} \in L^2(\partial\Omega)$ is called the trace of u and $(***)$ holds.

Remark: Not every element of $L^2(\partial\Omega)$ is the trace of some function in H_2^1.

Consider $u(x,y)=\sum_{k=1}^{\infty} k^{-2} r^{k!}\sin(k!)\varphi,\ \Omega=\{x,y \mid x^2+y^2<1\}$

$$=\{(r,\varphi) \mid 0\le r<1, 0\le\varphi\le 2\pi\}$$

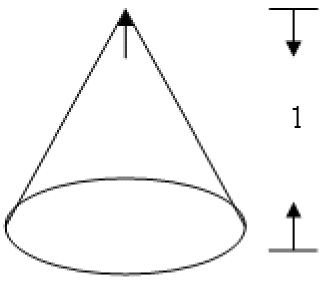
It can be shown (by harmonic analysis) that $\int |\nabla u|^2=\infty^{\notin H^1}$ and there does not exist any function in H^1 that has the same boundary values as u!

Theorem 9.

Trace inequality: Let Ω be Lipschtz domain $1\le p<\infty$. Then there is a constant c such that $\|u\|_{L^p(\partial\Omega)}\le c\|u\|_{L^p(\Omega)}^{1-\frac{1}{p}}\|u\|_{1,p}^{\frac{1}{p}}$ if $u\in H_p^1(\Omega)$

Trace theorem in practice:

Example: Suppose we erect a tent over a disk with radius $R=1$ such that its height $=1$ at the center. Find the shap if the tent.



It is well known the surface area is given by

$$S=\int_{Br}\sqrt{1+|\nabla u|^2}\,dx \quad \left(\sqrt{1+|\nabla u|^2}\approx 1+\frac{1}{2}|\nabla u|^2 \text{ when } |\nabla u|\ll 1\right)$$

To minimize S, we minimize $\int 1+\frac{1}{2}(\nabla u)^2 \equiv$ Consider the variational problem.

$\min_u \frac{1}{2}\int(\nabla u)^2\,dx$

subject to constrains

Case (I) $\begin{cases} u(0)=1 \\ u|_{\partial\Omega}=0 \end{cases}$

$$E.L.eq \Rightarrow \begin{cases} \Delta u+\underbrace{\lambda\delta_e(x)}_{\text{Lagrange mult}}=0 \\ u(0)=1 \\ u|_{\partial\Omega}=0 \end{cases}$$

Consider the singular solution $w_0(x)=\log\log\frac{eR}{r},\ r=|x|,\ R=1$

$$w_0'(r) = -\frac{1}{r(1-\log r)} \qquad w_\varepsilon(x) = \begin{cases} w_0(x) & \text{for } |x| \ge \varepsilon \\ \log\log\frac{eR}{\varepsilon} & \text{for } 0 \le |x| < \varepsilon \end{cases}$$

$$w_0''(r) = -\frac{\log r}{r^2(1-\log r)^2} \qquad \text{clearly } |w_\varepsilon|_{1,B_R} < |w_0|_{1,B_R} \text{ and } w_\varepsilon(0) \to \infty \quad \varepsilon \to 0$$

define $u_\varepsilon = \dfrac{w_\varepsilon(x)}{w_\varepsilon(0)}$ for $\varepsilon = 1, \frac{1}{2}, \frac{1}{3}, \cdots$

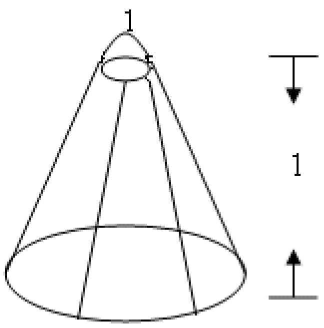
u_ε is a minimizing sequence $\underset{converge}{\rightarrow}$ 0 function

$\Rightarrow$ requirement $u(0) = 1$ is ignored.

Case(II)

The situation is different if we require $u = 1$ on a curve segment

(This is the case if the tent were attached to a ring)



In case (I): the pointwise constrain is meaningless in the weak sense

(i.e. evaluation function value in H^1 does not make sense)

$\Rightarrow (1)$ variational principle simply ignores the constrain and produce

wrong answer $\left(\text{variational principle gives} \begin{cases} \Delta u = 0 \\ u|_{\partial\Omega} = 0 \end{cases} \Rightarrow u = 0 \right)$

(2) minimizing sequence also wrong answer.

In case(II): evaluation of u on boundary segment (i.e. the ring) is possible due to the trace theorem

$\Rightarrow$ non-trivial solution can exist.

To solve this problem, one needs to introdue the constrained variational principle!

Now let's recall the original variational formulation:

V : be some Sobolev space $\left(\text{say } H_2^1\right)$

f : a quadratic functional $\left(f = \int_{\Omega} F\left(x, u, u'\right) dx \right)$

Suppose we want to seek the global minimizer u of f in V under the constrain $u\big|_{\partial\Omega} = \phi$

The admiisible space $V = \left\{ u \in H_2^1\left(\Omega\right) \middle| \; u\big|_{\partial\Omega} = \phi \right\}$

By trace theorem, the admissible is "ok" when $\phi \in L^2\left(\partial\Omega\right)$

But as mentioned before, not every $\phi \in L^2\left(\partial\Omega\right)$

$\exists u \in H_2^1$ such that $u\big|_{\partial\Omega} = \phi$!

$$\Rightarrow \quad \underset{\substack{\downarrow \\ cD^{(\partial)}u + du + \underset{\text{terms nor related to } u, u_x, u_y}{g}}}{-\frac{\partial F}{\partial u}} \quad \underbrace{-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{u_y}\right)}_{\substack{\downarrow (\text{F is quadratic}) \rightarrow \begin{cases} \frac{\partial F}{\partial u_x} \rightarrow \text{linear in } u_x \\ \frac{\partial F}{\partial u_y} \rightarrow \text{linear in } u_y \\ \frac{\partial F}{\partial u} \rightarrow \text{linear in } u \end{cases} \\ div\left(aD^{(\partial)}u + bu\right) + \underbrace{D^{(\partial)} f}_{\text{terms not related to } u, u_x, u_y}}} = 0$$

$\Rightarrow$ Differential equations:

$$L\left(u\right) = g + \underbrace{Df^{\,i}}_{div\left(\vec{f}\right)} \text{ here } L\left(u\right) = D_i\left(a^{ij} D_j u + b^i u\right) + c^i D_i u + du$$

here i, j are the Teusor notation (meaning summing over i & j induces)

So the question now is:

In what conditon of f or (what) f and boundary data, there exist a solution?

$\equiv$ In what condition of EL?

Existence & uniqueness of the solution of EL equation

(remember EL equation is derived from the condition $f' = 0$)

Consider the εL of in ex2 of p.10:

$F\left(x, y, u, u_x, u_y\right)$, ∂: $0 \le \left|\partial\right| \le 1$

Theorem 10. Suppose the operator L satisfies

(1) $a^{ij}\xi_i\xi_j \geq \lambda |J|^2$ (bounded below of the bilinear form $a(\xi_i, \xi_j) = \xi_i a^{ij}\xi_j$)

λ: some constant.

(2) $\sum |a^{ij}(x)|^2 \leq \Lambda^2$

and $\lambda^{-2}\left(\sum\left(|b^i|^2 + |i^i|^2\right) + \lambda^{-1}|d| \leq v^2\right)$

Λ and v are some constants >0 (bounded above of the operator L)

(3) $\int_\Omega dv - b^i D_i v dx \leq 0$ for all $v \geq 0$, $v \in c_0^1(\Omega)$

Then differential equation $L(u) = g + D_i f^i$ in Ω with dirichlet condition $u|_{\partial\Omega} = \phi$ is uniquely solvable.

Homogenization: consider $w = u - \varphi \Rightarrow w|_{\partial\Omega} = 0$

$$\Rightarrow L(w) = L(u) - L(\varphi) = g - i^i D_i\varphi - d\varphi + D_i\left(f^i - \underbrace{a^{ij} D_j\varphi - b^i\kappa}_{\in L^2}\right)$$

$$= \underbrace{g}_{\in L^2} + D_i \underbrace{f^i}_{\in L^2}$$

$\Rightarrow$ Only needs to understand the existence and unigueness of $L(u)$

$= g + D_i f$ wiht constrain $u|_{\partial\Omega} = 0$

$$\left|\int uL(u)\right| = \left|\int a^{ij}D^j u D^i v + \int (D^i u) b^i u + u i^i D^i u + duv\right|$$

$$\geq \left|\int a^{ij} D^j u D^i v\right| - \left|\int D^i v b^i u + v i^i D^i u + duv\right|$$

$$\underset{\left(\substack{\varsigma = |D^i u| \\ \text{Poincare'ineq.}}\right)}{\geq} \lambda|\varsigma|^2 - r\left(b^2 + c^2 + \lambda|d|\right)|\varsigma^2|$$

$$\geq \left(\lambda - r\lambda^2 v^2\right)|\varsigma|^2 > 0 \text{ when } \lambda \text{ small enough}$$

Simple example:

Consider Differential equation

$$\begin{cases} -\sum \dfrac{\partial}{\partial x_i}\left(a_{ij}\dfrac{\partial u}{\partial x_j}\right) + du = f \\ u|_{\partial\Omega} = 0 \end{cases}$$

If $a_{ij}, d \in L^\infty(\Omega)$, $[a_{ij}]$ is positive definite for all x and $f \in L^2(\Omega)$

It can be easily shown (2) and (3) holds when $d > 0$

To prove theorem 10, we need to ues the well-known

Lax-Molgram Theorem (see p.17 between 9.20~9.21)

(See D.Gilbary and N.S.Trudinger, Elliptic Partial differentail Equations of second order. (2*nd* edition) chapter 8.)

In the following, we will introduce some concepts and theorems in functional analysis that are important.