A complete normed linear space V equipped with inner product $(\ ,\ )$ is called a Hilbert space.

Hilbert space

The Sobolev space H_2^k with sobolev inner product $(u,v)_k = \sum_{0 \le |\partial| \le k} \left(u^{(\partial)}, v^{(\partial)}\right)_{L^2}$ is a Hilbert space.

Definition: Let H be a Hilbert space, and $S \subset H$ be a linear subset $(u,v \in S, \partial \in R)$ that is closed in H.

Then S is called a subspace of H.

Proposition: If S is a subspace of H, then $(S,(\ ,\ ))$ is also a Hilbert space.

Example:

(1) Let $T : H \to K$ be a contineous linear mapping.

$(H, K$ are linear space$)$

Then $\ker(T)$ is a subspace.

(2) $x^{\perp} = \{v \in H | (x,v) = 0\}$, $x \in H$ is a subspace.

$M \subset H$ is a subspace $\Rightarrow M^{\perp} = \{v \in H | (x,v) = 0$ for all $v \in M\}$ is also a subspace.

Theorem: Paralleogram law

$$\|v+w\|^2 + \|v-w\|^2 = 2\left(\|v\|^2 + \|w\|^2\right)$$

Proposition:

Let M be a subspace of H. Let $v \in H \setminus M$ and define $\delta = \inf\{\|v-w\| : w \in M\}$ $(\delta > 0)$

Then there exists $w_0 \in M$ such that

(i) $\|v - w_0\| = \delta$

(ii) $v - w_0 \in M^{+}$ (i.e. $(v - w_0, w) = 0\ \forall w \in M$)

Pf. Let $\{w_n\}$ be a minimizing sequence

(i) $\lim\limits_{n\to\infty}\|w_n - v\| = \delta$

Since $\|w_n - w_m\|^2 = \|(w_n - v)-(w_m - v)\|^2$

$\Rightarrow \frac{1}{2}\|w_n - w_m\|^2$

$\|(w_n - v)-(w_m - v)\|^2 + \|(w_n - v)+(w_m - v)\|^2$

$= 2\left(\|w_n - v\|^2 + \|w_m - v\|^2\right)$

$$\Rightarrow \|w_n - w_m\|^2 = 2\left(\|w_n - v\|^2 + \|w_m - v\|^2\right) - \|w_n + w_m - 2v\|^2$$

$$= 2\left(\|w_n - v\|^2 + \|w_m - v\|^2\right) - 4\left\|\frac{w_n + w_m}{2} - v\right\|^2$$

Since $\frac{1}{2}(w_n + w_m) \in H$, $\left\|\frac{w_n + w_m}{2} - v\right\| \geq \delta$

$$\Rightarrow \|w_n - w_m\|^2 \leq 2\left(\|w_n - v\|^2 + \|w_m - v\|^2\right) - 4\delta^2$$

$$\lim_{n,m\to\infty}\|w_n - w_m\|^2 \leq 2\left(\delta^2 + \delta^2\right) - 4\delta^2 = 0$$

Therefore $\{w_n\}$ is a Cauchy sequence.

Since M is closed, $\exists\, w_0 \in M$ such that $w_n \to w_0$ and

$\lim\limits_{n\to\infty}\|w_n - v\| = \|w_0 - v\| = \delta$ $\left(\|\cdot\| \text{ is continueous}\right)$

(ii) Let $z = v - w_0$ and $w \in H$. For $t \in R$, we have $w_0 + tw \in M$

$\Rightarrow \|z - tw\|^2 = \|v - (w_0 + tw)\|^2$ has an absolute minimum at $t = 0$

$$\Rightarrow \underbrace{\frac{d}{dt}\|z - tw\|^2}_{\frac{d}{dt}(z-tw,\,z-tw)}\bigg|_{t=0} = 0 \ \Rightarrow -2(z, w) = 0 \text{ for any } w \in M$$

$\Rightarrow z \in M^{+}$.

Remark:

Given $M \subset H$ and $v \in H$. One can decompose

$v = w_0 + w_1$ here $w_0 \in M$, $w_1 \in M^{\perp}$ "uniquely".

Furthermore, one can define projection operator

$$\begin{matrix} P_M : H \to M \\ P_M^{\perp} : H \to M^{\perp} \end{matrix} \quad \text{by} \quad \begin{matrix} P_M v = \begin{cases} v & \text{if } v \in M \\ w_0 & \text{if } v \in H \setminus M \end{cases} \\ P_M^{\perp} v = \begin{cases} 0 & \text{if } v \in M \\ w_1 & \text{if } v \in H \setminus M \end{cases} \end{matrix}$$

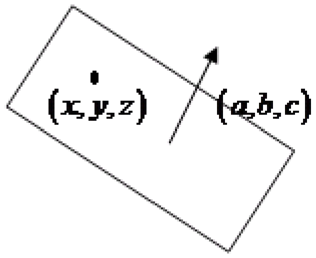
Moreover, uniqueness of the decomposition implies $P_M^{\perp} = P_{M^{\perp}}$

Proposition 2.

Given subspace H of M, there exists unigue decomposition $v = P_M v + P_{M^\perp} v$. In other word, $H = M \oplus M^\perp$.

Riesz-Representation theorem:

Any contineous linear functional L on a Hilbert space H can be represented uniquely as $L(v) = (u, v)$



In R^3, $L(0) = 0$

$L(x, y, z) = ax + by + cz$ linear functional can be represented by $L(x, y, z) = \underbrace{(a, b, c)}_{u} \cdot \underbrace{(x, y, z)}_{v}$

A plane cna be represented by $\{(x, y, z) | L(x, y, z) = 0\} = \ker(L)$

Pf.

(i) uniqueness: suppose $\exists\, u_1 \neq u_2 \in H$ such that

$L(v) = (u_1, v) = (u_2, v)$ for any v

$\Rightarrow (u_1 - u_2, v) = 0$

choose $v = u_1 - u_2$ $\Rightarrow (u_1 - u_2, u_1 - u_2) = 0$

$\Rightarrow u_1 - u_2 = 0$ $\rightarrow\leftarrow$

Therefore, the representation u of L is unique.

(ii) Define $M=\{v\in H \mid L(v)=0\}$. we want to show

$\exists\, u\in M^{\perp}$ such that $L(v)=(u,v)$ for all v.

(same idea as in the plane representation in R^3)

case(1): if $M^{\perp}=\{0\}$, then by the proposition 2 at p.38, it is clearly $H=M$. This means $L:H\to 0$

$\Rightarrow L\equiv 0 \;\Rightarrow \mathrm{L}(v)=(0,v)$ for all v.

case(2): if $M^{\perp}\neq 0$, then pick $z\in M^{\perp}$, $z\neq 0 \Rightarrow \mathrm{L}(z)\neq 0$

For $v\in H$, and $\beta=\dfrac{L(v)}{L(z)}$, we have $L(v-\beta z)=L(v)-\dfrac{L(v)}{L(z)}\cdot L(z)=0$

$\Rightarrow v-\beta z\in M$.

Since $v=\underbrace{(v-\beta z)}_{\in M}+\underbrace{(\beta z)}_{\in M}$, if $v\in M^{\perp}$ then $v-\beta z=0 \;\Rightarrow v=\beta z$

$\Rightarrow M^{\perp}$ is one dimensional.

Now choose $u=\dfrac{L(z)}{\|z\|^2}\cdot z \;\in M^{\perp}$

We have
$$
\begin{aligned}
(u,v)&=\Big(u, \underbrace{v-\beta z}_{\in M}+\underbrace{\beta z}_{\in M}\Big)\\
&=(u,\beta z)=\frac{L(z)}{\|z\|^2}\beta(z,z)\\
&=\beta L(z)=L(v) \text{ for all } v\in H
\end{aligned}
$$

Thus, u is the desired representation of L in H.

The Lax-Milgram Theorem

Lemma Contraction Mapping Principle:

Given a Banach Space V and a mapping $T:V\to V$ satisfies $\|Tv_1-Tv_2\|\le M\,\|v_1-v_2\|$ (T is contineous!) for all $v_1,v_2\in V$ and fixed $0\le \mathrm{M}<1$, then $\exists!\, u\in V$ such that $T(u)=u$.

T is called the contraction mapping and u is called a fixed point.

Pf. (sketch) Pick $v_0 \in V$, $v_k = Tv_{k-1}$, $k = 1, 2, 3, \cdots$

$$\|v_{k+1} - v_k\| = \|T(v_k) - T(v_{k-1})\| \le M \|v_k - v_{k-1}\|$$

$\Rightarrow \|v_{k+1} - v_k\| \le M^k \|v_1 - v_0\|$

$\Rightarrow$ For $N > n$,

$$\|v_N - v_n\| = \left\| \sum_{k=n}^{N-1} v_{k+1} - v_k \right\|$$

$$\le \left(\sum_{k=n}^{N-1} M^k \right) \cdot \|v_1 - v_0\| \le \left(\sum_{k=n}^{\infty} M^k \right) \|v_1 - v_0\|$$

$$\le \frac{M^n}{1-M} \|T(v_0) - v_0\| \to 0 \text{ as } n \to \infty$$

$\{v_n\}$ is a Cauchy sequence.

Since V is a complete space $\Rightarrow \exists u,\ v_n \to u$ and

by the continuity of T, $\lim_{n\to\infty} T(v_n) = T(u)$

$$\lim_{n\to\infty} T(v_n) = \lim_{n\to\infty} v_{n+1} = u$$

Hence, u is a fixed point.

uniqueness: suppose $\exists$ two fixed points u_1, u_2

$$\|T(u_1) - T(u_2)\| \le M \|u_1 - u_2\|$$

$\Rightarrow \|u_1 - u_2\| \le M \|u_1 - u_2\|$

This is contradict with the condition $0 \le M < 1$.

$\therefore$ the fixed point u is unique.

Theorem (Lax-Milgram) Given a Hilbert space $(v, (\ \cdot,\ \cdot))$

and a bilinear form a $(\ \cdot,\ \cdot)$ satisfies

(i) (continuity) $|a(u,v)| \le \beta \|u\|_v \|v\|_v$ for some $\beta > 0$

(ii) (coercivity) $|a(u,v)| \ge \rho \|u\|_v^2$ for some $\rho > 0$

and a continuous linear functional $F \in \underset{\left(\substack{\text{collection of linear} \\ \text{mapping from } V \to R}\right)}{V'}$ $(F : V \to R)$

Then, there exists a unique $u \in V$

such that

$$a(u,v) = F(v) \text{ for all } v \in V \quad \left(\|F\|_{v'} = \sup_{\substack{w \in V \\ w \neq 0}} \frac{L(w)}{\|v\|_v} \right)$$

Remark:

Consider (example in p.33)

$$a(u,v) \underset{\substack{\uparrow \\ \text{weak formulation} \\ \text{resulted from} \\ \text{veriational principle}}}{=} \underbrace{\int \underset{\substack{\uparrow \\ \text{linear in } u}}{\underbrace{\left(-\sum_i \left(a_{ij}\frac{\partial u}{\partial x_j}\right) + du\right)}} \underset{\text{linear in } v}{v} \, dx}_{\text{bilinear form}}$$

or more general example in p.31

$$\int \underbrace{\left(p_i\left(a^{ij} p_j u + b^i u\right) + i^i p_i u + du\right)}_{\text{linear in } u, D^{(\partial)}u} \underbrace{v}_{\text{linear in } v}$$

conditions (1)(2)(3) in Theorem 10 guarrantee $a(u,v)$ satisfies (i)&(ii) of the Lax-Milgram.