Pf. Sketch of proof:

(1) For any given u, consider a functional $Au(v) = a(u,v)$

show Au is linear & continuous from $V \to R$

$\Rightarrow Au \in V'$

(2) Consider A as a mapping from $V \to V'$

define by $A : u \to (Au)$

show A is also linear continuous by showing $\underbrace{\|A\|_{L(v,v')}}_{\sup_u \frac{\|Au\|}{\|u\|}\left(\|Au\| = \sup_{v\in V}\frac{\|A(u)(v)\|}{\|v\|}\right)} \le C$

(3) To find unique u such that

$Au(v) = F(v)$ for all $v \in V$

$\equiv$ find a unique Au such that $Au = F$ in V'

$\equiv$ find u $\tau(Au) = \tau(F)$ in V $\left\{\begin{array}{l}\text{Riesz representation, } \exists F^* \in V \text{ s.t. } (F^*, u) = F(u) \\ \qquad \exists a_u^* \in V \text{ s.t. } (a_u^*, v) = Au(v) \ \ v \in V \\ \text{define a mapping } \tau : V' \to V, \ (\tau\phi; v) = \phi(v) \ \ \phi \in v' \end{array}\right.$

(4) define a mapping $T : V \to V$

$Tv = v - \rho(\tau Av - \tau F) \ \ \forall v \in V$

if $\exists \rho \ne 0$ such that T is a contraction

then $\exists$ unique $u \in V$ such that

$Tu = u - \rho(\tau Au - \tau F) = u$

$\overset{\rho \ne 0}{\Rightarrow} \tau Au = \tau F$

(5) Try to find ρ

For any $v_1, v_2 \in V$

$$\|T(v_1) - T(v_2)\|^2 = \|v_1 - v_2 - \rho(\tau A v_1 - \tau A v_2)\|^2$$
$$= \|v - \rho(\tau A v)\|^2, \quad v = v_1 - v_2$$
$$= \|v\|^2 - 2\rho(\tau A v, v) + \rho^2 \|\tau A v\|^2$$
$$= \|v\|^2 - 2\rho A v(v) + \rho^2 (A v)(\tau A v)$$
$$= \|v\|^2 - 2\rho a(v, v) + \rho^2 a(v, \tau A v)$$
$$\le \|v\|^2 - 2\rho\alpha \|v\|^2 + \rho^2 \beta \|v\| \|\tau A v\|$$
$$\le \|v\|^2 - 2\rho\alpha \|v\|^2 + \rho^2 \beta^2 \|v\|^2$$
$$= (1 - 2\rho\alpha + \rho^2\beta^2) \|v_1 - v_2\|^2$$

$(\tau A v, \tau A v) = (A v)(\tau A v) = a(v, \tau A v)$

$\|\tau A v\|^2 = a(v, \tau A v) \le \beta \|v\| \|\tau A v\| \;\Rightarrow \|\tau A v\| \le \beta \|v\|$

one can choose $\rho \in \left(0, 2\dfrac{\alpha}{\beta^2}\right) \;\Rightarrow 1 - 2\rho\alpha + \rho^2\beta^2 < 1$

$\Rightarrow T$ is a contraction.

Finite element space

Part 1. Galerkin approximation

Given a finite dimensional subspace $V_n \subset (V, < \ , \ >)$

and $F \in V'$, find $u_n \in V_n$ such that

$a(u_n, v) = F(v)$ for all $v \in V_n$

Question:

1.Do there exist unique solution u_n?

2.What are the error estimates for $u - u_n$?

Ans:

(1) One only needs to show V_n is indeed a subspace of the normal linear space.

(By definition of subspace: V_n is linear & closed
$\Rightarrow V_n$ is complete $\Rightarrow V_n$ is a Hilbert space.)

$\overline{V_n} = V_n$ (every limit points are in V_n)

$\Rightarrow$ Lax-Milgram can be appied accumulative points.

i.e. show V_n is linear & V_n is closed under the given norm $\|\cdot\|_v$

(2) Estimates of the error

Theorem (Cea) Suppose the assumptions in L-M theorem hold and V_n is a linear subspace of $(V, <\ ,\ >)$

Suppose u_n is the solution of the Galerkin approximation

Then $\|u-u_n\|_v \le \frac{\beta}{\alpha} \min_{v\in V_n} \|u-v\|_v$

Pf. Since $a(u,v)=F(v)$ and $a(u_n,v)=F(v)$ for all $v\in V_n$

$\Rightarrow a(u-u_n,v)=0$ for all $v\in V_n$

$$\Rightarrow \alpha\|u-u_n\|_v^2 \le a(u-u_n,u-u_n)$$

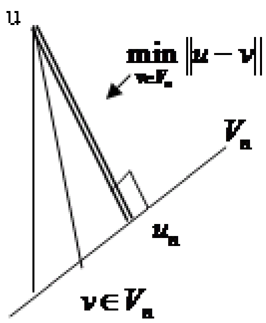
$$= a(u-u_n,u-v)+\underbrace{a(u-u_n,u-v_n)}_{=0} \quad \text{for all } v\in V_n$$

$$\le \beta\|u-u_n\|_v \|u-v\|_v$$

$$\Rightarrow \|u-u_n\|_v \le \frac{\beta}{\alpha}\|u-v\|_v \quad \text{for all } v\in V_n$$

$$\Rightarrow \|u-u_n\|_v \le \frac{\beta}{\alpha}\inf_v \|u-v\|_v$$

Geometric explanation:



$(1)\, u_n$ is almost the "best approximation" of u in V_n when $\dfrac{\beta}{\alpha} \approx 1$

(2) The conclusion in (1) is not exactly true when $\dfrac{\beta}{\alpha} \gg 1$

Example:

Consider $a(u,v) = \int_\Omega A\nabla u \nabla v + cuvdx,\quad \underset{(A, C \text{ are constant})}{A > 0,\ C > 0}$ (positive)

general inner produce $\langle u,v\rangle_A = \int A\nabla u\nabla v + cuv$ and the energy norm $|||u||| = \sqrt{\langle u,v\rangle_A}$

Let V be the Hilbert space $\left(H_2^1, <\ ,\ >_A\right)$

(Can you explain why V is a Hilbert space?)

It is clear that

$$\left.\begin{array}{r} a(u,v) = \langle u,v\rangle_A \le |||u|||\cdot|||v||| \\ \text{and } a(u,v) \ge |||u|||^2 \end{array}\right\} \Rightarrow \begin{array}{r} \text{the continuity const } \beta = 1 \\ \text{and the coercivity const } \alpha = 1 \end{array}$$

$\Rightarrow$ The cea theorem $\Rightarrow u_n$ is indeed the projection of u in V_n.

This also explains why we said the "energy" norm is a good norm in previous lectures.

Q: Consider $a(u,v) = \int_\Omega A\nabla u \nabla v + \underbrace{\beta\nabla u \cdot v}_{\substack{\text{skew symmetric} \\ \text{(thanks to integration by part)}}} + cuv$

$$\uparrow\left(\begin{array}{l} \text{can't define } \langle u,v\rangle_a = a(u,v) \text{ because} \\ \langle u,v\rangle_a \ne \langle v,u\rangle_a \\ \text{define } \langle u,v\rangle = \int A\nabla u\nabla v + \int cuv \end{array}\right)$$

Can you find the α and β in terms of $A,\ B$ and C? (not so trivial!)

You need to decide the "inner product" and the norm defined by the "inner product" and the space for V.

Poincare' inequality:

$$\int_{\Omega} u^2 \le \underset{\text{poincare constant}}{\underset{\uparrow}{c}} \int_{\Omega} |\nabla u|^2$$

here Ω is a Lipschtz domain and $u \in H^1_{2,0}$

Proof by example:

In 1-D: Consider $u(x) = u(0) + \int_0^x \left(\frac{du}{dx}\right) d\varsigma$ $\left(\text{let } u(0) = 0\right)$

$$u^2(x) = \left|\int_0^x \frac{du}{dx} d\varsigma\right|^2 \le \int_0^x 1^2 d\varsigma \cdot \int_0^x \left|\frac{du}{dx}\right|^2 d\varsigma$$

$$\Rightarrow \int_0^1 u^2(x) dx \le \int_0^1 x \cdot \int_0^x \left(\frac{du}{dx}\right)^2 d\varsigma dx$$

$$(0 \le x \le 1) \quad \le \int_0^1 \int_0^1 \left(\frac{du}{dx}\right)^2 d\varsigma dx = \int_0^1 \left|\frac{du}{dx}\right|^2 dx$$

$$\int_{\Omega} v^2 dx = \int_{\Omega} v^2 \Delta\phi dx \qquad \phi = \frac{1}{2d}|x|^2$$

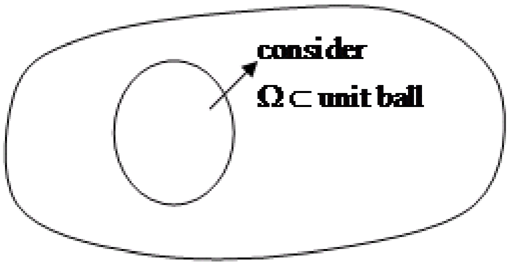
$$= 2v\nabla v \nabla\phi$$

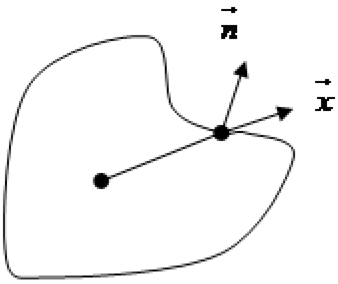
$$= \underbrace{\int_{\partial\Omega} v^2 \nabla\phi \cdot \vec{n}}_{(I)} - \underbrace{\int_{\Omega} 2v\nabla v(\nabla\phi)}_{(II)}$$

$$\nabla\phi = \frac{1}{2d}(\nabla x, x) \cdot 2 = \frac{1}{d} x$$

$$\Delta\phi = \nabla(\nabla\phi) = \frac{1}{d}(\nabla \cdot x) = \frac{1}{d} \cdot d = 1 \quad (d = \text{dimension of the domain})$$

$$\nabla\phi \cdot \vec{n} = \underset{\text{bounded above}}{\frac{1}{d} \vec{x} \cdot \vec{n}}$$





$$(I) \underset{\substack{\text{holder} \\ \text{inequality}}}{<} \frac{1}{d}\|v\|^2_{L^2(\partial\Omega)}$$

$$(II) \leq \int \frac{2}{d} v\left(\nabla v \cdot \vec{x}\right) dx$$

$$\leq \frac{2}{d}\left(\int |v|^2\right)^{\frac{1}{2}}\left(\int_{\Omega} |\nabla v \cdot x|^2\, dx\right)^{\frac{1}{2}}$$

$$\leq \frac{2}{d}\|v\|_{L^2(\Omega)}\left(\int_{\Omega} |\nabla v|^2 |x|^2\, dx\right)^{\frac{1}{2}}$$

$$\leq \frac{2}{d}\|v\|_{L^2(\Omega)}\left(\int_{\Omega} |\nabla v|^2\right)^{\frac{1}{2}}$$

Combine (I) & (II), one can imply

$$\Rightarrow \left(\int_{\Omega} v^2\right)^{\frac{1}{2}} \leq c\left(\int_{\partial\Omega} v^2 dx + \int_{\Omega} |\nabla v|^2\right)^{\frac{1}{2}}$$

For $v|_{\partial\Omega} = 0$ $\Rightarrow \left(\int_{\Omega} v^2\right)^{\frac{1}{2}} \leq c\left(\int_{\Omega} |\nabla v|^2\right)^{\frac{1}{2}}$

$(\partial < 1)$

$a^2 \le b^2 + \partial ac$

$\Rightarrow a^2 + \frac{\partial^2}{4} c^2 \le b^2 + \partial ac + \frac{\partial^2}{4} c^2$

$\Rightarrow \left(a - \frac{\partial}{2} c \right)^2 \le b^2 + \frac{\partial^2}{4} c^2$

$\Rightarrow a \le \frac{\partial}{2} c + \sqrt{b^2 + \frac{\partial^2}{4} c^2} < \tilde{c}\sqrt{b^2 + c^2}$

(1)

$$\underset{\text{coercivity}}{\Rightarrow} \int_\Omega \vec{\beta} \nabla u \cdot u = \int div\left(\vec{\beta} v\right) \cdot u$$

$$= -\int_\Omega \beta u \cdot \nabla u + \int_{\partial\Omega} \vec{\beta} u \cdot u \cdot \underset{>0}{\vec{n}} \Rightarrow \text{if } u = 0 \text{ on } \partial\Omega$$

$$\Rightarrow \int_\Omega \vec{\beta} \nabla u \cdot u = 0 \Rightarrow \mathrm{a}(u,u) > |||u||| \Rightarrow \text{coercivity astisfies.}$$

Error:

$$|||u - u_n||| \le \min_{v \in V_n} |||u - v|||$$

$$= \min_{v \in V_n} \sqrt{\int A \cdot \nabla (u - v) \cdot \nabla (u - v) + \int c (u - v)(u - v) dx}$$

$$\le c \min_{v \in V_n} \|u - v\|_{1,2} \underset{\substack{\uparrow \\ \text{poincare} \\ \text{inequality}}}{\le} c \left(\int_\Omega \left(\nabla (u - v) \right)^2 \right)^{\frac{1}{2}} dx$$

Now consider v as on interpolation of u over a given mesh (Think of Taylor formula)

$$\left(\begin{array}{l} \text{i.e. } v(x_i) = u(x_i) \\ u(x) = \underbrace{u(x_i) + u'(x_i)(x - x_i) + \frac{1}{2} u''(x_i)(x - x_i)^2}_{u} \end{array} \right)$$

we have $\left| (u - v)' \right| < h \cdot u''$

$$\Rightarrow |||u - u_n||| \le c \cdot h \quad \underset{= \left(\int_\Omega |D^2 u|^2 \right)^{\frac{1}{2}}}{|u|_{H_2^2}} \qquad , |u' - v'| \approx \left| u''(x_i) \underbrace{(x - x_i)}_{<h} \right|$$

by assuming regularity of u $\left(\begin{array}{l} a(u,v) = f(v) \ \ (\Delta u = f) \\ \text{assume } u \in H_2^2 \text{ and } \Delta u = f \\ |u|_{2,2} \le c \| f \|_{L^2} \end{array} \right)$

$$\Rightarrow |||u - u_n||| < c \cdot h \cdot \| f \|_{L^2(\Omega)}$$

Remark:

Regularity analysis is important! Even for the equation

$\Delta u = f$ where $f \in L^2$, one might not have $u \in H_2^2$!

Example 1.

Consider $\Omega = \left\{ (u,v) \middle| 0 \le v \le 1,\ 0 < \theta < \frac{\pi}{\beta} \right\}$

if $\frac{1}{2} < \beta < 1$ (Ω is not convex), consider $u = (1-r^2)\, v(r,\theta)$,

here $\underbrace{v(r,\theta) = r^{\beta} \sin \beta\theta}_{\Rightarrow \theta = 0,\, \theta = \frac{\pi}{\beta} \Rightarrow v(r,\theta) = 0} \quad \Rightarrow u|_{\partial\Omega} = 0$

$$\Delta u = (1-r^2)\Delta v + 2\nabla(1-r^2)\cdot\nabla v + v\Delta(1-r^2)$$

$$= -4r\frac{\partial v}{\partial r} - 4v = -(4\beta + 4)v = f$$

Since v is bounded, $f \in L^2(\Omega)$ (L-M theorem is satisfied)

However, $\frac{\partial^2 u}{\partial r^2} \approx r^{\beta - 2}$ near $r = 0$ $\Rightarrow \beta < 1 \Rightarrow \frac{\partial^2 u}{\partial r^2}$: unbounded

$\Rightarrow u \notin H_2^2$ (By the Sobolev inequality (2) Theorem 7)

What's known about regularity?

If $\partial\Omega$ is $\underset{(\partial\Omega \in C^2)}{\text{smooth}}$ (or convex), the solutions of elliptic P.D.E with pure Dirichlet B.C. or Neumann B.C has H^2 regularity.

Part II.

Finite element spaces for V_n

Consider V_n consists of piecewise polynomials.

We have

$$V_n \subset H^1(\Omega) \Leftrightarrow V_n \subset C^0(\overline{\Omega})$$

$$V_n \subset H^2(\Omega) \Leftrightarrow V_n \subset C^1(\overline{\Omega})$$

here $C^0(\overline{\Omega}) = \{ v | v \text{ is continuous on } \overline{\Omega} \}$

$$C^1(\overline{\Omega}) = \{ v | D^{\partial} \in C^0(\overline{\Omega}),\ |\partial| = 1 \}$$

To define a finite element space, we need to specify

(a) mesh T_n (generally a triangulation) on the domain Ω