Remark:

Regularity analysis is important! Even for the equation

$\Delta u = f$ where $f \in L^2$, one might not have $u \in H_2^2$!

Example 1.

Consider $\Omega = \left\{ (u,v) \middle| 0 \le v \le 1,\ 0 < \theta < \frac{\pi}{\beta} \right\}$

if $\frac{1}{2} < \beta < 1$ (Ω is not convex), consider $u = \left(1 - r^2\right) v(r,\theta)$,

here $\underbrace{v(r,\theta) = r^\beta \sin \beta\theta}_{\Rightarrow \theta = 0, \theta = \frac{\pi}{\beta} \Rightarrow v(r,\theta) = 0} \quad \Rightarrow u\big|_{\partial\Omega} = 0$

$$\Delta u = \left(1 - r^2\right)\Delta v + 2\nabla\left(1 - r^2\right)\cdot \nabla v + v\Delta\left(1 - r^2\right)$$

$$= -4r\frac{\partial v}{\partial r} - 4v = -(4\beta + 4)v = f$$

Since v is bounded, $f \in L^2(\Omega)$ (L-M theorem is satisfied)

However, $\frac{\partial^2 u}{\partial r^2} \approx r^{\beta - 2}$ near $r = 0 \quad \Rightarrow \beta < 1 \ \Rightarrow \frac{\partial^2 u}{\partial r^2}$: unbounded

$\Rightarrow u \notin H_2^2$ (By the Sobolev inequality (2) Theorem 7)

What's known about regularity?

If $\partial\Omega$ is $\underset{\left(\partial\Omega \in C^2\right)}{\text{smooth}}$ (or convex), the solutions of elliptic P.D.E with pure Dirichlet B.C. or Neumann B.C has H^2 regularity.

Part II.

Finite element spaces for V_n

Consider V_n consists of piecewise polynomials.

We have

$$V_n \subset H^1(\Omega) \ \Leftrightarrow V_n \subset C^0\left(\overline{\Omega}\right)$$

$$V_n \subset H^2(\Omega) \ \Leftrightarrow V_n \subset C^1\left(\overline{\Omega}\right)$$

here $C^0\left(\overline{\Omega}\right) = \left\{ v \middle| v \text{ is continuous on } \overline{\Omega} \right\}$

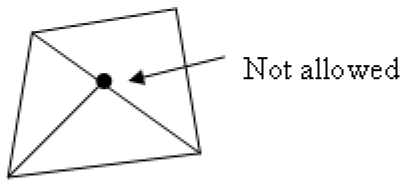
$C^1\left(\overline{\Omega}\right) = \left\{ v \middle| D^\partial \in C^0\left(\overline{\Omega}\right),\ |\partial| = 1 \right\}$

To define a finite element space, we need to specify

(a) mesh T_n (generally a triangulation) on the domain Ω

$\langle$condition on $J_n\rangle$: Any face of an element k in J_n is either a face of another element of a portion of the boundary $\partial\Omega$.

$\Rightarrow$ No hanging nodes are allowed!

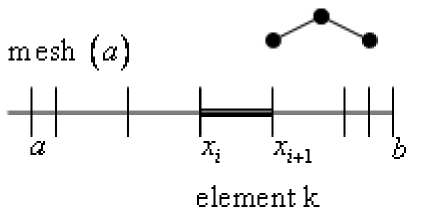


(b) the nature of the function v in V_n on each element $k \in T_n$

(c) the parameters to be used to describe the functions in V_n

Some finite element spcae:

For 1-D problems, one has



Example 1.

(a) each element contains 2 nodal points and 2 nodal value.

(b) piecewise linear function in element k.

(c) linear function $u_n(x)$ in k can be represented by

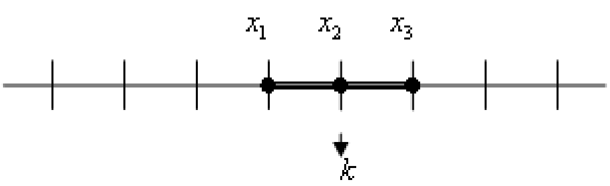
$$u_n(x) = \left(\frac{x_{i+1}-x}{x_{i+1}-x_i}\right) u_n(x_i) + \left(\frac{x_i - x}{x_i - x_{i+1}}\right) u_n(x_{i+1})$$

consider $s = \dfrac{x - x_i}{x_{i+1}-x_i}$ $\left(s \in [0,1] \text{ when } x \in [x_i, x_{i+1}]\right)$

$$\Rightarrow u_n(x) = \underbrace{(1-s)}_{\substack{\phi_1 \\ (\text{basis functions})}} \underbrace{u_n(x_i)}_{\substack{\text{nodal values} \\ (\text{unknowns})}} + \underbrace{s}_{\substack{\phi_2 \\ (\text{basis functions})}} \underbrace{u_n(x_{i+1})}_{\substack{\text{nodal values} \\ (\text{unknowns})}}$$

(i.e parameters used to describe the linear function)

Basis function satisfy $\phi_i(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$



Example 2.

(a) mesh

each element contains 3 nodal points, 3 nodal values.

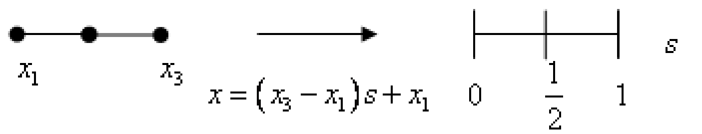
(b) quadratic functions in each element.

(c) quadratic function can be represented by

$$u_n(x) = \underbrace{\frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)}}_{\phi_1} u_n(x_1) + \underbrace{\frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)}}_{\phi_2} u_h(x_2) + \underbrace{\frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}}_{\phi_3} u_h(x_3)$$

$$= \phi_1(x) u_n(x_1) + \phi_2(x) u_n(x_2) + \phi_3(x) u_n(x_3)$$

parametrizing: (assume x_2 is the midpoint)

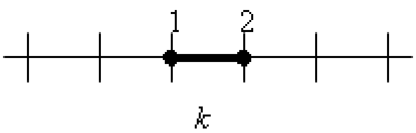


$$u_n(x) = \underbrace{\frac{\left(s-\frac{1}{2}\right)(s-1)}{\frac{1}{2}}}_{\phi_1(s)} u_n(x_1) + \underbrace{\frac{s(s-1)}{-\frac{1}{4}}}_{\phi_2(s)} u_n(x_2) + \underbrace{\frac{s\left(s-\frac{1}{2}\right)}{\frac{1}{2}}}_{\phi_3(s)} u_n(x_3)$$

basis function $\phi_i(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}, \quad i, j = 1 \sim 3$

Remark:

Example 1 & 2 are based on the Lagrange interpolation the functions on V_n are only $c(\overline{\Omega}) \Rightarrow V_n \subset H'(\Omega)$



Example 3.

(a) mesh

each element contains 2 nodal points and 4 nodal values

$\left(u_1, u_1', u_2, u_2'\right)$

(b) cubic Hermite polynomial in each element

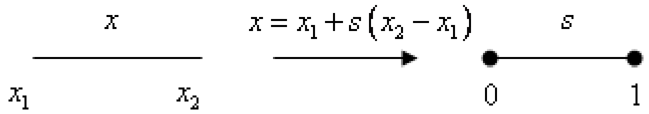
(c) $u_n(x) = \varphi_{1,0}(x)u_n(x_1) + \varphi_{2,0}(x)u_n(x_2) + \varphi_{1,1}(x)u_n'(x_1) + \varphi_{2,1}(x)u_n'(x_2)$

here $\begin{cases} \varphi_{i,0}(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} & \varphi_{i,0}'(x_j) = 0 \\ \varphi_{i,1}(x_j) = 0, & \varphi_{i,1}'(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \end{cases}$

$$\begin{cases} \varphi_{1,0} = \left(\dfrac{x - x_2}{x_1 - x_2}\right)^2 \left(1 - 2\dfrac{x - x_1}{x_1 - x_2}\right) \\ \varphi_{2,0} = \left(\dfrac{x - x_1}{x_2 - x_1}\right)^2 \left(1 - 2\dfrac{x - x_2}{x_2 - x_1}\right) \end{cases}$$

$$\begin{cases} \varphi_{1,1} = \left(\dfrac{x - x_2}{x_1 - x_2}\right)^2 (x - x_1) \\ \varphi_{2,1} = \left(\dfrac{x - x_1}{x_2 - x_1}\right)^2 (x - x_2) \end{cases}$$

parametrizing:



$\varphi_{1,0}(s) = (s-1)^2(1+2s) = 2s^3 - 3s^2 + 1$

$\varphi_{2,0}(s) = s^2(1 - 2(s-1)) = -2s^3 + 3s^2$

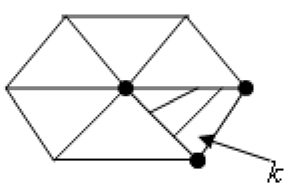
$\varphi_{1,1}(s) = (s-1)^2 s = s^3 - 2s^2 + s$

$\varphi_{2,1}(s) = s^2(s-1) = s^3 - s^2$

(Those are same as the shap functions of a beam element)

Remark:

The functions in V_n (here) are in $c(\overline{\Omega}) \Rightarrow V_n \subset H^2(\Omega)$



For 2-D problems:

Example 4.

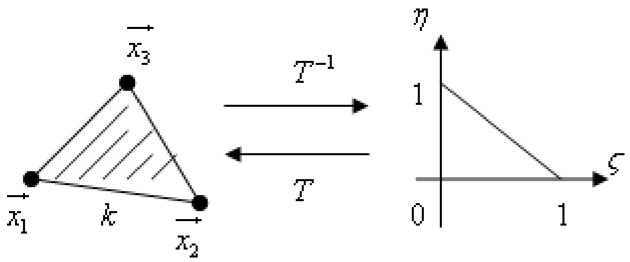
(a) mesh

each element contains 3 nodal points and 3 nodal values

(b) piecewise linear function in each element

(c) (i) parametrization

$$T(0,0)=\overrightarrow{x_1}$$
$$T(1,0)=\overrightarrow{x_2} \qquad \Rightarrow \Phi=\left(\overrightarrow{x_2}-\overrightarrow{x_1},\overrightarrow{x_3}-\overrightarrow{x_1}\right) \text{ in matrix form}$$
$$T(0,1)=\overrightarrow{x_3}$$



consider an affine mapping (linear)

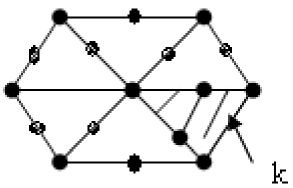
$$\begin{pmatrix} x \\ y \end{pmatrix}=T\begin{pmatrix} \varsigma \\ \eta \end{pmatrix}\equiv\Phi\cdot\begin{pmatrix} \varsigma \\ \eta \end{pmatrix}+\left(\overrightarrow{x_1}\right)$$

(ii) $u_n(x)=\sum_{i=1}^{3}\phi_i(\varsigma,\eta)u_n\left(\overrightarrow{x_i}\right)$

$$\text{here } \begin{matrix} \phi_1(\varsigma,\eta)=1-\varsigma-\eta \\ \phi_2(\varsigma,\eta)=\varsigma \\ \phi_3(\varsigma,\eta)=\eta \end{matrix} \qquad \Rightarrow \phi_i\left(T^{-1}\left(x_j\right)\right)=\begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$$

$$V_n=\left\{u \,\middle|\, u|_k=p_1(k),\ \forall k\in \underbrace{J_n}_{\text{Collection of all elements=mesh}} \text{ and } u \text{ is contineous at nodes}\right\}$$

$$\subset c^0\left(\overline{\Omega}\right)$$



Example 5.

(a) mesh

each element has 6 nodes and 6 nodal values

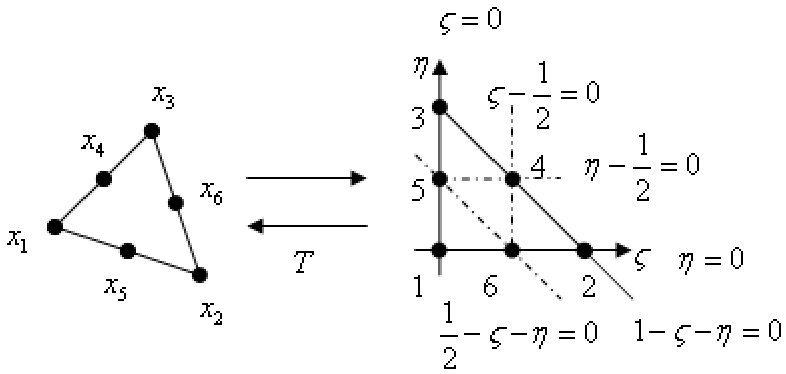
(b) quadratic function in each element $($6 coefficients in quadratic polynomial$)$

(c) assuming the points on each edge is the mid-point

(i) Parametrization:

$$\Phi = \left[\overrightarrow{x_2} - \overrightarrow{x_1}, \overrightarrow{x_3} - \overrightarrow{x_1}\right]$$

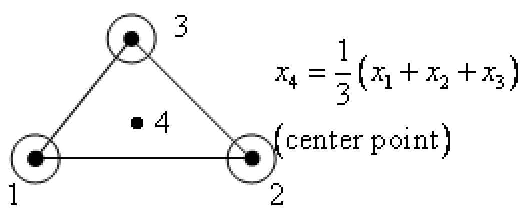
$$\begin{pmatrix} x \\ y \end{pmatrix} = \Phi \begin{pmatrix} \varsigma \\ \eta \end{pmatrix} + \overrightarrow{x_1}$$



$$(ii)\; u_n(x) = \sum_{i=1}^{6} \phi_i(\varsigma, \eta)\, u_n\left(\overrightarrow{x_i}\right)$$

$$\begin{cases} \phi_1 = (1 - 2\varsigma - 2\eta)(1 - \varsigma - \eta) \\ \phi_2 = (1 - 2\varsigma)\varsigma \\ \phi_3 = (1 - 2\eta)\eta \\ \phi_4 = 4\varsigma\eta \\ \phi_5 = 4(1 - \varsigma - \eta)\eta \\ \phi_6 = 4(1 - \varsigma - \eta)\varsigma \end{cases}$$

$$V_n = \left\{u \middle|\; u|_k = p_2(k)\ \forall k \in J_n,\ u \text{ is contineous at nodes}\right\} \subset c^0\left(\overline{\Omega}\right)$$



Example 6.

(a) mesh

each element has 4 nodal points 3 values (u, u_x, u_y) at each node $i\,(i=1,2,3)$ 1 value u at node 4.

(b) cubic polynomial $(10$ coefficients$)$

(c) (i) parametrization

$$(ii)\, u_n(x) = \sum_{i=1}^{3} \phi_i u_n(x_i) + \sum_{i=1}^{3} \phi_i^x \left(\frac{\partial u_n(x_i)}{\partial x} \right) + \sum_{i=1}^{3} \phi_i^y \left(\frac{\partial u_n(x_i)}{\partial y} \right) + \phi_4 u_n(x_4)$$

here $\phi_i(x_j) = \delta_{ij}$ $, j = 1,2,3,4,$ $\frac{\partial}{\partial \varsigma}\phi_i(x_j) = 0,$ $\frac{\partial}{\partial \eta}\phi_i(x_j) = 0,$ $i, j = 1 \sim 3$

$\phi_i^x(x_j) = 0$ $, j = 1,2,3,4,$ $\frac{\partial}{\partial \varsigma}\phi_i^x(x_j) = \delta_{ij},$ $\frac{\partial}{\partial \eta}\phi_i^x(x_j) = 0,$ $i, j = 1 \sim 3$

$\phi_i^y(x_j) = 0$ $, j = 1,2,3,4,$ $\frac{\partial}{\partial \varsigma}\phi_i^y(x_j) = 0,$ $\frac{\partial}{\partial \eta}\phi_i^y(x_j) = \delta_{ij},$ $i, j = 1 \sim 3$

$\phi_4(x_j) = \delta_{4,j}$ $, j = 1,2,3,4,$ $\frac{\partial}{\partial \varsigma}\phi_4(x_j) = \frac{\partial}{\partial \eta}\phi_4(x_j) = 0,$ $i, j = 1 \sim 3$

How to determine basis functions?

Consider $l_1 = 1 - \varsigma - \eta,\ l_2 = \varsigma,\ l_3 = \eta,\ l_4 = \phi_4 = (l_1 \cdot l_2 \cdot l_3) \cdot 27,\ l_i(x_j) = \delta_{ij}$

$$\phi_1 = \underbrace{l_1^2(1 + 2(l_2 + l_3))}_{L_1} - L_1\left(\frac{1}{3}, \frac{1}{3}\right)\phi_4$$

$$\phi_2 = \underbrace{l_2^2(1 + 2(l_1 + l_3))}_{L_2} - L_2\left(\frac{1}{3}, \frac{1}{3}\right)\phi_4$$

$$\phi_3 = \underbrace{l_3^2(1 + 2(l_1 + l_2))}_{L_3} - L_3\left(\frac{1}{3}, \frac{1}{3}\right)\phi_4$$

$$\phi_1^x = \underbrace{l_1^2(l_2)}_{L_1^x} - L_1^x\left(\frac{1}{3}, \frac{1}{3}\right)\phi_4$$

$$\phi_2^x = \underbrace{-(l_1 + l_3)l_2^2}_{L_2^x} - L_2^x\left(\frac{1}{3}, \frac{1}{3}\right)\phi_4$$

$$\phi_3^x = \underbrace{l_3^2(l_2)}_{L_3^x} - L_3^x\left(\frac{1}{3}, \frac{1}{3}\right)\phi_4$$

$$\phi_1^y = \underbrace{l_1^2 (l_3)}_{L_1^y} - L_1^y\left(\frac{1}{3},\frac{1}{3}\right)\phi_4$$

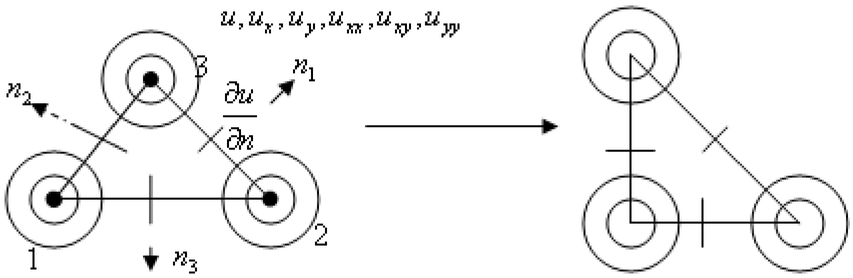
$$\phi_2^y = \underbrace{l_2^2 (l_3)}_{L_2^y} - L_2^y\left(\frac{1}{3},\frac{1}{3}\right)\phi_4$$

$$\phi_3^y = \underbrace{-(l_1 + l_2) l_3^2}_{L_3^y} - L_3^y\left(\frac{1}{3},\frac{1}{3}\right)\phi_4$$

$$\phi_4 = 27 l_1 l_2 l_3$$

Remark:

the function space $V_n \subset c^0\left(\overline{\Omega}\right)$ and u and ∇u are contineous at nodal points 1,2,3 but $V_n \not\subset c^1\left(\overline{\Omega}\right)$



Example 7. Argyris triangle

(a) mesh

21 nodal values $\leftrightarrow$ polynomial of deg 5 $\Rightarrow$ 21 coefficients

$$\left(\underbrace{\sum_{k=0}^{5} x^k y^{5-k}}_{6} \underbrace{\sum_{k=0}^{4} x^k y^{4-k}}_{5} \cdots \right)$$

(b) $p_5(k)$: polynomial of degree 5 in each element

(c) The coefficients of $p_5(k)$ can be uniguely determined

if one can show

$$\begin{pmatrix}\text{linear 21}\\ \text{equation}\end{pmatrix}\begin{cases} p_5(x_i)=0 \quad i=1\sim 3 & \dfrac{\partial^2}{\partial x^2}p_5(x_i)=0 \quad i=1\sim 3 \\ \dfrac{\partial}{\partial x}p_5(x_i)=0 \quad i=1\sim 3 & \dfrac{\partial^2}{\partial y^2}p_5(x_i)=0 \quad i=1\sim 3 \\ \dfrac{\partial}{\partial y}p_5(x_i)=0 \quad i=1\sim 3 & \dfrac{\partial^2}{\partial x\partial y}p_5(x_i)=0 \quad i=1\sim 3 \\ \left[\dfrac{\partial p_5}{\partial x}\left(\dfrac{x_i+x_j}{2}\right), \dfrac{\partial p_5}{\partial y}\left(\dfrac{x_i+x_j}{2}\right)\right]\cdot n_k=0 \quad i\neq j\neq k & \end{cases}$$

$\Rightarrow p_5=0$

$$\left(\text{Parametrization} \Rightarrow \text{we only need to substitute } x_1=\begin{pmatrix}0\\0\end{pmatrix},\ x_2=\begin{pmatrix}0\\1\end{pmatrix},\ x_3=\begin{pmatrix}0\\1\end{pmatrix} \text{ into the above equation}\right)$$

To find the coefficients of $p_5(k)$ in terms of nodal values, one solve

$$\begin{cases} p_5(x_i)=u_i & \dfrac{\partial^2}{\partial x^2}p_5(x_i)=\dfrac{\partial^2}{\partial x^2}u_i \quad i=1\sim 3 \\ \dfrac{\partial}{\partial x}p_5(x)=\dfrac{\partial}{\partial x}u_i & \dfrac{\partial^2}{\partial y^2}p_5(x_i)=\dfrac{\partial^2}{\partial y^2}u_i \quad i=1\sim 3 \\ \dfrac{\partial}{\partial y}p_5(x)=\dfrac{\partial}{\partial y}u_i & \dfrac{\partial^2}{\partial x\partial y}p_5(x_i)=\dfrac{\partial^2}{\partial x\partial y}u_i \quad i=1\sim 3 \\ \left[\dfrac{\partial p_5}{\partial x}\left(\dfrac{x_i+x_j}{2}\right), \dfrac{\partial p_5}{\partial y}\left(\dfrac{x_i+x_j}{2}\right)\right]\cdot\overrightarrow{n_k}, \quad i\neq j\neq k & \end{cases}$$

(d) one can show that $V_n = \{u \mid u|_k \in p_5(k),\ k \in J_n\} \subset c^1(\overline{\Omega})$

observation: consider $k_1, k_2 \in J_n$ with common edges

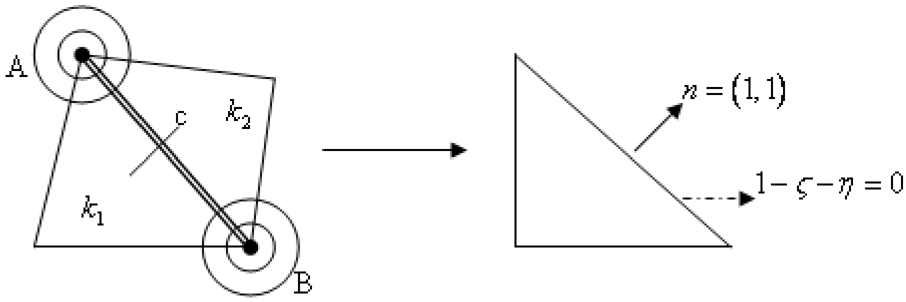
let $v_1 \in p_5(k_1)$ and $v_2 \in p_5(k_2)$

we have $w = v_1 - v_2$ satisfying

$$\begin{cases} w = 0 \\ w_x = 0 \\ w_y = 0 \qquad \text{at } A, B \ \rightarrow 12 \text{ equations} \\ w_{xx} = 0 \\ w_{yy} = 0 \\ \dfrac{\partial w}{\partial x} = 0,\ \dfrac{\partial w}{\partial y} = 0 \text{ at } c \quad \rightarrow 2 \text{ equations} \end{cases}$$

$$\frac{\partial w}{\partial n} = \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} \quad \rightarrow 21 - \underbrace{\underbrace{6}_{\text{\# of terms of deg 5}} - \underbrace{1}_{\text{constant terms}}}_{\text{gone after differentiation}} = 14 \text{ coefficients left}$$

$$\Rightarrow \frac{\partial w}{\partial n} = 0 \quad \text{on } s.$$

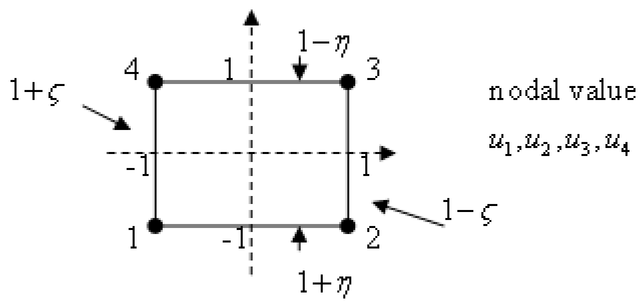


Moreover, since $w|_s = 0 \Rightarrow \dfrac{\partial w}{\partial s} = 0,$ as a result,

we have $\underset{(Dw)}{\nabla w} = 0$ on S

$\Rightarrow \underset{(Dv_1)}{\nabla v_1} = \underset{(Dv_2)}{\nabla v_2}$ on S $\Rightarrow$ The function DV is contineous across S.

Therefore, one can conclude $V_n \subset c^1(\overline{\Omega})$



Example 8. Pectangular meshes

(1) four nodes element

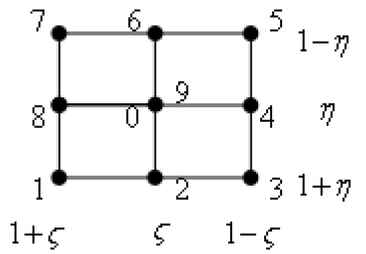
$$\phi_1 = \frac{1}{4}(1-\varsigma)(1-\eta)$$

$$\phi_2 = \frac{1}{4}(1+\varsigma)(1-\eta)$$

$$\phi_3 = \frac{1}{4}(1+\varsigma)(1+\eta)$$

$$\phi_4 = \frac{1}{4}(1-\varsigma)(1+\eta)$$

$$\begin{cases} \text{1-d element} \underset{\text{tensor}}{\otimes} \text{1-d element} \\ \underset{\text{basis in 1-d variable } \varsigma}{(1-\varsigma, 1+\varsigma)} \otimes \underset{\text{basis in 1-d variable } \eta}{(1-\eta, 1+\eta)} \\ \Rightarrow \underset{\uparrow \text{basis in 2-d}}{\left((1-\varsigma)(1-\eta), (1-\varsigma)(1+\eta), (1+\varsigma)(1-\eta), (1+\varsigma)(1+\eta)\right)} \end{cases}$$



nodal value $u_1 \sim u_9$

(2) nine nodes element

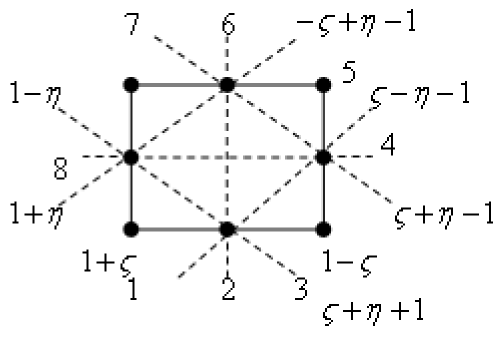
$$\phi_1 = \frac{1}{4}\varsigma\eta(1-\varsigma)(1-\eta)$$

$$\phi_2 = \frac{-1}{2}(1+\varsigma)(1-\varsigma)\eta(1-\eta)$$

$$\phi_3 = \frac{1}{4}(1+\varsigma)\varsigma\eta(1-\eta)$$

$$\vdots$$

$$\phi_9 = (1-\varsigma)(1+\varsigma)(1-\eta)(1+\eta)$$



(3) 8 nodes element

$$\phi_1 = -\frac{1}{4}(1-\varsigma)(1-\eta)(1+\varsigma+\eta) \qquad \phi_5 = -\frac{1}{4}(1+\varsigma)(1+\eta)(1-\varsigma-\eta)$$

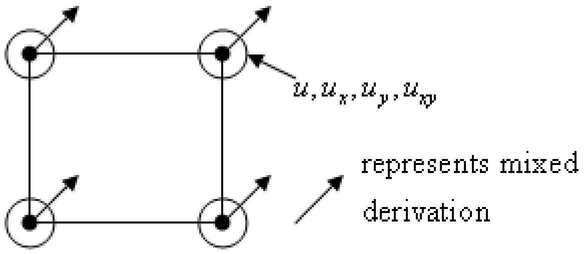
$$\phi_2 = \frac{1}{2}(1-\varsigma)(1+\varsigma)(1-\eta) \qquad \phi_6 = \frac{1}{2}(1-\varsigma)(1+\varsigma)(1+\eta)$$

$$\phi_3 = -\frac{1}{4}(1+\varsigma)(1-\eta)(1-\varsigma+\eta) \qquad \phi_7 = -\frac{1}{4}(1-\varsigma)(1+\eta)(1+\varsigma-\eta)$$

$$\phi_4 = \frac{1}{2}(1+\varsigma)(1-\eta)(1+\eta) \qquad \phi_8 = \frac{1}{2}(1-\varsigma)(1-\eta)(1+\eta)$$

(4) Bogner-Fox-Schmit element

It can be shown BFS gives $V_n \subset c^1(\overline{\Omega})$!



For 3-D problems:

(1) Tetrahedron element (4-nodes)

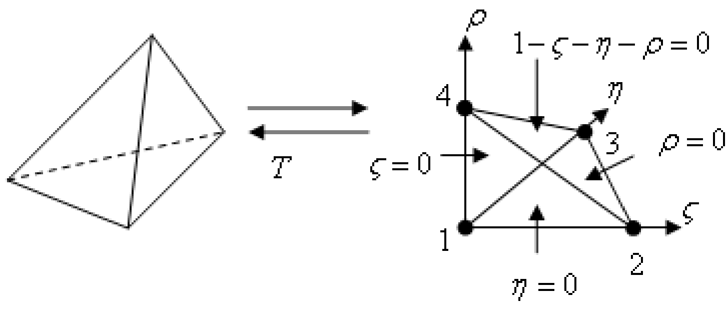
$$\phi_1 = 1-\varsigma-\eta-\rho \quad | \quad \phi_1 = \frac{1}{8}(1-\varsigma)(1-\eta)(1-\rho) \quad \phi_5 = \frac{1}{8}(1-\varsigma)(1-\eta)(1+\rho)$$

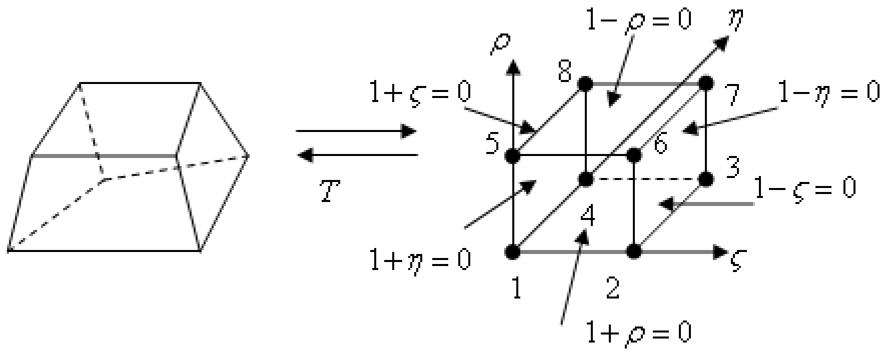
$$\phi_2 = \varsigma \quad | \quad \phi_2 = \frac{1}{8}(1+\varsigma)(1-\eta)(1-\rho) \quad \phi_6 = \frac{1}{8}(1+\varsigma)(1-\eta)(1+\rho)$$

$$\phi_3 = \eta \quad | \quad \phi_3 = \frac{1}{8}(1+\varsigma)(1+\eta)(1-\rho) \quad \phi_7 = \frac{1}{8}(1+\varsigma)(1+\eta)(1+\rho)$$

$$\phi_4 = \rho \quad | \quad \phi_4 = \frac{1}{8}(1-\varsigma)(1+\eta)(1-\rho) \quad \phi_8 = \frac{1}{8}(1-\varsigma)(1+\eta)(1+\rho)$$

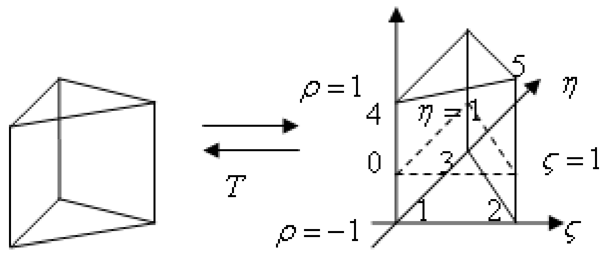
$$\left(\sum_{i=1}^{4} \phi_i = 1\right)$$





(2) Hexahedron

Quadrilateral element basises can be obtained by tensor of linear element in ρ and 4-nodes element in the ς-η plane.



(3) prisinatic element

the element basised can be obtained from tensor product of linear element of ρ (2-nodes) and linear element of ς-η plane (3 nodes)

$(1-\rho, 1+\rho)\otimes(\varsigma, \eta, 1-\varsigma-\eta)$

$$\Rightarrow \phi_1 = \frac{1}{2}(1-\varsigma-\eta)(1-\rho) \qquad \phi_4 = \frac{1}{2}(1-\varsigma-\eta)(1+\rho)$$

$$\phi_2 = \frac{1}{2}\varsigma(1-\rho) \qquad \phi_5 = \frac{1}{2}\varsigma(1+\rho)$$

$$\phi_3 = \frac{1}{2}\eta(1-\rho) \qquad \phi_5 = \frac{1}{2}\eta(1+\rho)$$

Approximation theory for FEM.