Approximation theory for FEM

The question to be answered here is how big the error $u-u_n$ is ?

Here u_h belongs to given finite element space.

First, let's consider $V_h =$ piecewise linear functions in R^2

For $k \in J_h$,

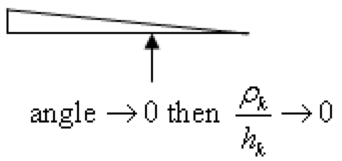
$$\begin{cases} h_k = \text{the diameter of } k = \text{longest side of } k \\ \rho_k = \text{the diameter of the circle inschibed in } k \\ h = \max\limits_{h_k \in J_k} h_k \end{cases}$$

we assume $\exists$ a constant β such that

$$\frac{\rho_k}{h_k} \geq \beta \quad \forall k \in J_h \quad (\text{meaning no triangle in } J_h \text{ is degenrated!})$$

Let $N_i,\ i = 1 \sim n,$ be the nodes of J_h and define the interpolant

$$(\pi_h u)(N_i) = u(N_i) \quad \left(\begin{array}{l}\text{meaning } V_h \text{ consists of nodal interpolation} \\ \text{functions. This is not the case when using sin,} \\ \text{cos as basis functions.}\end{array}\right)$$



Theorem 1.

Let $k \in J_h$ be an element with vertices $a^i,\ i = 1 \sim 3$. Given $v \in C^0(k)$,

let the interpolant $\pi_h v \in P_1(k)$ be defined by $\pi v(a^i) = v(a^i),\ i = 1 \sim 3$.

Then $(1) \|v - \pi_h v\|_{L^\infty(k)} \leq 2h_k^2 \max\limits_{|\partial|=2} \|D^\partial v\|_{L^\infty(k)}$

Recall Theorem 7(*)

$$(2) \max_{|\partial|=1} \|D^\partial (v - \pi_h v)\|_{L^\infty(k)} \leq 6 \cdot \frac{h_k^2}{\rho_k} \max_{|\partial|=2} \|D^\partial v\|_{L^\infty(k)}$$

where $\|v\|_{L^\infty(k)} = \max\limits_{x \in k} |v(x)|$

(*)The Sobolev inequality(2) C^0 can be embedded in H^2 in R^2.

Now, since $\left|a_j^i - x_j\right| \le h_k$, for $i = 1,2,3,\ j = 1,2,$

$\left|R_i(x)\right| \le 2h_k^2 \max\limits_{|\partial|=2} \left\|D^{\partial} v\right\|_{L^\infty(k)}$

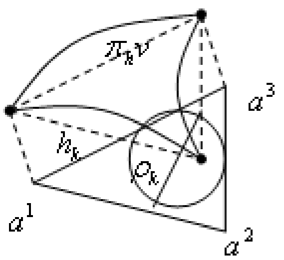
Combine (1) & (2)

$$\Rightarrow \pi_h v = \sum_{i=1}^{2} v(x)\phi_i(x) + \sum_{i=1}^{3} P_i(x)\phi_i(x) + \sum_{i=1}^{3} R_i(x)\phi_i(x)$$

$$= v(x)\sum_{i=1}^{3}\phi_i(x) + \sum_{i=1}^{3} P_i(x)\phi_i(x) + \sum_{i=1}^{3} R_i(x)\phi_i(x) \quad -(3)$$

Observation (1) $\sum\limits_{i=1}^{3}\phi_i(x) = 1$

$$\Rightarrow \left|\pi_h v - v(x)\right| \le \left|\sum_{i=1}^{3} P_i(x)\phi_i(x)\right| + \left|\sum_{i=1}^{3} R_i(x)\phi_i(x)\right| \quad -(*)$$



proof:

Let $\phi_i,\ i = 1 \sim 3,$ be the basis functions

For $w \in P_1(k)$, we have $w(x) = \sum\limits_{i=1}^{3} W\left(a^i\right)\phi_i(x)$ for $x \in k$

$$\Rightarrow (\pi_h v)(x) = \sum_{i=1}^{3} v\left(a^i\right)\phi_i(x) \quad \text{for } x \in k \quad -(1)$$

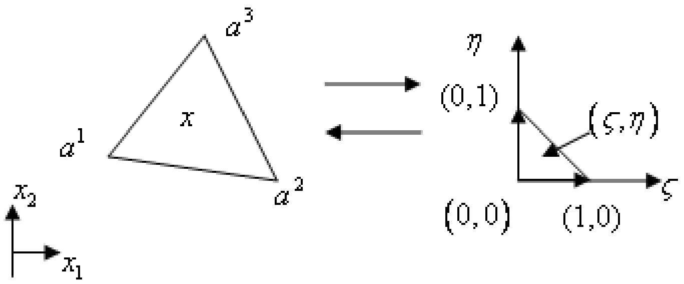
By Taylor expansion, one has $x = (x_1, x_2),\ y = (y_1, y_2)$

$$v(y) = v(x) + \sum_{j=1}^{2} \frac{\partial v(x)}{\partial x_j}\left(y_j - x_j\right) + R(x,y)$$

here $R(x,y) = \dfrac{1}{2}\sum\limits_{i,j=1}^{2} \dfrac{\partial^2 v(\varsigma)}{\partial x_i \partial x_j}(y_i - x_i)\left(y_j - x_j\right)$

Now choose $y = a^i$, we have $v\left(a^i\right) = v(x) + P_i(x) + R_i(x) \quad -(2)$

$$\begin{cases} P_i(x) = \underbrace{\sum\limits_{j=1}^{2} \dfrac{\partial v}{\partial x_j}\left(a_j^i - x_j\right)}_{=\nabla \vec{v}\cdot\left(a^i - x\right)} & \left(a^i = \left(a_1^i, a_2^i\right)\right) \\ R_i(x) = R\left(x, a^i\right) \end{cases}$$



Now consider the parametrization

It can be shown $x = (1-\varsigma-\eta)a^1 + \varsigma a^2 + \eta a^3$

and $\sum_{i=1}^{3} P_i(x)\phi_i(x) = \sum_{i=1}^{3} \nabla v\left(\left(a^i - x\right)\cdot \phi_i(x)\right)$

$$\underset{\text{parametrization}}{=} \sum \nabla v\left(a^i - \left[(1-\varsigma-\eta)a^1 + \varsigma a^2 + \eta a^3\right]\cdot \phi_i(\varsigma,\eta)\right)$$

$$= \nabla v \cdot \begin{Bmatrix} \left(a^1 - \left[(1-\varsigma-\eta)a^1 + \varsigma a^2 + \eta a^3\right]\right)\cdot(1-\varsigma-\eta) + \\ \left(a^2 - \left[(1-\varsigma-\eta)a^1 + \varsigma a^2 + \eta a^3\right]\right)\cdot \varsigma + \\ \left(a^3 - \left[(1-\varsigma-\eta)a^1 + \varsigma a^2 + \eta a^3\right]\right)\cdot \eta \end{Bmatrix}$$

$$= 0 \quad (\text{By direct computation}) \qquad -(4)$$

Therefore,

$$(*) \Rightarrow |\pi_h v - v| \le \left|\sum_{i=1}^{3} R_i(x)\phi_i(x)\right| \le \sum_{i=1}^{3} \max_{x\in k} |R_i(x)|\phi_i(x)$$

$$\le \max_{\substack{x\in k \\ i=\{2\}}} |R_i(x)| \le 2h_k^2 \max_{|\partial|=2} \left\|D^{\partial} v\right\|_{L^\infty(k)}$$

(1) is proved.

To prove (2), let's differentiate (3)

$$\overset{(3)\&(4)}{\Rightarrow} D^{(\partial)}\pi_h v = D^{(\partial)}v(k)\left(\sum_{i=1}^{3}\phi_i(x)\right) + v(x)\sum_{i=1}^{3}D^{(\partial)}\phi_i(x) + D^{(\partial)}\sum_{i=1}^{3}R_i(x)\phi_i(x) \quad -(5)$$

it can be shown

$$\sum_{i=1}^{3}D^{(\partial)}\phi_i(x) = 0 \quad \overset{\text{by parametrization}}{\leftrightarrow} \quad \begin{cases} \frac{\partial}{\partial\varsigma}(1-\varsigma-\eta) + \frac{\partial}{\partial\varsigma}(\varsigma) + \frac{\partial}{\partial\varsigma}(\eta) = 0 \\ \frac{\partial}{\partial\eta}(1-\varsigma-\eta) + \frac{\partial}{\partial\eta}(\varsigma) + \frac{\partial}{\partial\eta}(\eta) = 0 \end{cases}$$

This implies

$$D^{(\partial)}\pi_h v - D^{(\partial)}v = D^{(\partial)}\sum_{i=1}^{3}R_i(x)\phi_i(x) = D^{(\partial)}\sum_{i=1}^{3}\left(a^i - x\right)^T[H]\left(a^i - x\right)\cdot\phi_i(x)$$

here $[H] = \left[\frac{\partial^2 v(\xi)}{\partial x_i \partial x_j}\right]$.

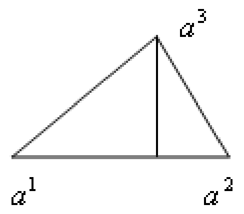
$$\because D^{\partial}\left(\left(a^i - x\right)H\left(a^i - x\right)\right) = 2[H]\left(a^i - x\right)$$

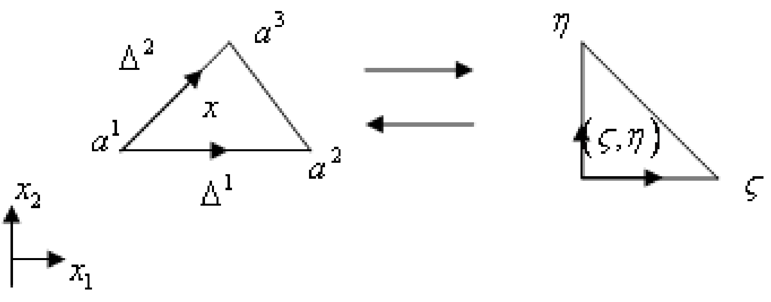
$$\Rightarrow \sum_{i=1}^{3}\left(D^{(\partial)}R_i(x)\right)\phi_i(x) = 2[H]\sum_{i=1}^{3}\left(a^i - x\right)\cdot\phi_i(x) = 0 \ \left(\text{by } (4)\right)$$

$$\Rightarrow \left|D^{(\partial)}\pi_h v - D^{\partial}v\right| = \left|\sum_{i=1}^{3}R_i(x)D^{(\partial)}\phi_i(x)\right| \le \sum_{i=1}^{3}\left|R_i(x)\right|\left|D^{(\partial)}\phi_i(x)\right| \quad -(7)$$

Again, by parametrization, we can show

$$\left|D^{(\partial)}\phi_i(x)\right| = \begin{cases} \frac{\partial\varsigma}{\partial x_1}\cdot\frac{\partial\phi_i}{\partial\varsigma} + \frac{\partial\eta}{\partial x_1}\cdot\frac{\partial\phi_i}{\partial\eta} \\ \frac{\partial\varsigma}{\partial x_2}\cdot\frac{\partial\phi_i}{\partial\varsigma} + \frac{\partial\eta}{\partial x_2}\cdot\frac{\partial\phi_i}{\partial\eta} \end{cases} = \begin{bmatrix} \frac{\partial\varsigma}{\partial x_1} & \frac{\partial\eta}{\partial x_1} \\ \frac{\partial\varsigma}{\partial x_2} & \frac{\partial\eta}{\partial x_2} \end{bmatrix}\begin{bmatrix} \frac{\partial\phi_i}{\partial\varsigma} \\ \frac{\partial\phi_i}{\partial\eta} \end{bmatrix}$$



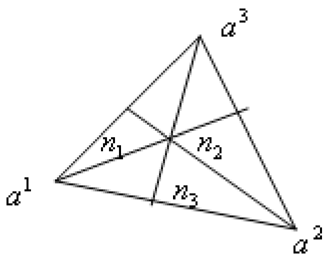


Recall:

$$T(\varsigma,\eta)=\left(\underbrace{a^2-a^1}_{\Delta^1},\underbrace{a^3-a^1}_{\Delta^2}\right)\begin{pmatrix}\varsigma\\ \eta\end{pmatrix}+\overrightarrow{a_1}=\vec{x}$$

$$\Rightarrow\begin{pmatrix}\varsigma\\ \eta\end{pmatrix}=\frac{1}{\underbrace{\left|\left(\Delta^1,\Delta^2\right)\right|}_{(\text{Jacobian})}}\begin{pmatrix}\Delta_2^2 & -\Delta_1^2\\ -\Delta_2^1 & \Delta_1^1\end{pmatrix}\left(\vec{x}-\overrightarrow{a^1}\right)$$

$$\Rightarrow\begin{pmatrix}\frac{\partial\varsigma}{\partial x_1} & \frac{\partial\eta}{\partial x_1}\\ \frac{\partial\varsigma}{\partial x_2} & \frac{\partial\eta}{\partial x_2}\end{pmatrix}=\frac{1}{|\Delta|}\begin{pmatrix}\Delta_2^2 & -\Delta_2^1\\ -\Delta_1^2 & \Delta_1^1\end{pmatrix}$$



$$\left\|\frac{1}{|\Delta|}\begin{pmatrix}\Delta_2^2 & -\Delta_2^1\\ -\Delta_1^2 & \Delta_1^1\end{pmatrix}\begin{pmatrix}\frac{\partial\phi_2}{\partial\varsigma}\\ \frac{\partial\phi_2}{\partial\eta}\end{pmatrix}\right\|=\frac{1}{|\Delta|}\left\|\begin{pmatrix}\Delta_2^2\\ -\Delta_1^2\end{pmatrix}\right\|=\frac{\left\|a^3-a_1\right\|}{|\Delta|}=\frac{1}{h_2}$$

$$\left\|\frac{1}{|\Delta|}\begin{pmatrix}\Delta_2^2 & -\Delta_2^1\\ -\Delta_1^2 & \Delta_1^1\end{pmatrix}\begin{pmatrix}\frac{\partial\phi_3}{\partial\varsigma}\\ \frac{\partial\phi_3}{\partial\eta}\end{pmatrix}\right\|=\frac{1}{|\Delta|}\left\|\begin{pmatrix}-\Delta_2^1\\ \Delta_1^1\end{pmatrix}\right\|=\frac{\left\|a^2-a_1\right\|}{|\Delta|}=\frac{1}{h_3}$$

$$\left\|\frac{1}{|\Delta|}\begin{pmatrix}\Delta_2^2 & -\Delta_2^1\\ -\Delta_1^2 & \Delta_1^1\end{pmatrix}\underset{\begin{pmatrix}-1\\ -1\end{pmatrix}}{\underset{\|}{\begin{pmatrix}\frac{\partial\phi_1}{\partial\varsigma}\\ \frac{\partial\phi_1}{\partial\eta}\end{pmatrix}}}\right\|=\frac{1}{|\Delta|}\left\|\begin{pmatrix}\Delta_2^2-\Delta_2^1\\ \Delta_1^1-\Delta_1^2\end{pmatrix}\right\|=\frac{1}{|\Delta|}=\frac{1}{h_2}\left\|\begin{pmatrix}\Delta_2^1-\Delta_2^2\\ \Delta_1^1-\Delta_1^2\end{pmatrix}\right\|$$

$$=\frac{\left\|a^2-a^3\right\|}{|\Delta|}=\frac{1}{h_1}$$

$$\Rightarrow\left\|D^{(\partial)}\phi_i\right\|_k<\frac{1}{\rho_k}$$

As a result, (7) becomes

$$\left|D^{(\partial)}\pi_h v - D^{(\partial)} v\right| \le \frac{1}{\rho_k}\sum_{i=1}^{3}\left|R_i\left(x\right)\right| \le 6\frac{h_k^2}{\rho_k}\max_{|\partial|=2}\left\|D^{\partial} v\right\|_{L^\infty(k)}$$

This proves (2).

Let's define the semi-norm

$$|v|_{H_2^r} = \left(\sum_{|\partial|=r}\int\left|D^{\partial} v\right|^2\right)^{\frac{1}{2}}$$

(because $|v|_{H_2^r} = 0$ even if $v \ne 0$ (e.g. $v \equiv 1, r \ge 1$))

Poincare inequality$\left(\|u\|_{L^2} < \|\nabla u\|_{L^2}\right)$ shows that it is suffice to estimate the H_2^r semi-norm for obtaining the upper bound of the H_2^r-norm.

Theorem 2. Under the assumption of theorem 1,
there is a constant c such that

$$\|v - \pi_h v\|_{L_2(k)} \le c h_k^2 |v|_{H^2(k)}$$

$$|v - \pi_h v|_{H^1(k)} \le c\frac{h_k^2}{\rho_k}|v|_{H^2(k)}$$

(Pf: skip.)

Global error estimation

Theorem 3. Under the assumption of Theorem 1,
we have $\|v - \pi_h v\|_{L^2(\Omega)} \le c h^2 |u|_{H^2(\Omega)}$

$$|v - \pi_h v|_{H^1(\Omega)} \le c h |u|_{H^2(\Omega)}$$

Pf: $\|v - \pi_h v\|_{L^2(\Omega)}^2 = \sum_{k\in J_n}\|v - \pi_h v\|_{L^2(\Omega)}^2 \le \sum_{k\in J_n} c^2 h_k^4 |u|_{H^2(k)}^2 \le c^2 h^4 |u|_{H^2(\Omega)}^2$

$$|v - \pi_h v|_{H^1}^2 \le \sum_{k\in J_n} c^2\frac{h_k^4}{\rho_k^2}|u|_{H^2(k)}^2 \underset{\left(\frac{\rho_k}{h_k}\ge\beta\right)}{\le} \sum_{k\in J_n} c^2\frac{h_k^2}{\beta^2}|u|_{H^2(k)}^2 \le c(\beta) h^2 |u|_{H^2(\Omega)}^2$$

Remark(2) error of FEM solution satisfies

$$\left\|\left|u - u_h\right|\right\| \underset{(\text{Cea})}{\le} \min_{v\in V_h}\left\|\left|u - v\right|\right\| \underset{(\text{Poincare})}{\le} c\min_{v\in V_h}|u - v|_{H^1} \underset{\left(\substack{\text{interpolation}\\ \text{error}}\right)}{\le} ch|u|_{H^2(\Omega)}$$

↑ energy norm

Remark(3) Error estimation in L^2-norm

Consider $a(u,v)=\int A\nabla u\nabla v+cuvdx$. Suppose u and u_h are the weak solution and the FEM solution respectively. $\left(\text{i.e } \begin{matrix} a(u,v)=F(v) & v\in V \\ a(u_n,v)=F(v) & v\in V_h \end{matrix}\right)$

Let $e=u-u_h$. Clearly one has $a(u-u_h,v)=0 \Rightarrow a(e,v)=0 \quad v\in V_h$

Consider φ the solution of $\begin{cases} a(v,\varphi)=(v,e) \\ \varphi|_\Omega=0 \end{cases} \quad -(*)$

$((*)$ is called the adjoint problem$)$

For any $\varphi_h\in V_h$ and $v=u-u_h$, we have

$$a(v,\varphi-\varphi_h)=a(u-u_h,\varphi)+a(u-u_h,\varphi_h)$$

$$=a(u-u_h,\varphi)=\langle u-u_h,e\rangle=a(u-u_h,\varphi-\varphi_h)$$

$$\Rightarrow \|u-u_h\|_{L^2}^2 \overset{\text{Holdering}}{\leq} c\|u-u_h\|_{H^1(\Omega)}\|\varphi-\varphi_h\|_{H^1(\Omega)}$$

$$\leq \tilde{c}h\|u-u_h\|_{H^1}|\varphi|_{H^2(\Omega)} \quad (\text{by Remark}(2))$$

When the adjoint problem$(*)$ has H^2-regularity(for example Ω is convex)

we have $|\varphi|_{H^2(\Omega)}<\|e\|_{L^2(\Omega)}$

Using this inequality, one has

$$\|u-u_h\|_{L^2(\Omega)}^2\leq\tilde{c}h\|u-u_h\|_{H^1(\Omega)}\cdot\|u-u_h\|_{L^2(\Omega)}$$

$$\Rightarrow \|u-u_h\|_{L^2(\Omega)}\leq\tilde{c}h^2|u|_{H^2(\Omega)} \quad (\text{by remark}(2))$$

The above argument is the so-called "duality argument".

Remark(4): Interpolation with poly of higher degree r ≥ 1

$$\|u-\pi_h u\|_{L^2(\Omega)}\leq ch^{r+1}|u|_{H^{r+1}(\Omega)}$$

$$\|u-\pi_h u\|_{H^1(\Omega)}\leq ch^{r}|u|_{H^{r+1}(\Omega)}$$

$$\|u-\pi_h u\|_{H^2(\Omega)}\leq ch^{r-1}|u|_{H^{r+1}(\Omega)}$$

If u doesn't have regularity of H^{r+1}, then suppose $u\in H^s$, $1\leq s\leq r+1$,

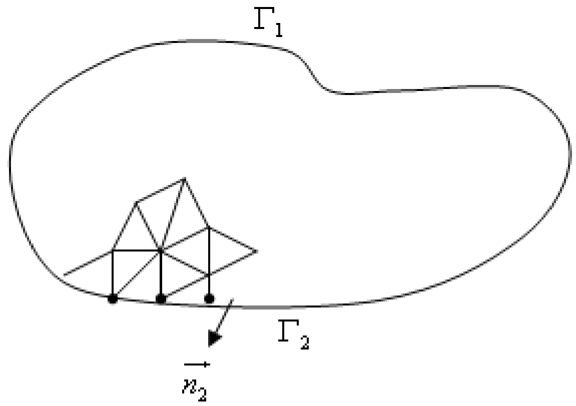
we have $\|u-\pi_h u\|_{L^2(\Omega)}\leq ch^{s}|u|_{H^{s}(\Omega)}$

$$\|u-\pi_h u\|_{H^1(\Omega)}\leq ch^{s-1}|u|_{H^{s}(\Omega)}$$

General FEM Procedure for solving P.D.E

Consider the following P.D.E

$$\begin{cases} -\text{div}(A_{2\times2}\cdot\nabla u)+B\nabla u+cu=f \\ u|\Gamma_1=g; \quad \underset{(\text{normal flux})}{A\nabla u\cdot\vec{n}}=r \end{cases}$$



Step 1: write down the weak formulation

$$\int_\Omega v\left[\left(-\mathrm{div}\left(A\nabla u\right)+B\cdot\nabla u+cu\right)\right]dx=\int_\Omega vfdx \quad \text{,here } v\big|_{\Gamma_1}=0$$

$$\overset{\text{divergence theorem}}{\Rightarrow}\ \underbrace{\int_\Omega \nabla vA\nabla u}_{(I)}+\underbrace{\int_\Omega v\left(B\nabla u\right)}_{(II)}+\underbrace{\int_\Omega cvu}_{(III)}$$

$$=\int_\Omega v\cdot f+\int_{\Gamma_2} v\left(A\nabla u\right)\cdot \underbrace{\vec{n}_2}_{\text{normal vector of }\Gamma_2} \qquad -(8)$$

$$=\underbrace{\int_\Omega v\cdot fdx}_{(IV)}+\underbrace{\int_{\Gamma_2} v\cdot rds}_{(V)}$$

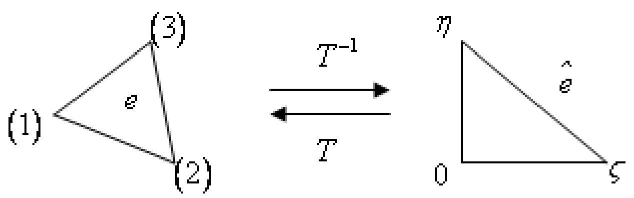
Step 2. FEM discretization

(a) generate mesh J_h for Ω

(b) consider (2)

$$\int_\Omega \nabla vA\nabla udx=\sum_{e\in J_h(\Omega)}\int_e \nabla vA\nabla udx$$

parametrization for e:



$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}^{(1)} + T(\varsigma, \eta)$$

$$= \begin{pmatrix} x \\ y \end{pmatrix}^{(1)} (1 - \varsigma - \eta) + \begin{pmatrix} x \\ y \end{pmatrix}^{(2)} \varsigma + \begin{pmatrix} x \\ y \end{pmatrix}^{(3)} \eta$$

$$= \begin{pmatrix} x \\ y \end{pmatrix}^{(1)} \underbrace{\begin{pmatrix} \Delta_2 x & \Delta_3 x \\ \Delta_2 y & \Delta_3 y \end{pmatrix}}_{=\Phi} \begin{pmatrix} \varsigma \\ \eta \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix}^{(1)}$$

$$\Rightarrow \begin{pmatrix} \varsigma \\ \eta \end{pmatrix} = T^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \Phi^{-1} \begin{pmatrix} x \\ y \end{pmatrix} + \Phi^{-1} \begin{pmatrix} x \\ y \end{pmatrix}^{(1)}$$

We have

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \varsigma} \frac{\partial \varsigma}{\partial x} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \varsigma} \frac{\partial \varsigma}{\partial y} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial y}$$

$$\Rightarrow \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial \varsigma}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \varsigma}{\partial y} & \frac{\partial \eta}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial \varsigma} \\ \frac{\partial u}{\partial \mu} \end{pmatrix} = \left(\Phi^{-1}\right)^T \begin{pmatrix} \frac{\partial u}{\partial \varsigma} \\ \frac{\partial u}{\partial \mu} \end{pmatrix}$$

Let $\Phi = \left(\Phi^{-1}\right)^T$

Choose FEM spaces and discretizatin

$$(c) \int_e \nabla v A_{2\times 2} \nabla u dx = \int_{\hat{e}} \left\lfloor \frac{\partial v}{\partial \varsigma} \quad \frac{\partial v}{\partial \eta} \right\rfloor \Phi^T A(\Phi) \begin{pmatrix} \frac{\partial u}{\partial \varsigma} \\ \frac{\partial u}{\partial \eta} \end{pmatrix} |J| d\varsigma d\eta$$

Let $\phi_1 \sim \phi_m$ be the basis functions on the reference element satisfying $\phi_i(x_j) = \delta_{ij}$ and $v_1 \sim v_m$ be the nodal values

$$v = \sum_{i=1}^{m} v_i \phi_i, \; u = \sum_{j=1}^{m} u_j \phi_j$$

$$\Rightarrow \begin{pmatrix} \frac{\partial u}{\partial \varsigma} \\ \frac{\partial u}{\partial \eta} \end{pmatrix} = \underbrace{\begin{bmatrix} \frac{\partial \phi_1}{\partial \varsigma} & \cdots & \frac{\partial \phi_m}{\partial \varsigma} \\ \frac{\partial \phi_1}{\partial \eta} & \cdots & \frac{\partial \phi_m}{\partial \eta} \end{bmatrix}_{2\times m}}_{D} \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}$$

$$\left\lfloor \frac{\partial v}{\partial \varsigma}, \frac{\partial v}{\partial \eta} \right\rfloor = \lfloor v_1 \cdots \; \cdots \; v_m \rfloor D_{m\times 2}^T$$

$$\int_e \nabla v A \nabla u dx = \int_{\hat{e}} \lfloor v_1 \cdots \ \cdots \ v_m \rfloor D^T \cdot \Phi^T A(\Phi) D |J| \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} dx$$

$$= \overrightarrow{v_e} \cdot K_e \cdot \overrightarrow{u_e} \quad -(9)$$

here $\overrightarrow{v_e} = \lfloor v_1 \cdots \ \cdots \ v_m \rfloor$, $\overrightarrow{u_e} = \lfloor u_1 \cdots \ \cdots u_m \rfloor$,

$$K_e = \int_{\hat{e}} D^T \Phi^T A(\Phi) D |J| d\varsigma d\eta$$ $\leftarrow$ need Numerical Integration

K_e : stiffness matrix.

Notes: Closer look at computing Jacobian $|J|$

$$|J| = \det \begin{pmatrix} \frac{\partial x}{\partial \varsigma} & \frac{\partial y}{\partial \varsigma} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{pmatrix}$$

let $\varphi_1 = 1 - \varsigma - \eta,\ \varphi_2 = \varsigma,\ \varphi_3 = \eta$ (basis of the mapping functions)

one has $x = x_1\varphi_1 + x_2\varphi_2 + x_3\varphi_3$

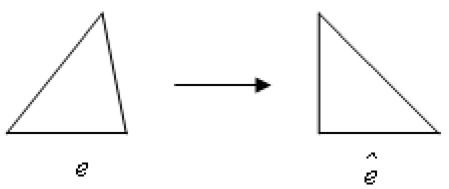
$$y = y_1\varphi_1 + y_2\varphi_2 + y_3\varphi_3$$

$$\Rightarrow \frac{\partial x}{\partial \varsigma} = \sum_{i=1}^{3} x_i \frac{\partial \varphi_i}{\partial \varsigma},\ \frac{\partial y}{\partial \varsigma} = \sum_{i=1}^{3} \frac{\partial \varphi_i}{\partial \varsigma} y_i$$

$$\frac{\partial x}{\partial \eta} = \sum_{i=1}^{3} x_i \frac{\partial \varphi_i}{\partial \eta},\ \frac{\partial y}{\partial \mu} = \sum_{i=1}^{3} \frac{\partial \varphi_i}{\partial \eta} y_i$$

$$\Rightarrow |J| = \begin{vmatrix} \frac{\partial \vec{\varphi}}{\partial \varsigma} \circ \vec{x} & \frac{\partial \vec{\varphi}}{\partial \varsigma} \circ \vec{y} \\ \frac{\partial \vec{\varphi}}{\partial \eta} \circ \vec{x} & \frac{\partial \vec{\varphi}}{\partial \eta} \circ \vec{y} \end{vmatrix} \qquad \begin{aligned} \vec{\varphi} &= \langle \varphi_1, \varphi_2, \varphi_3 \rangle \\ \vec{x} &= \langle x_1, x_2, x_3 \rangle \\ \vec{y} &= \langle y_1, y_2, y_3 \rangle \end{aligned}$$

$\circ$: inner product.

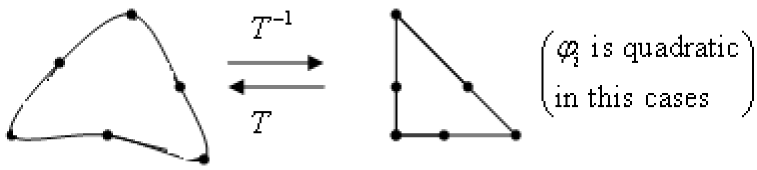


In linear basis function, $|J|$ is a constant.

More general case, let $\varphi_1 \sim \varphi_m$ be basis of the mapping function

$$\begin{pmatrix} x \\ y \end{pmatrix} = \sum_{i=1}^{m} \begin{pmatrix} x_i \\ y_i \end{pmatrix} \varphi_i$$

$$\Rightarrow |J| = \begin{vmatrix} \frac{\partial \vec{\varphi}}{\partial \varsigma} \circ \vec{x} & \frac{\partial \vec{\varphi}}{\partial \varsigma} \circ \vec{y} \\ \frac{\partial \vec{\varphi}}{\partial \eta} \circ \vec{x} & \frac{\partial \vec{\varphi}}{\partial \eta} \circ \vec{y} \end{vmatrix} \rightarrow \text{Jacobian needs to be approximated in numerical integration.}$$



(d) Similar discretization process applied on

$$(II) \int_{\Omega} v(B \cdot \nabla u) = \sum_{e \in J_h} \int_{e} v \underset{\substack{\uparrow \\ (\text{vector})}}{B} \cdot \nabla u$$

$$\int_{e} vB \cdot \nabla u dx = \int_{\hat{e}} \lfloor v_1 \cdots \cdots v_m \rfloor \cdot \underbrace{\begin{bmatrix} \phi_1 \\ \vdots \\ \phi_m \end{bmatrix}_{m \times 1} [B_1, B_2](\Phi) D|J|}_{W_e} \underbrace{\begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}}_{\vec{u_e}} d\varsigma d\eta$$

$$(III) \int_{\Omega} cvudx = \sum_{e \in J_h} \int_{e} cvudx$$

$$\int_{\Omega} cvudx = \int_{\hat{e}} \lfloor v_1 \cdots \cdots v_m \rfloor \underbrace{\begin{pmatrix} \phi_1 \\ \vdots \\ \phi_m \end{pmatrix} (\phi_1 \cdots \cdots \phi_m) |J|}_{M_e (\text{mass matrix})} \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} dx$$

Step 3. Assembling the element matrices K_e, W_e, M_e into global matrices.

$$\sum_{e \in J_h(\Omega)} K_e = K \quad , \quad \sum_{e \in J_h(\Omega)} W_e = W, \quad \sum_{e \in J_h(\Omega)} M_e = M \text{ and } \sum_{e \in J_h(\Omega)} \vec{u}_e = \vec{u}$$

here $\vec{u}$, $\vec{v}$ are the global nodal vector.

The left hand side of (8) is discretized into matrix form

$$\vec{v}(K + W + M)\vec{u} = (I) + (II) + (III) \text{ in } (8)$$

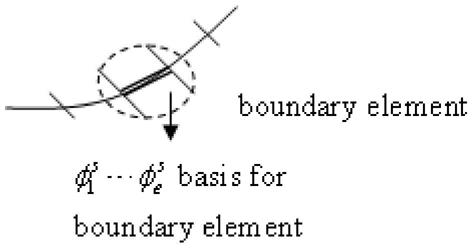
Moreover the righthand side of (8) can also be discretized as

$$(IV)-(8)=\vec{v}\cdot M\cdot\vec{f}$$

$$(V)-(8)=\int_{\Gamma_2} v\cdot r\,ds=\sum_{e\in J_h(\Gamma_2)}\vec{v_e}\cdot \underbrace{M_e^s}_{\text{mass matrix for boundary element}}\ \vec{r_e}$$

$$M_e^s=\int_{\hat{e}}\begin{pmatrix}\phi_1^s\\ \vdots\\ \phi_e^s\end{pmatrix}\left(\phi_1^s\cdots\ \cdots\phi_e^s\right)\underbrace{\left|J^s\right|}_{\text{Jacobian of surface integral}}\ ds=M^s\cdot\vec{r}^{\,s}$$

$$\Rightarrow\vec{v}\left(M\vec{f}+M^s\vec{r}^{\,s}\right)=(IV)+(V)\ \text{in}\ (8)$$



Step 4. Solve the linear system

$$(K+W+M)\vec{u}=M\vec{f}+M^s\vec{r}^{\,s}$$

Example: Given $\Omega=[0,\pi]\times[0,\pi]$, solve

$$\begin{cases}-\Delta u+2u=0\\ u\big|_{\partial\Omega}=0\end{cases}\quad\text{using rectangular 4 nodals element.}$$

where each boundary segment is decomosed into 4,8,16,32 uniform partition.

Compare your answer with the exact solution $u(x,y)=\sin x\sin y$.