Remark: We don't want the error from solving the ODE system
to exceed the error bound in the above theorem.
Otherwise, the error from time discretization will be the
dominant error source.

By Cholesky decomposition, we have $M = L^T L$

$$(4) \Rightarrow \begin{cases} \dot{\psi} + A\psi = F \\ \psi_0 = U_0 \end{cases} \quad \text{here} \begin{cases} \psi = L\psi \\ A = L^{-T} K L^{-1} \\ F = L^{-T} F \\ U_0 = L^{-T} U_0 \end{cases} \quad -(5)$$

$\Rightarrow$ general solution of (5)

$$\psi = e^{-At}\psi_0 + \int_0^t e^{-A(t-s)} F(s)\, ds \quad -(5)^*$$

$$\dot{\psi} = -Ae^{-At}\psi_0$$

$$\psi = e^{-At}\psi + e^{-At}\int_0^t F(s)\, ds + \int_0^t e^{As} F(s)\, ds$$

$$\frac{d}{dt}\int_0^t e^{-A(t-s)} F(s)\, ds = \frac{d}{dt}\int_0^t e^{-At} \cdot e^{As} F(s)\, ds$$

$$= \frac{d}{dt} e^{-At} \int_0^t e^{As} F(s)\, ds$$

$$= -Ae^{-At}\int_0^t e^{As} F(s,d) + e^{-At}\left(e^{At} F(t)\right)$$

$$= -A\int_0^t e^{-A(t-s)} \widetilde{F(s)}\, ds + \widetilde{F(t)}$$

$$\widetilde{\dot{\psi} = -} A \underbrace{\left(e^{-At}\widetilde{\psi_0 + \int_0^t} e^{-A(t-s)} \widetilde{F(s)}\, ds\right)}_{\psi} + \widetilde{F(t)}$$

$$\overset{\substack{\text{Homogeneous} \\ \text{problem } F=0}}{\Rightarrow} \quad \|\psi(t)\| < \|\psi(0)\|$$

$$\left\|\overset{\circ}{\psi}(t)\right\| \underset{(*)}{<} \frac{c}{t}\|\psi(0)\| \qquad c \text{ depends on considiton of } A$$

$(*)$ Prove using $\left(\sum \frac{\Lambda_{\min}^n(A)}{n!} < \left\|e^A\right\| < \sum \frac{\Lambda_{\max}^n(A)}{n!}\right)$

Backward Euler

$$\Rightarrow \frac{\psi^{n+1}-\psi^{n}}{\Delta t_n} A\psi^{n+1} = F_{n+1}\left(=F\left(t_{n+1}\right)\right) \quad -(6)$$

Crank-Nicolson

$$\Rightarrow \frac{\psi^{n+1}-\psi^{n}}{\Delta t_n} + \frac{1}{2}A\left(\psi^{n+1}+\psi^{n}\right) = \frac{1}{2}\left(F_{n+1}+F_{n}\right) \quad -(7)$$

Consider the exact solution of (5) satisfies

$$\frac{\psi\left(t_{n+1}\right)-\psi\left(t_n\right)}{\Delta t_n} + A\psi\left(t_{n+1}\right) = F\left(t_{n+1}\right)+\delta_n \quad \left(\delta_n = o\left(\Delta t_n\right)\right) \ (\text{truncation error}) \ -(8)$$

$$(6)-(8) \Rightarrow \frac{e_{n+1}-e_n}{\Delta t_n} + Ae_{n+1} = \delta_n$$

$$\Rightarrow \left(I+\Delta t_n A\right)^{-1} e_{n+1} = e_n + \Delta t \delta_n$$

$$\Rightarrow e_{n+1} = \left(I+\Delta t_n A\right)^{-1} e_n + \left(I+\Delta t A\right)^{-1} \widetilde{\delta_n \ \left(\delta_n = \Delta t \delta_n\right.} = o\left(\Delta t^2\right)\Big)$$

Since A is positive definite $\Rightarrow \Lambda_{evalue}\left(I+\Delta t A\right) > 1$

$\left(\delta_n = o\left(\Delta t_n\right)\right.$ depends on $\widetilde{\psi''}$ if $\widetilde{\psi'}$ changes smoothly than the constant in $o\left(\Delta t_n\right)$ can be small. $\Big)$

$$\left.\begin{aligned} &e_{n+1} - re_n < re\delta_n \\ &re_n - r^2 e_{n-1} < r^2 \delta_{n-1} \\ &r^2 e_{n-1} - r^3 e_{n-2} < r^3 \delta_{n-2} \\ &r^n e_1 - r^{n+1} e_0 < r^{n+1}\delta \end{aligned}\right\} \Rightarrow e_{n+1} - r^{n+1}e_0 < \left(\sum_{k=1}^{n} r^r\right)\delta = r\left(\frac{1-r^{n+1}}{1-r}\right)\delta$$

$$\Rightarrow \left\|e_{n+1}\right\| < \left(\left\|e_n\right\| + \delta_n\right)\cdot r \qquad r = \frac{1}{1+\|A\|\Delta t} < 1.$$

for stability: one can assume $\delta_n \ll e_n \left(\because \delta_n = o\left(\Delta t^2\right) \text{ and } e_n = o\left(\Delta t\right)\right)$

$\Rightarrow \left\|e_{n+1}\right\| < \left\|e_n\right\|!$

For global error estimate, we have

$$\Rightarrow \left\|e_{n+1}\right\| < r^{n+1}\left\|e_0\right\| + r\left(\frac{1-r^{n+1}}{1-r}\right) \underset{\substack{\uparrow \\ \text{truncation error } o\left(\Delta t^2\right)}}{\widetilde{\delta}}$$

$$\approx \mathrm{r}^{n+1}\left\|e_0\right\| + \frac{1-r^{n+1}}{\|A\|\Delta t}\widetilde{\delta}$$

$$\approx r^{n+1}\left\|e_0\right\| + \left(\frac{1-r^{n+1}}{\|A\|\Delta t}\right)\cdot \underset{\substack{\uparrow \\ \text{truncation } o\left(\Delta t\right)}}{\delta}$$

as $n \to \infty$, we have error $\approx \frac{1}{\|A\|\Delta t}\delta$

$\Rightarrow$ meaning the accumulated error from time discretization is controlled by the truncation error.

Similarly, one can prove the same result for the Crank-Nicolson scheme.

Exercise: prove this.

Remark:

(1) From the error estimation in Theorem 1, we want $\Delta t \approx o(h^2)$ for the Euler method. In this choice of the step size, we have $\delta =$ truncation error (in time) $= o(h^2)$ and $\|A\|\Delta t \approx o(h^{-2})o(h^2) \approx o(1) \Rightarrow$ the accumulated error from the discretization in the same order as in Theorem 1.

(2) For smoother solution $\|A\| \approx o(h^2)$ larger time step is possible, can you explain why?

Exercise: How to choose Δt when forward Euler method is employed and explain your reason at the stability point of view.

Theorem 1.

There is a constant c such that $I \in [0,T]$

$$\max_{t\in I}\|u(t)-u_h(t)\| \le c\left(1+\left|\log\frac{T}{h^2}\right|\right)\max_{t\in I} h^2\|u(t)\|_{H^2(\Omega)}$$

pf: Consider the auxiliary problem $\left(a(u,v)=\int \mu\nabla u\nabla v dx\right)$

$$(++)\begin{cases} -\left(\dot{\varphi}_h(s),v\right)+a\left(\varphi_h(s),v\right)=0 & \forall v\in V_h, s\in(0,t) \\ \varphi_h(t)=e_h(t) \end{cases}$$

let $e_h(t)=u_h(t)-\widetilde{u_h(t)}\left(e_h(s)=u_h(s)-\widetilde{u_h(s)}\right)$ and $\widetilde{u_h(s)}$ satisfies

$(+++)\ a\left(u(s)-\widetilde{u_h,v}\right)=0 \quad$ for all $v\in V_h, s\in(0,t)$

let $w_h(t)=u-u_h=\left(u-\widetilde{u_h}\right)-\left(u_h-\widetilde{u_h}\right)$ and $\theta(s)=u(s)-\widetilde{u_h(s)}$

$\Rightarrow e_h(s)=\theta_h(s)-w_h(s)$

$$\|e_h\|^2 = \int_0^t -\left(\dot{\varphi}_h, e_h\right) + a\left(\varphi_h, e_h\right) ds + \left(\varphi_h(t), e_h(t)\right)$$

$$\overset{\text{integration by parts}}{=} \int_0^t \left(\varphi_h, \dot{e}_h\right) + a\left(\varphi_h, e_h\right) ds - \left(\varphi_h, e_h\right)\Big|_0^t + \left(\varphi_h(t), e_h(t)\right)$$

$$\overset{e_h=\theta_h - w_h}{=} \int_0^t \left(\varphi_h, \dot{\theta}_h\right) + a\left(\varphi_h, \theta_h\right) - \underbrace{\left[\left(\varphi_h, \dot{w}_h\right) + a\left(\varphi_h, w_h\right)\right]}_{\substack{=0 \\ \text{orthogonally of FEM error} \\ w_h = \text{FEM error}}} ds + \varphi_h(0) e_h(0)$$

$$\overset{\substack{a(\varphi_h,\theta_h)=0 \text{ by}(+++) \\ \text{and } w_h(0)=0 \\ (\text{initial error}=0)}}{=} \int_0^t \left(\dot{\varphi}_h, \theta_h\right) ds + \left(\varphi_h(t), \theta_h(t)\right)$$

$$\Rightarrow \|e_h\|^2 \le \max_{s\in(0,t)} \|\theta_h(s)\| \left(\int_0^t \left\|\dot{\varphi}_h\right\| + \|\varphi_h\|\right)$$

Since φ_h is the condition of $(++)$ with homogeneous right hand side,

by $(5)^*$, it can be shown $\left|\int_0^t \left\|\dot{\varphi}_h\right\| + \frac{1}{t}\|\varphi_h\| ds\right| < c\left(1 + \log\left(\frac{t}{h^2}\right)\right) \underset{\substack{\uparrow \\ \text{initial of } (++)}}{\|e_h(t)\|}$

$$\Rightarrow \|e_h\| < c\left(1 + \log\frac{t}{h^2}\right) \cdot \max_{s\in(0,t)} \|\theta_h(s)\|$$

Finally, since $u - u_h = u - \widetilde{u_h + u_h} - u_h$

$$\Rightarrow \|u - u_h\| \le \left\|u - \widetilde{u_h}\right\| + \|e_h\|$$

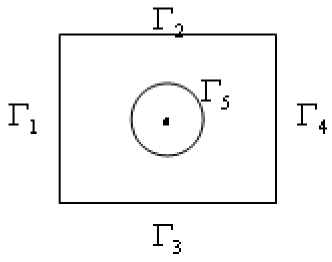
$$\overset{\substack{\text{(by the } L^2\text{-error} \\ \text{estimation of } (+++))}}{\le} ch^2 \max_{t\in I} \|u(t)\|_{H^2} + c\left(1 + \log\frac{t}{h^2}\right) h^2 \max_{t\in I} \|u(t)\|_{H^2}$$

$$< c\left(1 + \log\frac{t}{h^2}\right) h^2 \max_{t\in I} \|u(t)\|_{H^2}$$

Exercise:

Solve the Heat equation
$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = 0 \quad \text{on } \Omega = [-1,1]\times[-1,1] \setminus C, \\ \qquad\qquad C: \text{circle center at } (0,0) \text{ wiht radius } \dfrac{1}{2} \\ u\big|_{\Gamma_1} = 10, \ u\big|_{\Gamma_5} = 0 \\ \dfrac{\partial u}{\partial n}\Big|_{\Gamma_i} = 0, \ i = 2 \sim 4 \end{cases}$$

with initial $u(x,y) = \begin{cases} 10 & x = -1 \\ 0 & \text{elsewise} \end{cases}$



(1) using Forward Euler, back Euler, Crank-Nicolson method for

time-discretization of $\frac{\partial u}{\partial t}$.

(2) using linear triangular element to discretize Δu.

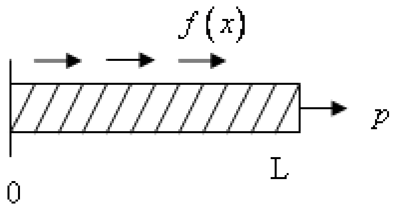
(3) compare your answer at t $=1,5,10,20$ over diff mesh sizes.

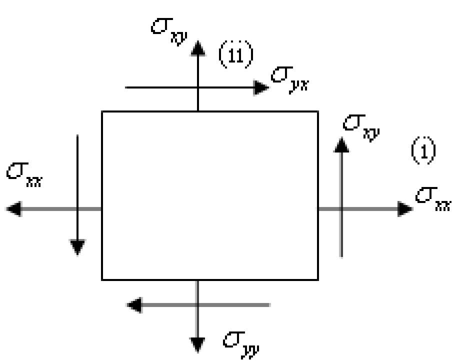
Can you confine the conclusion in Theorem 1?

Recall the FEM modeling of 1-D solid (bar, beam, fream) and structure,
the virtural work principle leads to the following equation:

$$(*)\int_0^L \underbrace{\underset{\substack{\uparrow \\ \text{virtual strain}}}{\frac{\partial \delta u}{\partial x}}\ \underbrace{\underset{\substack{\uparrow \\ \text{Hooke's constant}}}{E}\ \underbrace{\frac{\partial}{\partial x}\underset{\substack{\uparrow \\ \text{(diplacement)}}}{u}}_{\text{strain}}\cdot \underset{\text{(area of cross section)}}{A(x)}}_{\text{stress}}}_{\text{interal virtual work}}\, dx = \underbrace{\delta u(L)\cdot p+\int_0^L \delta u(x)\, f(x)\,dx}_{\text{external virtual work}}$$

(obtained from th stress-strain relationship $\sigma_{xx}=E\varepsilon_{xx}$)





In 2-D, the stress-strain relationship for a plate is as following

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix} = \underbrace{\frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}}_{C} \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{pmatrix}, \text{ here}$$

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}$$

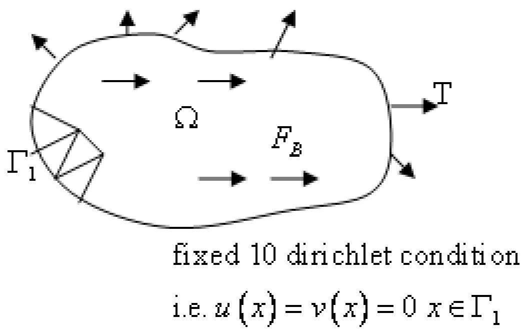
$$\varepsilon_{yy} = \frac{\partial v}{\partial y} \qquad , \ (u,v) = \text{displacement vector}$$

$$\varepsilon_{xy} = \underbrace{\frac{\partial u}{\partial y}}_{\substack{\text{shear stram:stram} \\ \text{in } v \text{ along } y \text{ direction}}} + \frac{\partial v}{\partial x}$$

σ_{xx} : stress along x-direction (due to strain in x-direction)

σ_{yy} : stress along y-direction (due to strain in y-direction)

σ_{xx} : stress along y-direction due to shear strain in x-direction (i) + stress along x-direction due to shear strain in y-direction (ii) } shear stress



the equation $(*)$

$$\Rightarrow \underbrace{\int_{\Omega} \lfloor \delta\varepsilon_{xx}, \delta\varepsilon_{yy}, \delta\varepsilon_{xy} \rfloor \cdot C \cdot \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{pmatrix} dA}_{(\text{I})} = \underbrace{\int_{\Omega} \lfloor \delta u, \delta v \rfloor \cdot \overrightarrow{F_B} dA}_{(\text{III})} + \underbrace{\int_{\partial\Omega} \lfloor \delta u, \delta v \rfloor \cdot \vec{T} de}_{(\text{II})}$$

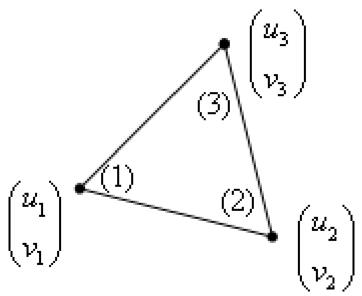
$$\Rightarrow (\text{I}) = \int \lfloor \delta u, \delta v \rfloor \underbrace{\begin{bmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \cdot C \cdot \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}}_{\nwarrow \text{互為transport} \nearrow} \begin{pmatrix} u \\ v \end{pmatrix} dA$$

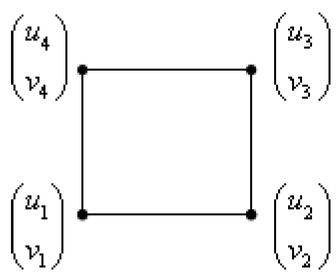
parametrizing:

$$\text{Consider } \begin{pmatrix} \frac{\partial u}{\partial \zeta} \\ \frac{\partial u}{\partial \eta} \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{pmatrix}}_{J} \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} = \frac{1}{\det J} \begin{bmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial y}{\partial \zeta} \\ -\frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta} \end{bmatrix} \begin{pmatrix} \frac{\partial u}{\partial \zeta} \\ \frac{\partial u}{\partial \eta} \end{pmatrix} = \underbrace{\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}}_{\left(\substack{\text{Y can be determind by} \\ \text{the map functions by} \\ \text{considering } \binom{x}{y} = \sum_i \binom{x}{y}_i \psi_i}\right)}$$

$$\Rightarrow \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{pmatrix} = \underbrace{\begin{bmatrix} Y_{11} & Y_{12} & 0 & 0 \\ 0 & 0 & Y_{21} & Y_{22} \\ Y_{21} & Y_{22} & Y_{11} & Y_{12} \end{bmatrix}}_{\Phi} \begin{pmatrix} \frac{\partial u}{\partial \zeta} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \zeta} \\ \frac{\partial v}{\partial \eta} \end{pmatrix}$$





Choose FEM space $\left(\text{Suppose linear element}\right)$

$u^{(i)} = \sum_i N_i u_i^{(i)},\quad N_1 \sim N_4$ are basis functions $\left(\text{shape}\right)$

$$\Rightarrow \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} \end{pmatrix} = \Phi_{3\times 4}\cdot \underbrace{\begin{bmatrix} \frac{\partial N_1}{\partial \zeta} & 0 & \frac{\partial N_2}{\partial \zeta} & 0 & \frac{\partial N_3}{\partial \zeta} & 0 & \frac{\partial N_4}{\partial \zeta} & 0 \\ \frac{\partial N_1}{\partial \eta} & 0 & \frac{\partial N_2}{\partial \eta} & 0 & \frac{\partial N_3}{\partial \eta} & 0 & \frac{\partial N_4}{\partial \eta} & 0 \\ 0 & \frac{\partial N_1}{\partial \zeta} & 0 & \frac{\partial N_2}{\partial \zeta} & 0 & \frac{\partial N_3}{\partial \zeta} & 0 & \frac{\partial N_4}{\partial \zeta} \\ 0 & \frac{\partial N_1}{\partial \eta} & 0 & \frac{\partial N_2}{\partial \eta} & 0 & \frac{\partial N_3}{\partial \eta} & 0 & \frac{\partial N_4}{\partial \eta} \end{bmatrix}_{4\times 8}}_{D} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix}$$

$$\Rightarrow (\mathrm{I}) = \underbrace{\sum_{\tau\in J_h} \int_{\hat{\tau}}}_{\substack{\text{FEM assembling} \\ \text{over triangles or} \\ \text{quadriletral(rectangle)}}} \lfloor \delta u_1 \delta v_1 \cdots \delta u_4 \delta v_4 \rfloor \cdot \underbrace{D^* \cdot \Phi^* \cdot C \cdot \Phi D \cdot |J|}_{K^e} \begin{bmatrix} u_1 \\ v_1 \\ \vdots \\ u_4 \\ v_4 \end{bmatrix} d\zeta\, d\eta$$

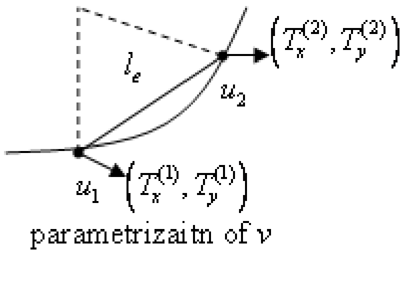
$$(\mathrm{II})\,\partial\Omega = \bigcup_{e\in\partial\Omega} \int_{\partial\Omega} \lfloor \delta u, \delta v \rfloor \cdot \vec{T} de = \sum_{\substack{e\in J(\partial\Omega) \\ \uparrow}} \int \lfloor \delta u, \delta v \rfloor \cdot \overrightarrow{T^e} de$$

assembling over boundary line segments

$$\int_e \lfloor \delta u, \delta v \rfloor \cdot \begin{pmatrix} T_x^e \\ T_y^e \end{pmatrix} ds = \int_{\hat{e}} \lfloor \delta u_1 \delta v_1 \delta u_2 \delta v_2 \rfloor \begin{pmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \end{pmatrix} \cdot \begin{bmatrix} N_1 & 0 & N_2 & 0 \\ 0 & N_1 & 0 & N_2 \end{bmatrix} \begin{pmatrix} T_x^{(1)} \\ T_y^{(1)} \\ T_x^{(2)} \\ T_y^{(2)} \end{pmatrix} \cdot |e|\, ds$$

$$= \lfloor \overrightarrow{\delta u} \rfloor \int_0^1 \begin{pmatrix} N_1^2 & 0 & N_1 N_2 & 0 \\ 0 & N_1^2 & 0 & N_1 N_2 \\ N_2 N_1 & 0 & N_2^2 & 0 \\ 0 & N_2 N_1 & 0 & N_2^2 \end{pmatrix} |e|\, ds \quad N_1 = 1-s, N_2 = s$$

(III) Skip when zero body force



parametrizaitn of v

Exercise:

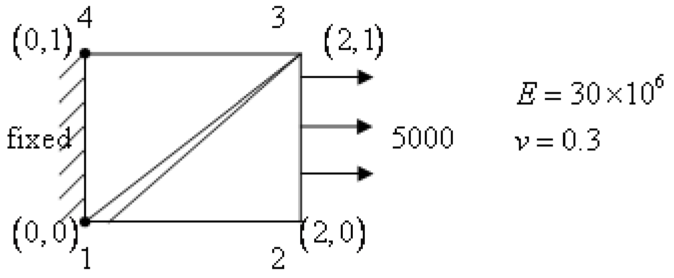
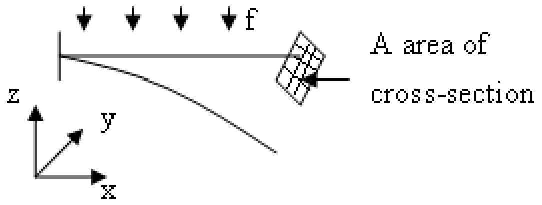


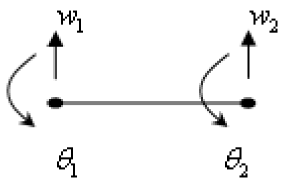
Plate Bending:

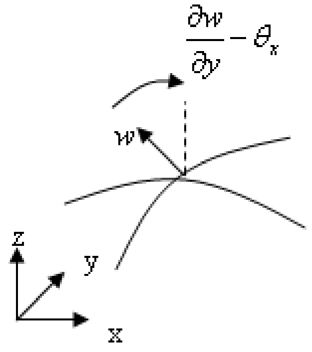
Recall in 1-D Beam bending, the virtual work principle leads to

$$\int_0^L EI \frac{\partial^2 \delta w}{\partial x^2}\frac{\partial^2 w}{\partial x^2} dx + \int_0^L v \overline{f} dx = 0 \quad \overline{f} = \int_A f dA,\ I = \int_A z^2 dydz$$

The state varibles are w, θ where $\theta = -\frac{\partial w}{\partial x}$ and the shap functions are $N_{w_1}, N_{\theta_1}, N_{w_2}, N_{\theta_2}$







In 2-D, similarly, we have the strain-stress relationship

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \\ \sigma_{zz} \\ \sigma_{zx} \\ \sigma_{zy} \end{pmatrix} = C \cdot \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \\ \varepsilon_{zz} \\ \varepsilon_{zx} \\ \varepsilon_{zy} \end{pmatrix} \qquad \begin{aligned} \varepsilon_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \varepsilon_{zx} &= \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \\ \varepsilon_{zy} &= \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \end{aligned}$$

here we assume $\begin{cases} \varepsilon_{xz} \text{ very small} \\ \varepsilon_{yz} \text{ very small} \quad (\text{thin plate}) \\ \varepsilon_{zz} \text{ very small} \end{cases}$

Kirchhoff theory assume $\begin{matrix} \varepsilon_{xz} = \varepsilon_{yz} = 0 \\ \frac{\partial u}{\partial z} = -\frac{\partial w}{\partial x}, \ \frac{\partial v}{\partial z} = -\frac{\partial w}{\partial y} \end{matrix} \Rightarrow \begin{matrix} \theta_y = -\frac{\partial w}{\partial x} \\ \theta_x = -\frac{\partial w}{\partial y} \end{matrix}$

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \overset{\left(u \approx z \cdot \frac{\partial u}{\partial z} = z\left(\frac{-\partial w}{\partial x}\right)\right)}{=} -z\frac{\partial \theta_y}{\partial x} = -z\frac{\partial^2 w}{\partial x^2}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} \overset{\left(v \approx z \cdot \frac{\partial v}{\partial z} = z\left(\frac{-\partial w}{\partial y}\right)\right)}{=} -z\frac{\partial \theta_x}{\partial y} = -z\frac{\partial^2 w}{\partial y^2} \quad \Rightarrow \text{consider state varibles} \begin{pmatrix} w \\ \theta_x \\ \theta_y \end{pmatrix}$$

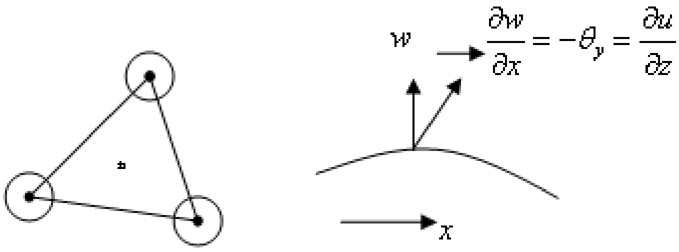
$$\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -2z\frac{\partial^2 w}{\partial x \partial y}$$

$\Rightarrow$ Total number of unknowns over a triangle 9 (rectanguler) 12

$\Rightarrow$ The strain Energy (internal energy)

$$U = \iint \left(\delta\varepsilon_{xx}, \delta\varepsilon_{yy}, \delta\varepsilon_{xy}\right) \cdot C \cdot \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{pmatrix} dA \quad -(*)$$

here $C = \begin{bmatrix} D & vD & 0 \\ vD & D & 0 \\ 0 & 0 & \frac{D(1-v)}{2} \end{bmatrix}$ where $D = \frac{t^3}{12}\frac{E}{1-v^2}$, t is the thickness of the plate.



To discretize $(*)$, one can use the finite element space in example 6 at p.58 which is the piecewise cubic polynomial spaces.
(with 10 coefficients for each polynomial)
Since we only have 9 state variables $\Rightarrow$ the state varibles can't be determined uniquely. By ignoring the center node $\Rightarrow$ Gives us an "incomplete" cubic polynomial space. Now we have 9 nodal basis $\phi_1\ \phi_2\ \phi_3,\ \phi_1^x\ \phi_2^x\ \phi_3^x,\ \phi_1^y\ \phi_2^y\ \phi_3^y$.
The coefficients can now be determined uniquely.
This special Finite element is called the BCIE element.
(Bazeley, Cheung, Irons and Zienkiewicz)

$$\Rightarrow U = \iint \begin{bmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \cdot C \cdot \underbrace{\begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{pmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \end{pmatrix}}_{D} dxdy = \sum_{\tau \in J_h} \iint_{\hat{\tau}}$$

$$\begin{pmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{pmatrix} \begin{pmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \end{pmatrix} = \Phi_{3\times 4} \cdot \begin{bmatrix} \frac{\partial}{\partial \zeta} & 0 \\ \frac{\partial}{\partial \eta} & 0 \\ 0 & \frac{\partial}{\partial \zeta} \\ 0 & \frac{\partial}{\partial \eta} \end{bmatrix}_{4\times 2} \begin{bmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \end{bmatrix}$$

$$= \underbrace{\Phi_{3\times 4} \cdot \begin{bmatrix} \frac{\partial}{\partial \zeta} & 0 \\ \frac{\partial}{\partial \eta} & 0 \\ 0 & \frac{\partial}{\partial \zeta} \\ 0 & \frac{\partial}{\partial \eta} \end{bmatrix} \cdot \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}}_{B} \cdot \begin{bmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \end{bmatrix}$$

$$= B\cdot\begin{pmatrix}\dfrac{\partial}{\partial\zeta}\sum_{i=1}^{9}w_i\phi_i\\ \dfrac{\partial}{\partial\eta}\sum_{i=1}^{9}w_i\phi_i\end{pmatrix} = \begin{pmatrix}\dfrac{\partial}{\partial\zeta}\phi_1 & \cdots & \dfrac{\partial}{\partial\zeta}\phi_9\\ \dfrac{\partial}{\partial\eta}\phi_1 & \cdots & \dfrac{\partial}{\partial\eta}\phi_9\end{pmatrix}\begin{pmatrix}w_1\\ \vdots\\ w_9\end{pmatrix} = B\cdot D$$

$$= \underbrace{\Phi_{3\times4}\underbrace{\begin{bmatrix}Y_{11} & 0 & Y_{12} & 0\\ 0 & Y_{11} & 0 & Y_{12}\\ Y_{21} & 0 & Y_{22} & 0\\ 0 & Y_{21} & 0 & Y_{22}\end{bmatrix}}_{Y}}_{\widetilde{\Phi}}\underbrace{\begin{bmatrix}\dfrac{\partial}{\partial\zeta} & 0\\ \dfrac{\partial}{\partial\eta} & 0\\ 0 & \dfrac{\partial}{\partial\zeta}\\ 0 & \dfrac{\partial}{\partial\eta}\end{bmatrix}\cdot\begin{pmatrix}\dfrac{\partial}{\partial\zeta}\\ \dfrac{\partial}{\partial\eta}\end{pmatrix}}\cdot[\phi_1\cdots\phi_9]$$

$$\Vert$$

$$\underbrace{\begin{bmatrix}\dfrac{\partial^2}{\partial\zeta^2}\\ \dfrac{\partial}{\partial\zeta}\dfrac{\partial}{\partial\eta}\\ \dfrac{\partial}{\partial\eta}\dfrac{\partial}{\partial\zeta}\\ \dfrac{\partial^2}{\partial\eta^2}\end{bmatrix}[\phi_1\cdots\phi_9]}_{D}$$

$$\Rightarrow U = \sum_{\tau\in J_h}\iint_{\hat{\tau}}\delta\vec{w}\underbrace{D^*\overset{*}{\widetilde{\Phi}}C\widetilde{\Phi}D}_{K^e}\cdot\vec{w}$$

Remark:

(1) This element does not satisfy the conforming property

$\Big($i.e. the state variables are not continuous $\Rightarrow$ in fact the state variables may not continuous at element edges.$\Big)$

but the state variables are continuous at nodal points.

(2) To ensure continuity of the state variables (i.e ensure FEM space is C′ in this case) conforming elements such as Argyris triangle or the so called Clough-Tocker element are needed. More unknown coefficients (involving second derivatives) are required

$\Big($Agyris triangle: 21 unknowns
Clough-Tocker: 30 unknowns $\Rightarrow$ Program usually is too complicated.