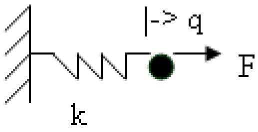
Finite Element Method



(A) A SDOF system

k : spring constant (measurement of stiffness)

F : force

q : displacement of DOF (degree of freedom)

(1) $k \cdot g = F$ (equilibrium)

(B) For MDOF system (suppose n degree of freedom)

n : # of DOF

$K_{n\times n}$: stiffness matrix

$\vec{F}_{n\times 1}$: load vector (external force)

$$\vec{q}_{n\times 1} = \begin{Bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{Bmatrix} = \text{DOF vector or displacement vector}$$

(2) $K \cdot \vec{q} = \vec{F}$ (equilibrium)

Physical structure $(A, B) \underset{\text{FEM}}{\Rightarrow} Kq = F$

Remark : In general, the equilibrium is described by PDEs, FEM is a powerful method for discretization of PDEs.

FEM modeling of solids and structures can be carried out by the virtual work principle which is directly link to the energy variation of the associated PDEs.

Virtual Work Principle $\equiv$ Variational Principle

Consider

$$Kg - F = 0 \quad ---- \; (3) \quad (\text{force balance})$$

By introducing arbitrary displacement $\delta q \neq 0$ called virtual displacement and multiplying δq to eq.(3). We have

$$\delta q(Kq - F) = 0 \quad ----- (4) \quad (\text{energy balance})$$

Observation:

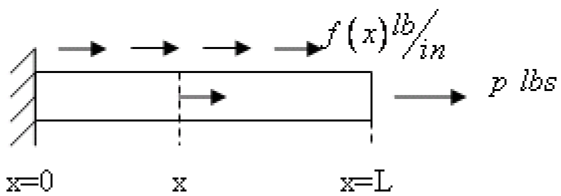
(i) (3) and (4) are equivalent. (ii) $\begin{cases} (3) \text{ is in vectorial form.} \\ (4) \text{ is in scalar form.} \end{cases}$

Note:

$\delta q \cdot F$ is called the external virtual work.

$\delta q \cdot K \cdot q$ is called the internal virtual work.

Example: A bar under axial force

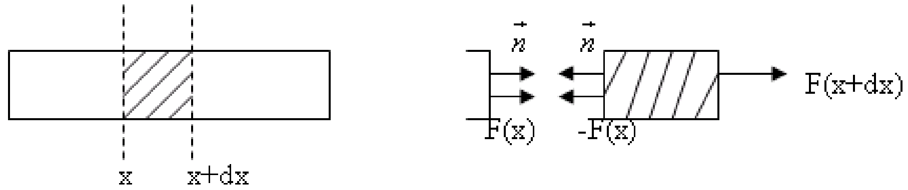


$u(x)$: actual displacement

$f(x)$: applied force per unit length

$A(x)$: cross-sectional area

p: applied force at $x = L$

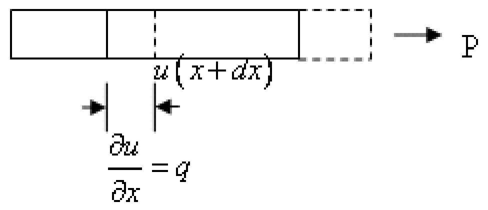


By the mean-value theorem, We have $F(x+dx) - F(x) + f(x+\varepsilon)dx = 0$

$$\Rightarrow \left[\frac{\partial F}{\partial x}(x) + f(x+\varepsilon) \right] dx + \mathrm{O}(dx^2) = 0$$

Let $dx \to 0,\ \varepsilon \to 0$

$$\Rightarrow \frac{\partial F}{\partial x} + f(x) = 0 \quad ----(5) \quad \begin{pmatrix} \text{internal force change} & + & \text{external force} & \\ \partial F / \partial x & + & f & = 0 \end{pmatrix}$$



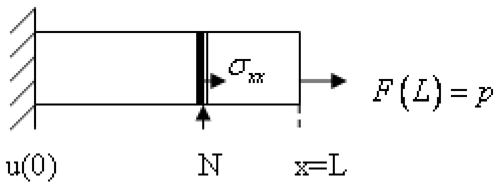
Let σ_{xx} be the inner stress. Let ε_{xx} be the strain satisfies $\varepsilon_{xx} = \frac{\partial}{\partial x} u(x)$

Hooke's law gives the Stress-Strain relationship:

$$\sigma_{xx} = E\varepsilon_{xx} \quad ----(6)$$

Moreover, the actual force is

$$F(x) = \int_s \sigma_{xx} dA \underset{\substack{\text{assume } \sigma_{xx} \\ \text{constant long} \\ \text{cross-section}}}{=} \sigma_{xx} A(x)$$



$$(5) \quad \Rightarrow \quad \frac{\partial}{\partial x}\left(\sigma_{xx}(x) A(x)\right) + f = 0 \quad -----(7) \quad (\text{PDE}).$$

To solve (7), one requires boundary conditions:

boundary conditions $\begin{cases} u(0) = 0 \\ F(L) = \sigma_{xx}(L) A(L) = p \end{cases}$, in this example.

Now, one can rewrite (7) in weak formulation by multiplying virtual strain δu to (7) as following:

$$\int_0^L \delta u\left(\frac{\partial}{\partial x}\sigma_{xx}A(x)\right)+\int_0^L \delta uf(x)dx=0$$

$$\underset{\text{by part}}{\overset{\text{integration}}{\Rightarrow}}$$

$$\int_0^L \left(\frac{\partial \delta u}{\partial x}\right)\sigma_{xx}A(x)dx=\delta u(x)\sigma_{xx}A(x)\Big|_{x=L}-\delta q(x)\sigma_{xx}A(x)\Big|_{x=0}+\int_0^L \delta u(x)f(x)dx$$

Let $\delta u(0)=0$ and apply the boundary condition, we have

$$(6)\Rightarrow \underbrace{\int_0^L \frac{\partial \delta u}{\partial x}E\frac{\partial}{\partial x}u(x)\cdot A(x)dx}_{\text{internal work }\delta U}=\underbrace{\delta u(L)\cdot p+\int_0^L \delta u(x)f(x)dx}_{\text{external work }\delta W}\quad ----(8)$$

Observation: (1) the unknown is $u(x)$

(2) the left hand side is a bilinear operator in variables δu and u.

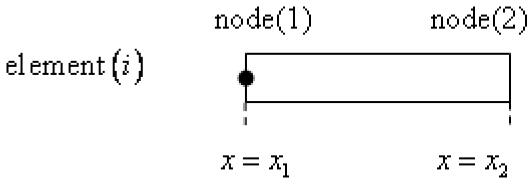
Finite element modeling:

Idea: Divide the bar into segments called "element"

$$(8)\quad\Rightarrow\quad \left.\begin{aligned}\delta U&=\sum_{i=1}^{3}\int_{(i)} EA\frac{\partial \delta u}{\partial x}\frac{\partial u}{\partial x}dx\\ \delta W&=\sum_{i=1}^{3}\int_{(i)} f\delta u dx+\delta u\cdot p\end{aligned}\right\}\quad -----(9)$$

Step1: Choose a FEM space:

Assume $u(x)$ is linear within each element.



Introducing non-dimensional local coordinates "s"
defined such that

$$\begin{matrix} s = 0 \text{ at node } (1) \\ s = 1 \text{ at node } (2) \end{matrix} \Rightarrow s = \frac{x - x_1}{x_2 - x_1} \Rightarrow x = (1-s)x_1 + sx_2$$

Let $N_1(s) = 1 - s,\ N_2(s) = s$ (called the mapping function)
We have

$$x = N_1(s)x_1 + N_2(x)x_2 \quad -----(10)$$

$$\Rightarrow dx = (x_2 - x_1)ds = l \cdot ds \quad -----(11) \quad (l = \text{element length})$$

Moreover, since u is linear, one has $u = c_1 + c_2 x$ (or $u = a_1 + a_2 s$ in the reference coordinate s)

Suppose $\begin{cases} u = u_1 \text{ at node } (1) \\ u = u_2 \text{ at node } (2) \end{cases}$, we have $a_1 = u_1,\ a_2 = u_2 - u_1$

$$\Rightarrow u = (1-s)u_1 + su_2$$

rewrite

$u = \tilde{N}_1 u_1 + \tilde{N}_2 u_2$ where $\tilde{N}_1 = 1 - s,\ \tilde{N}_2 = s$ are called the shap functions $-----(12)$

Remark : In case, mapping functions $\equiv$ shap functions, the FEM formulation is called "isoparametirc formulation".

Step2: Discretization over each element

For $\delta u_{(i)} = \int_{(i)} EA \frac{\partial \delta u}{\partial x} \cdot \frac{\partial u}{\partial x} dx$

Since

(i) $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial x} = \frac{1}{l}(-u_1 + u_2) = \frac{1}{l}[-1,1]\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

(ii) For arbitrary δu, choose $\delta u = (1-s)\delta u_1 + s\delta u_2$ (the so called Galerkin formulation)

$$\frac{\partial \delta u}{\partial x} = \frac{1}{l}(-\delta u_1 + \delta u_2) = \frac{1}{l}(\delta u_1, \delta u_2)\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\begin{aligned}
\Rightarrow \quad \delta u_{(i)} &= \int_{x_1}^{x_2} EA \frac{\partial \delta u}{\partial x} \frac{\partial u}{\partial x} dx \\
&\underset{\substack{\text{change varible} \\ x \to s}}{=} \int_0^1 EA \frac{1}{l^2} (\delta u_1, \delta u_2)\begin{pmatrix} -1 \\ 1 \end{pmatrix}(-1,1)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \cdot l ds \\
&= [\delta u_1, \delta u_2]\left(\frac{1}{l}\int_0^1 EA ds\right) \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\
&= [\delta u_1, \delta u_2] K^{(i)} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},
\end{aligned}$$

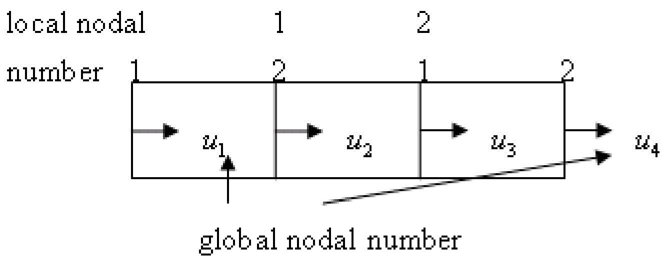
here $K^{(i)} = \frac{1}{l}\int_0^1 EA ds \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ is called the element stiffness and $K^{(i)}$ is symmetric.

For $\delta W_{(i)} = \int_{(i)} f \delta u dx = \int_0^1 \left((1-s)\delta u_1 + s\delta u_2\right) \cdot f(s) \cdot l ds$

$$= [\delta u_1, \delta u_2]\begin{pmatrix} \int_0^1 (1-s) f(s) l ds \\ \int_0^1 s f(s) l ds \end{pmatrix} = [\delta u_1, \delta u_2]\begin{pmatrix} Q_1^{(i)} \\ Q_2^{(i)} \end{pmatrix},$$

here $\begin{pmatrix} Q_1^{(i)} \\ Q_2^{(i)} \end{pmatrix}$ is called the element load vector.

Step3: Summing or assembling element vectors and matrices



	local nodal number	global nodal number
element 1:	$u_1 \rightarrow u_1$	$\delta u_1 \rightarrow \delta u_1$
	$u_2 \rightarrow u_2$	$\delta u_2 \rightarrow \delta u_2$
element 2:	$u_1 \rightarrow u_2$	$\delta u_1 \rightarrow \delta u_2$
	$u_2 \rightarrow u_3$	$\delta u_2 \rightarrow \delta u_3$
element 3:	$u_1 \rightarrow u_3$	$\delta u_1 \rightarrow \delta u_3$
	$u_2 \rightarrow u_4$	$\delta u_2 \rightarrow \delta u_4$

$$\Rightarrow \begin{matrix} \text{elem\#1} \\ \cdots \\ \text{elem\#2} \\ \cdots \\ \text{elem\#3} \\ \cdots \end{matrix} \begin{pmatrix} 1 \\ 2 \\ \overline{2} \\ 3 \\ \overline{3} \\ 4 \end{pmatrix}$$

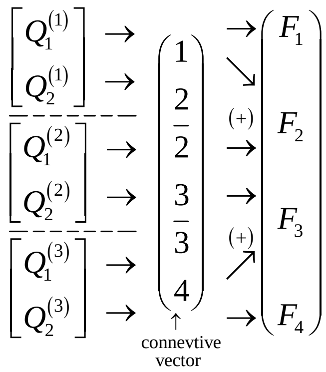
This vector array contains global dof corresponding to element dof is called "connectivity vector"

(i) Assembly of the global load vector

$$\delta W = \sum_{i=1}^{3} \delta W_i = \sum_{i=1}^{3} \left[\delta u_1^{(i)}, \delta u_2^{(i)} \right] \begin{bmatrix} Q_1^{(i)} \\ Q_2^{(i)} \end{bmatrix}$$

$$= \left[\delta u_1, \delta u_2 \right] \begin{bmatrix} Q_1^{(1)} \\ Q_2^{(1)} \end{bmatrix} + \left[\delta u_2, \delta u_3 \right] \begin{bmatrix} Q_1^{(2)} \\ Q_2^{(2)} \end{bmatrix} + \left[\delta u_3, \delta u_4 \right] \begin{bmatrix} Q_1^{(3)} \\ Q_2^{(3)} \end{bmatrix}$$

$$= \left[\delta u_1, \delta u_2, \delta u_3, \delta u_4 \right] \begin{bmatrix} Q_1^{(1)} \\ Q_2^{(1)} + Q_1^{(2)} \\ Q_2^{(2)} + Q_1^{(3)} \\ Q_2^{(3)} \end{bmatrix} = \overrightarrow{\delta q} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} \nwarrow \text{global load vector} \qquad -----(13)$$



(ii) Assembly of the global stiffness matrix

$$\delta U = \sum_{i=1}^{3} \delta U_i = \sum_{i=1}^{3} \left[\delta u_1^{(i)}, \delta u_2^{(i)} \right] \begin{bmatrix} K_{11}^{(i)} & K_{12}^{(i)} \\ K_{21}^{(i)} & K_{22}^{(i)} \end{bmatrix} \begin{pmatrix} u_1^{(i)} \\ u_2^{(i)} \end{pmatrix}$$

$$= \sum_{i=1}^{3} \sum_{j,k=1}^{2} K_{j,k}^{(i)} \delta u_j^{(i)} u_k^{(i)} \quad ----(A)$$

$$\overset{\text{after}}{\underset{\text{assenbly}}{=}} \left[\delta u_1, \delta u_2, \delta u_3, \delta u_4 \right] \begin{bmatrix} K_{11} & \cdots & K_{14} \\ \vdots & \ddots & \vdots \\ K_{41} & \cdots & K_{44} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

$$= \sum_{i,j=1}^{4} K_{ij} \delta u_i u_j \quad ----(B)$$

Compare $(A)\&(B)$ term by term through using the connective vector, we have

element #1 $\begin{cases} K_{1,1}^{(1)} \delta u_1^{(1)} u_1^{(1)} \to K_{1,1} \delta u_1 \cdot u_1 \\ K_{1,2}^{(1)} \delta u_1^{(1)} u_2^{(1)} \to K_{1,2} \delta u_1 \cdot u_2 \\ K_{2,1}^{(1)} \delta u_2^{(1)} u_1^{(1)} \to K_{2,1} \delta u_2 \cdot u_1 \\ K_{2,2}^{(1)} \delta u_2^{(1)} u_2^{(1)} \to K_{2,2} \delta u_2 \cdot u_2 \end{cases}$, etc,

element#2 $\begin{cases} K_{1,1}^{(2)} \delta u_1^{(2)} u_1^{(2)} \to K_{2,2} \delta u_2 \cdot u_2 \\ K_{1,2}^{(2)} \delta u_1^{(2)} u_2^{(2)} \to K_{2,3} \delta u_2 \cdot u_3 \\ K_{2,1}^{(2)} \delta u_2^{(2)} u_1^{(2)} \to K_{3,2} \delta u_3 \cdot u_2 \\ K_{2,2}^{(2)} \delta u_2^{(2)} u_2^{(2)} \to K_{3,3} \delta u_3 \cdot u_3 \end{cases}$

$$\text{Assembling} \Rightarrow K_{4\times 4} = \left[\begin{array}{c|c|c|c} K_{1,1}^{(1)} & K_{1,2}^{(1)} & 0 & 0 \\ \hline K_{1,2}^{(1)} & K_{2,2}^{(1)} + K_{1,1}^{(2)} & K_{1,2}^{(2)} & 0 \\ \hline 0 & K_{2,1}^{(2)} & K_{2,2}^{(2)} + K_{1,1}^{(3)} & K_{1,2}^{(3)} \\ \hline 0 & 0 & K_{2,1}^{(3)} & K_{2,2}^{(3)} \end{array}\right]$$

$$\Rightarrow \delta U = \left[\delta u_1, \delta u_2, \delta u_3, \delta u_4\right] \cdot K_{4\times 4} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \quad -(14)$$

Together with $(8),(9),(13)$ and (14), we have

$$\left[\delta u_1, \delta u_2, \delta u_3, \delta u_4\right] \cdot K_{4\times 4} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \left[\delta u_1, \delta u_2, \delta u_3, \delta u_4\right] \left(\begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right)$$

$$\underset{\substack{\text{boundary} \\ \text{condition} \\ u(0)=0 \\ (\text{assume } p=0)}}{\overset{\text{apply}}{\Rightarrow}} \qquad \overrightarrow{\delta u} \cdot \left(K \vec{u} - \vec{F}\right) = 0 \quad -(15)$$

The discrete system can now be solved.