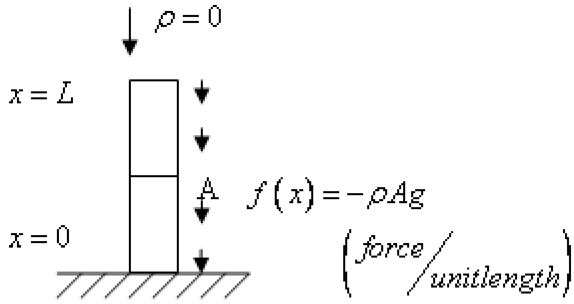
Example 2. A column under own weight (a Hagpole problem)

Assume A is constant

$\rho =$ density $\left({}^{weight}\!/\!{}_{volumn} \right)$

$g =$ gravity

Find u and σ_{xx}



Ans. Exact solution

$$\frac{\partial F}{\partial x} + f = 0 \quad \text{where } F = \sigma_{xx} A$$

$$\Rightarrow \frac{\partial F}{\partial x} = -f(x) = \rho A g \quad \Rightarrow F(x) = \rho A g (x + c_1)$$

By force boundary condition $F(L) = 0,$ we have $c_1 = -L$

$$\Rightarrow F(x) = \rho A g (x - L)$$

Moreover, by $F(x) = \sigma_{xx} A$ and $\sigma_{xx} = E\varepsilon_{xx} = E\frac{\partial u}{\partial x}$

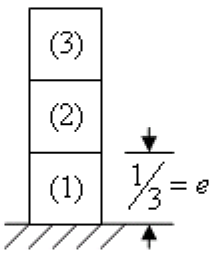
$$\Rightarrow \sigma_{xx} = \rho g (x - L) = E\frac{\partial u}{\partial x}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\rho g}{E}(x - L) \quad \Rightarrow u(x) = \frac{1}{E}\rho g\left(\frac{1}{2}x^2 - Lx + c_2\right)$$

By geometric boundary condition $u(0) = 0$

we have $c_2 = 0$

$$\Rightarrow u(x) = \frac{1}{E}\rho g\left(\frac{1}{2}x^2 - Lx\right) \quad -\text{exact solution}$$



Finite Element modeling

Let's consider elements of equal length.

For element $\#i$,

$$\begin{cases} K_{22}^{(i)} = \dfrac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{ and} \\ \begin{pmatrix} Q_1^{(i)} \\ Q_2^{(i)} \end{pmatrix} = \begin{pmatrix} \int_0^1 (1-s)(-\rho A g l)\,ds \\ \int_0^1 s(-\rho A g l)\,ds \end{pmatrix} = \dfrac{-1}{2}\rho A g l \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{cases}$$

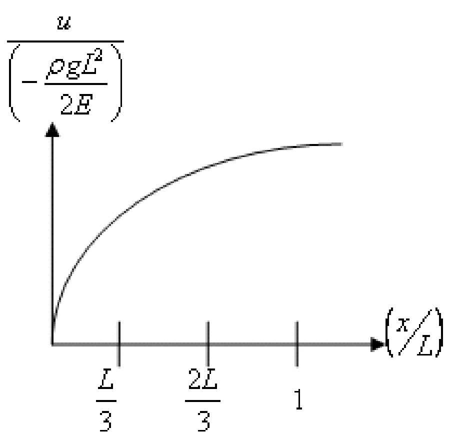
$$\underset{\text{assembly}}{\overset{FEM}{\Rightarrow}} K_{44} = \begin{bmatrix} 1 & -1 & & 0 \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ 0 & & -1 & 1 \end{bmatrix} \cdot \frac{EA}{l},\ F_{4\times 1} = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix} \cdot \left(-\frac{\rho A g l}{2}\right)$$

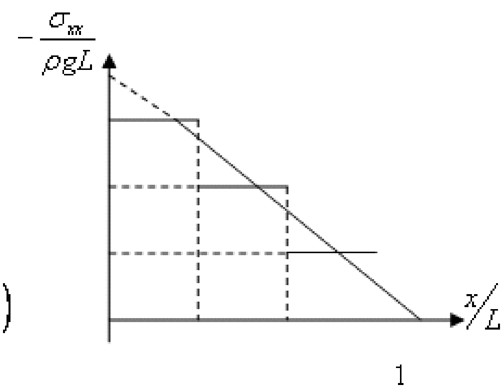
apply boundary conditions $\begin{cases} u(0) = 0 \\ \rho = 0 \end{cases}$

$$\Rightarrow \frac{EA}{l}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{pmatrix} u_2 \\ u_3 \\ u_4 \end{pmatrix} = -\frac{\rho A g l}{2}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{pmatrix} u_2 \\ u_3 \\ u_4 \end{pmatrix} = -\frac{\rho g l^2}{2E}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

$$\underset{\left(l=\frac{L}{3}\right)}{\Rightarrow} \begin{pmatrix} u_2 \\ u_3 \\ u_4 \end{pmatrix} = -\frac{\rho g L^2}{2E}\begin{pmatrix} \frac{5}{9} \\ \frac{8}{9} \\ 1 \end{pmatrix}$$





Calculation of Stress

$\sigma_{xx} = E\varepsilon_{xx}$

Within element $\#i$: $\sigma_{xx} = E\frac{1}{l}(u_2 - u_1)$

element(1): $\sigma_{xx} = \frac{E}{\left(\frac{L}{2}\right)}\left(\frac{-5\rho g L^2}{18E}\right) = -\frac{5}{6}\rho g L$

element(2): $\sigma_{xx} = \frac{E}{\frac{L}{3}}\left(-\left(\frac{8}{18} - \frac{5}{18}\right)\frac{\rho g L^2}{2E}\right) = -\frac{1}{2}\rho g L$

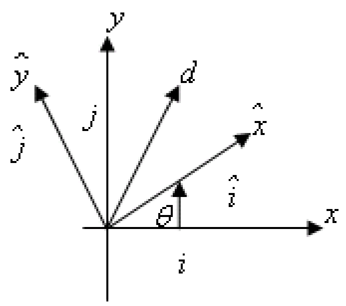
element(3): $\sigma_{xx} = -\frac{1}{6}\rho g L$

(Simple linear algebra coordinate transformation)

Transformation of vectors in 2 dimensions

Let $\{x, y\}$ be the global coordinate system,

$\{\hat{x}, \hat{y}\}$ be the local coordinate system.



The displacement vector:

$$d = d_x i + d_y j = \widehat{d_x}\,\hat{i} + \widehat{d_y}\,\hat{j}, \text{ and}$$

$$\begin{pmatrix} \widehat{d_x} \\ \widehat{d_y} \end{pmatrix}_{\{\hat{i},\hat{j}\}} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{pmatrix} d_x \\ d_y \end{pmatrix}_{\{i,j\}} \qquad c = \cos\theta,\ s = \sin\theta$$

$\Rightarrow$ For a 2-DOF system on a plan, we have

$$\begin{pmatrix} \widehat{d_x^{(1)}} \\ \widehat{d_y^{(1)}} \end{pmatrix} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{pmatrix} d_x^{(1)} \\ d_y^{(1)} \end{pmatrix}; \quad \begin{pmatrix} \widehat{d_x^{(2)}} \\ \widehat{d_y^{(2)}} \end{pmatrix} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{pmatrix} d_x^{(2)} \\ d_y^{(2)} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \widehat{d_x^{(1)}} \\ \widehat{d_y^{(1)}} \\ \widehat{d_x^{(2)}} \\ \widehat{d_y^{(2)}} \end{pmatrix} = \underbrace{\begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix}}_{=T} \begin{pmatrix} d_x^{(1)} \\ d_y^{(1)} \\ d_x^{(2)} \\ d_y^{(2)} \end{pmatrix} \quad \Rightarrow \begin{array}{l} \hat{\vec{d}} = T\vec{d} \\ \vec{d} = T^{*}\hat{\vec{d}} \end{array}$$

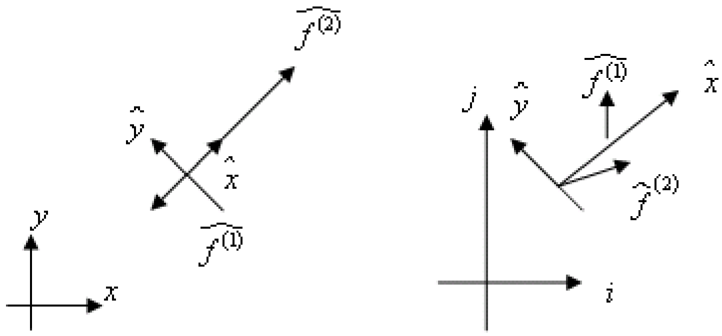
Similarly, the force vector transformed in the same way

$$\begin{pmatrix} \widehat{f_x^{(1)}} \\ \widehat{f_y^{(1)}} \\ \widehat{f_x^{(2)}} \\ \widehat{f_y^{(2)}} \end{pmatrix} = [T] \begin{pmatrix} f_x^{(1)} \\ f_y^{(1)} \\ f_x^{(2)} \\ f_y^{(2)} \end{pmatrix} \quad \Rightarrow \begin{array}{l} \hat{\vec{f}} = T\vec{f} \\ \vec{f} = T^{*}\hat{\vec{f}} \end{array}$$

Since we have

$$\underbrace{\widehat{K}}_{\substack{\text{element} \\ \text{stiffness} \\ \text{matrix}}} \cdot \hat{\vec{d}} = \hat{\vec{f}}$$

$$\underset{\text{transformation}}{\overset{\text{under}}{\Rightarrow}} \widehat{K} \cdot T\vec{d} = T\vec{f} \quad \Rightarrow \underbrace{T^{*}\widehat{K}T}_{K}\vec{d} = \vec{f}$$



For the bar element, force and displacement are only assumed on the axial direction

The element stiffness matrix can now be generalized as following $K = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \left(\frac{EA}{l} \right)$ (assuming constant E and A)

$$\widehat{K} = \frac{EA}{l} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(Notice: displacement in $\hat{y}$ direction does not have any influence to the calculation of $\underset{\delta U = \delta \vec{u} K \vec{u}}{\delta U^{I}}$ (the internal energy))

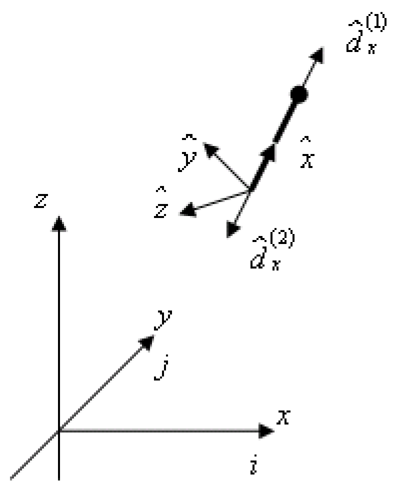
$$\underset{\text{Transformation}}{\Rightarrow} K = T^{*} \widehat{K} T = \frac{EA}{l} \begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix} \quad -(16)$$

Remark:

Assembling process and stress calculation remains the same

$$\sigma_{xx} = E \frac{\partial u}{\partial x} = E \cdot [-1,1] \begin{pmatrix} \widehat{d_x^{(1)}} \\ \widehat{d_x^{(2)}} \end{pmatrix} = E \cdot [-1,1] \begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix} \begin{pmatrix} d_x^{(1)} \\ d_y^{(1)} \\ d_x^{(2)} \\ d_y^{(2)} \end{pmatrix}$$

Example 3 (P.55 example 4.3)



Transformation matrix in 3D

$$\widehat{d_x}\hat{i}+\widehat{d_y}\hat{j}+\widehat{d_z}\hat{k}=d_x i+d_y j+d_z k$$

directional cosine

$$c_x=\hat{i}\cdot i \quad c_y=\hat{i}\cdot j \quad c_z=\hat{i}\cdot k$$

$$\Rightarrow \widehat{d_x}=c_x d_x+c_y d_y+c_z d_z$$

$$\Rightarrow \begin{pmatrix}\widehat{d_x^{(1)}}\\ \widehat{d_x^{(2)}}\end{pmatrix}=\underbrace{\begin{bmatrix}c_x & c_y & c_z & 0 & 0 & 0\\ 0 & 0 & 0 & c_x & c_y & c_z\end{bmatrix}}_{T}\begin{bmatrix}d_x^{(1)}\\ d_y^{(1)}\\ d_z^{(1)}\\ d_x^{(2)}\\ d_y^{(2)}\\ d_z^{(2)}\end{bmatrix}$$

$$\Rightarrow K=T^*\widehat{K}T \quad \left(\hat{k}=\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\cdot\frac{AE}{l}\right)$$

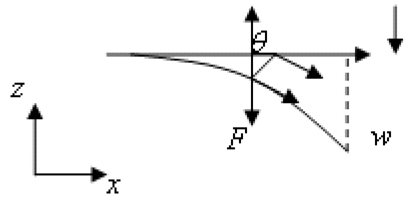
$$\Rightarrow K=\frac{AE}{l}\left[\begin{array}{ccc|ccc} c_x^2 & c_xc_y & c_xc_z & -c_x^2 & -c_xc_y & -c_xc_z\\ & c_y^2 & c_yc_z & -c_xc_y & -c_y^2 & -c_yc_z\\ * & & c_z^2 & -c_yc_z & -c_yc_z & -c_z^2\\ \hline & & & c_x^2 & c_xc_y & c_xc_z\\ & * & & & c_y^2 & c_yc_z\\ & & & * & & c_z^2\end{array}\right] \quad -(17)$$

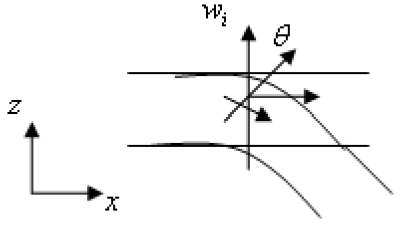
$$\sigma_{xx}=E\frac{\partial\hat{u}}{\partial\hat{x}}=\frac{E}{l}[-1,1]\begin{bmatrix}c_x & c_y & c_z & 0 & 0 & 0\\ 0 & 0 & 0 & c_x & c_y & c_z\end{bmatrix}\begin{bmatrix}d_x^{(1)}\\ d_y^{(1)}\\ d_z^{(1)}\\ d_x^{(2)}\\ d_y^{(2)}\\ d_z^{(2)}\end{bmatrix}$$

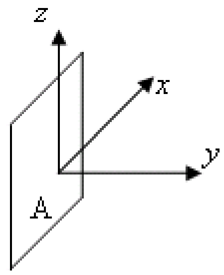
Beam bending :

Assume bending in the $x-z$ plane

$$u(x)=z\cdot\theta \text{ and } \theta=-\frac{\partial^2 w}{\partial x^2}$$







Recall $\sigma_{xx} = E\varepsilon_{xx} = E\dfrac{\partial u}{\partial x} = Ez\left(-\dfrac{\partial^2 w}{\partial x^2}\right) = E\int_s z$

Moreover, $\dfrac{\partial F}{\partial x} + f = 0$

$\Rightarrow \int_A \int_0^L v\dfrac{\partial F}{\partial x} + \int_0^L vf = 0$

$\Rightarrow -\int_A \int_0^L F\cdot\dfrac{\partial v}{\partial x}dx + \int_A \int_0^L vf = 0$

assume $v = z\cdot\delta\theta,\ \delta\theta = -\dfrac{\partial\delta w}{\partial x}$ (Galenkin formulation)

$\Rightarrow \int_0^L \int_A (-Ez)\dfrac{\partial^2 w}{\partial x^2} - z\dfrac{\partial^2\delta w}{\partial x^2} + \int_0^L \int_A vf = 0$

$\Rightarrow \int_0^L E\cdot I\dfrac{\partial^2\delta w}{\partial x^2}\dfrac{\partial^2 w}{\partial x^2}dx + \int_0^L v\overline{f(x)}dx = 0$ (F : average force at x)

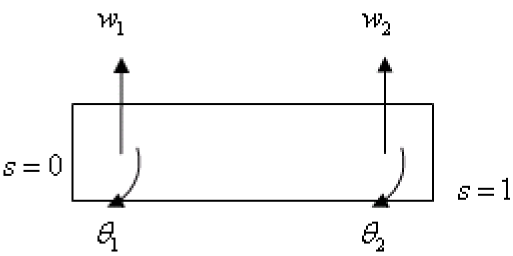
$I = \int_A z^2 dydz \qquad -(18)$

For a beam element:

state varibles: θ, w

elem nodal DOF:

$$\begin{cases} w_1 \\ \theta_1 = \left(-\dfrac{\partial w}{\partial x}\right)_1 \\ w_2 \\ \theta_2 = \left(-\dfrac{\partial w}{\partial x}\right)_2 \end{cases}$$



Notice: θ, w must be contineous!

Assume $w(x) = a_1 + a_2 s + a_3 s^2 + a_4 s^3 \quad -(19)$

$$(i)\frac{\partial w}{\partial x} = \frac{\partial w}{\partial s} \cdot \frac{1}{\left(\frac{\partial x}{\partial s}\right)} = \frac{1}{l}\frac{\partial w}{\partial s}$$

$$\theta = -\frac{\partial w}{\partial x} = -\frac{1}{l}\left(a_2 + 2a_3 s + 3a_4 s^2\right)$$

at node (1), $s = 0$

$$\begin{aligned} w_1 &= a_1 \\ \theta_1 &= -\frac{1}{l}a_2 \end{aligned} \quad \Rightarrow \quad \begin{aligned} a_1 &= w_1 \\ a_2 &= -l\theta_1 \end{aligned}$$

at node (2)

$$w_2 = a_1 + a_2 + a_3 + a_4 = w_1 - l\theta_1 + a_3 + a_4 \;\Rightarrow a_3 + a_4 = l\theta_1 - w_1 + w_2$$

$$\theta_2 = \frac{-1}{l}\left(a_2 + 2a_3 + 3a_4\right) = -\frac{1}{l}\left(-l\theta_1 + 2a_3 + 3a_4\right) \Rightarrow 2a_3 + 3a_4 = la_1 - l\theta_2$$

$\Rightarrow$ solve for $a_3,\ a_4$

$$\begin{aligned} a_3 &= 3w_2 - 3w_1 + 2l\theta_1 + l\theta_2 \\ a_4 &= -2w_2 - 2w_1 - l\theta_1 - l\theta_2 \end{aligned} \quad \text{, plug into } (19)$$

we have

$$w(x) = N_{w_1} w_1 + N_{\theta_1}\theta_1 + N_{w_2} w_1 + N_{\theta_2}\theta_2$$

here

$$\left.\begin{aligned} N_{w_1} &= 1 - 3s^2 + 2s^3 \\ N_{\theta_1} &= \left(-s + 2s^2 - s^3\right)l \\ N_{w_2} &= 3s^2 - 2s^3 \\ N_{\theta_2} &= \left(s^2 - s^3\right)l \end{aligned}\right\} \text{shap function} \quad -(20)$$

Remark: $\int N_w ds = \frac{1}{2}$

Recall that the mapping function is

$$x = (1 - s)x_1 + sx_2$$

The shap function $\neq$ map function

$\Rightarrow$ We obtain a non-isoparametric formulation

FEM formulation for (18)

$$\text{Let } w = \underbrace{\left[N_{w_1}\ N_{\theta_1}\ N_{w_2}\ N_{\theta_2}\right]}_{\vec{N}} \cdot \underbrace{\begin{pmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{pmatrix}}_{\text{element dof vector}} = \overrightarrow{N_l} \cdot \overrightarrow{q^e}$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial s}\left(\frac{\partial}{\partial s} w \cdot \frac{1}{\left(\frac{\partial x}{\partial s}\right)}\right)\frac{1}{\left(\frac{\partial x}{\partial s}\right)} = \frac{1}{l^2}\frac{\partial^2}{\partial s^2} w$$

$$= \frac{1}{l^2}\begin{bmatrix} \frac{\partial^2}{\partial s^2} N_{w_1} & \frac{\partial^2}{\partial s^2} N_{\theta_1} & \frac{\partial^2}{\partial s^2} N_{w_2} & \frac{\partial^2}{\partial s^2} N_{\theta_2} \end{bmatrix} \cdot \begin{pmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{pmatrix}$$

$$= \frac{1}{l^2}\underbrace{\begin{bmatrix} 6s-6 & (4-6s)l & 6-6s & (2-6s)l \end{bmatrix}}_{B\ (\text{linear in } s)} \vec{q}^{\,e}$$

Similarly, $\frac{\partial^2}{\partial x}\delta w = B\delta\vec{q}^{\,e} = \delta\vec{q}^{\,eT} B^T$

For an element, the internal virtual work,

$$\delta U_I = \int_{x_1}^{x_2} EI \frac{\partial^2 \delta w}{\partial x^2}\frac{\partial^2 w}{\partial x^2} dx$$

$$= \int_0^1 EI \frac{1}{l^4}\delta q^{eT} B^T B q^e \cdot l ds$$

$$= \delta q^{eT}\underbrace{\left(\frac{EI}{l^3}\int_0^1 B^T B ds\right)}_{K^e} q^e$$

here

$$K^e = \frac{2EI}{l^3}\begin{bmatrix} 6 & -3l & -6 & -3l \\ -3l & 2l^2 & 3l & l^2 \\ -6 & 3l & 6 & 3l \\ -3l & l^2 & 3l & 2l^2 \end{bmatrix} \quad -(21)$$

Element load vector (external virtual work)

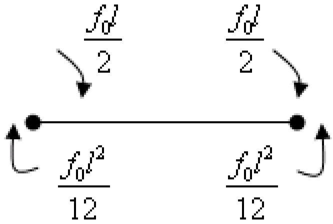
$$\delta w^2 = \int_{x_1}^{x_2} \delta v \cdot f dx$$

$$= \int_0^1 \left[\delta w_1 \ \ \delta\theta_1 \ \ \delta w_2 \ \ \delta\theta_2 \right] \begin{pmatrix} N_{w_1} \\ N_{\theta_1} \\ N_{w_2} \\ N_{\theta_2} \end{pmatrix} f \cdot l ds$$

$$= \left(\delta g^e \right)^T Q^e_{4\times1}$$

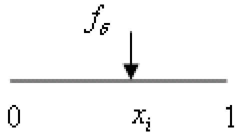
$$Q^e = \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{pmatrix} = \begin{pmatrix} l\int_0^1 N_{w_1} \cdot fds & \leftarrow \text{nodal force} \\ l\int_0^1 N_{\theta_1} \cdot fds & \leftarrow \text{nodal moment} \\ l\int_0^1 N_{w_2} \cdot fds & \leftarrow \text{nodal force} \\ l\int_0^1 N_{\theta_2} \cdot fds & \leftarrow \text{nodal moment} \end{pmatrix}$$

(1) uniform force



$$Q^e = \begin{bmatrix} -f_0 \dfrac{l}{2} \\ -f_0 \dfrac{l^2}{12} \\ -f_0 \dfrac{l}{2} \\ f_0 \dfrac{l^2}{12} \end{bmatrix}$$ equivalent to nodal force & moment

(2) nodal point load



f_δ satisfies $\int_0^1 N(x) f_\delta dx = f_\delta$

Standard FEM Assembling process gives the global stiffness system

$$K\vec{q} = \vec{F}$$

$\underset{\text{applying boundary condition}}{\Rightarrow}$ solving for $\vec{q}$

To compute stress in each element

$$\sigma_{xx} = E\varepsilon_{xx} = E\frac{\partial u}{\partial x} = Ez\frac{\partial \theta}{\partial x} = -Ez\frac{\partial^2 w}{\partial x^2} = -zE\frac{1}{l^2}Bq^e$$

Remark: If the state varibles are defined

$$\begin{cases} Q_1 = \left(\dfrac{\partial w}{\partial x}\right)_1 \\ Q_2 = \left(\dfrac{\partial w}{\partial x}\right)_2 \end{cases}, \text{ one obtains } \begin{cases} \widetilde{N}_{1w} \\ \widetilde{N}_{1\theta} \\ \widetilde{N}_{2w} \\ \widetilde{N}_{2\theta} \end{cases} \text{ from } (20) \text{ and}$$

$$K^e = \frac{2EI}{l^3}\begin{bmatrix} 6 & 3l & -6 & 3l \\ 3l & 2l^2 & -3l & l^2 \\ -6 & -3l & 6 & -3l \\ 3l & l^2 & -3l & 2l^2 \end{bmatrix}$$

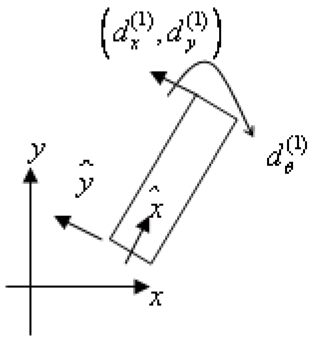
Transformation matrix:

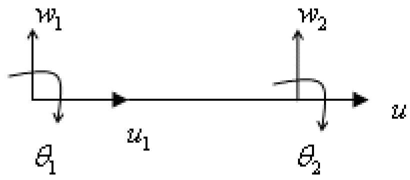
$$\begin{pmatrix} \widehat{d_x} \\ \widehat{d_z} \end{pmatrix} = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}\begin{pmatrix} d_x \\ d_z \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \widehat{d_z^{(1)}} \\ \widehat{d_\theta^{(1)}} \\ \widehat{d_z^{(2)}} \\ \widehat{d_\theta^{(2)}} \end{pmatrix} = \underbrace{\begin{bmatrix} -s & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{T}\begin{pmatrix} d_x^{(1)} \\ d_y^{(1)} \\ d_\theta^{(1)} \\ d_x^{(2)} \\ d_y^{(2)} \\ d_\theta^{(2)} \end{pmatrix}$$

For the stiffness matirx, let's denote $K^e = \hat{k}$ in the local coordinate $(\hat{x}, \hat{y})$.

The stiffness matrix in global coordinate equals to $K = T^* \hat{k} T$





planar Frame element:

state variables: u, w, θ

$$\text{elem nodal dof } \begin{Bmatrix} u_1 \\ w_1 \\ \theta_1 \\ u_2 \\ w_2 \\ \theta_2 \end{Bmatrix}, \ \theta_1 = \left(\frac{\partial w_1}{\partial x}\right)_1, \ \theta_2 = \left(\frac{\partial w}{\partial x}\right)_2$$

internal virtual work

$$\delta U^e = \left[\delta w_1, \delta\theta_1, \delta w_2, \delta\theta_2\right] \cdot K^e_{\text{beam}} \begin{pmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{pmatrix} + \left[\delta u_1, \delta u_2\right] \underbrace{K^e_{\text{bar}}}_{\left(\frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right)} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\overset{a=\frac{EI}{l^3}, k=\frac{EA}{e}}{=} \left[\delta u_1 \ \delta w_1 \ \delta\theta_1 \ \delta u_2 \ \delta w_2 \ \delta\theta_2\right] \cdot \begin{bmatrix} k & 0 & 0 & -k & 0 & 0 \\ 0 & 12a & 6al & 0 & -12a & 6al \\ 0 & 6al & 4al^2 & 0 & -6al & 2al^2 \\ -k & 0 & 0 & k & 0 & 0 \\ 0 & -12a & -6al & 0 & 12a & -6al \\ 0 & 6al & 2al^2 0 & 0 & -6al & 4al^2 \end{bmatrix} \begin{pmatrix} u_1 \\ w_1 \\ \theta_1 \\ u_2 \\ w_2 \\ \theta_2 \end{pmatrix}$$

Transformation matrix

$$T = \left[\begin{array}{ccc|ccc} c & s & 0 & & & \\ -s & c & 0 & & 0 & \\ 0 & 0 & 1 & & & \\ \hline & & & c & s & 0 \\ & 0 & & -s & c & 0 \\ & & & 0 & 0 & 1 \end{array}\right] \Rightarrow K = T^* \hat{k} T$$