4. Brownian Motions

The origination of Brownian motions is based on botanist Brown's observation of pollen grain pending on the water, of which motion is determined by the net effect of bombardment of water molecules. In this section, we introduce the mathematical framework in the one-dimensional case and discuss their properties. Thereafter, $(X_t)_{t\ge 0}$ refers to a stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where X_t is non-degenerate for $t > 0$. For convenience, we assume that $\mathbb{P}$ is complete.

4.1. **Definitions.** A stochastic process $(X_t)_{t\ge 0}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathbb{R}$ is a *Brownian motion* if $X_0 = 0$ and

- (A1) *(Independency of increments)* X_{t_0}, $X_{t_1} - X_{t_0}$,..., $X_{t_n} - X_{t_{n-1}}$ are independent for any $t_0 < t_1 < \cdots < t_n$ and $n \ge 1$.
- (A2) *(Stationarity of increments)* For any $s \ge 0$, the distribution of $X_{t+s} - X_t$ is independent of t.
- (A3) *(Continuity)* For any $\delta > 0$, one has

$$\lim_{s\downarrow 0} \frac{\mathbb{P}(|X_{t+s} - X_t| > \delta)}{s} = 0.$$

Remark 4.1. By the $\pi - \lambda$ lemma, (A1) is equivalent to the independence of $\mathcal{F}(X_{t+s} - X_t)$ and $\mathcal{F}(X_r, r \le t)$ for any $s, t \ge 0$.

Proposition 4.1. *Let $(X_t)_{t\ge 0}$ be a Brownian motion. Then, there are $\mu \in \mathbb{R}$ and $\sigma > 0$ such that X_t is normal with mean μt and variance $\sigma^2 t$ for $t > 0$.*

To prove this proposition, we recall the following lemma.

Lemma 4.2. *For $n \ge 1$, let $X_{n,1}, ..., X_{n,n}$ be i.i.d. non-degenerate random variables and set $S_n = \sum_{i=1}^n X_{n,i}$ and $M_n = \max\{|X_{n,i}| : 1 \le i \le n\}$. Assume that S_n converges in distribution to X. Then, X is normal if and only if M_n converges to 0 in distribution.*

Proof of Proposition 4.1. By the stationarity of increments, (A3) is equivalent to

$$\lim_{s\downarrow 0} \frac{\mathbb{P}(|X_s| > \delta)}{s} = 0, \quad \forall \delta > 0.$$

Fix $t > 0$. By (A1) and (A2), X_t is infinitely divisible since

$$X_t = \sum_{k=1}^n (X_{tk/n} - X_{t(k-1)/n}).$$

Let $M_n = \max\{|X_{tk/n} - X_{t(k-1)/n}| : 1 \le k \le n\}$. Note that, for $\delta > 0$,

$$\begin{aligned}\mathbb{P}(M_n > \delta) &= 1 - \mathbb{P}(M_n \le \delta) = 1 - \mathbb{P}(|X_{t/n}| \le \delta)^n \\ &= 1 - [1 - \mathbb{P}(|X_{t/n}| > \delta)]^n = 1 - [1 - o(t/n)]^n \to 0\end{aligned}$$

as $n \to \infty$. By Lemma 4.2, this implies that X_t is normal. Set $\phi(t) = \mathbb{E}X_t$ and $\varphi(t) = \mathrm{Var}(X_t)$. Clearly, one has

$$\phi(t + s) = \mathbb{E}X_{t+s} = \mathbb{E}(X_{t+s} - X_t) + \mathbb{E}X_t = \phi(s) + \phi(t)$$

and

$$\varphi(t + s) = \mathrm{Var}(X_{t+s} - X_t + X_t) = \mathrm{Var}(X_{t+s} - X_t) + \mathrm{Var}(X_t) = \varphi(s) + \varphi(t).$$

Following these computations, we may conclude that $\phi(t) = \phi(1)t$ and $\varphi(t) = \varphi(1)t$ for $t \in \mathbb{Q} \cap [0, \infty)$. As X_{t+s} converges in distribution to X_t, we have $\phi(t + s) \to \phi(t)$ and $\varphi(t + s) \to \varphi(t)$. This implies that ϕ and φ are right-continuous and, hence, $\mathbb{E}X_t = t\mathbb{E}X_1$ and $\mathrm{Var}(X_t) = t\mathrm{Var}(X_1)$ for $t > 0$. ☐

Definition 4.1. A Brownian motion is a process $(X_t)_{t\ge 0}$ satisfying $X_0 = 0$ and

(1) For $t_0 < t_1 < \cdots < t_n$, $X_{t_0}, X_{t_1} - X_{t_0}, ..., X_{t_n} - X_{t_{n-1}}$ are independent.
(2) There are $\mu \in \mathbb{R}$ and $\sigma^2 > 0$ such that $X_{t+s} - X_t$ is normally distributed with mean $s\mu$ and variance $s\sigma^2$ for all $t \ge 0$ and $s > 0$.

If $\mu = 0$ and $\sigma = 1$, then $(X_t)_{t\ge 0}$ is called a normalized Brownian motion.

Remark 4.2. μ is called the *drift* of a Brownian motion.

Exercise 4.1. Let $(X_t)_{t\ge 0}$ be a Brownian motion and set $Y_t = (X_t - t\mathbb{E}X_1)/\sqrt{\mathrm{Var}(X_1)}$. Show that $(Y_t)_{t\ge 0}$ is a normalized Brownian motion.

Exercise 4.2. Suppose $X_1, ..., X_n$ are random variables such that $X_1, X_2 - X_1, ..., X_n - X_{n-1}$ are independent. Prove that if F_k is the distribution function of $X_k - X_{k-1}$, then the joint distribution function F of $X_1, ..., X_n$ is given by

$$F(x_1, x_2, ..., x_n) = \int_{(-\infty, x_1]} F_1(dy_1) \int_{(-\infty, x_2 - y_1]} F_2(dy_2) \times \cdots \times \int_{(-\infty, x_n - y_{n-1} - \cdots - y_1]} F_n(dy_n).$$

Note that one may build up a probability space for the existence of a process with given finite-dimensional distributions. Without the regularity of sample pathes, there might be events of interests but non-measurable, e.g.

$$A = \{X_t = 0 \text{ for some } t \in I\} = \bigcup_{t\in I}\{X_t = 0\}, \quad B = \{|X_t| \le c, \forall t \in I\} = \bigcap_{t\in I}\{|X_t| \le c\}.$$

However, if the process is continuous almost surely, then the above events turn into measurable sets. For instance, let $I = [0, 1]$ and $(Y_t)_{t\in I}$ be a continuous process. Define, for $\epsilon > 0$ and $n \ge 1$,

$$A_\epsilon = \{|Y_t| < \epsilon \text{ for some } t \in I\}, \quad A_{n,\epsilon} = \{|Y_{k/2^n}| < \epsilon \text{ for some } k \ge 0\}.$$

Clearly, $A_{n,\epsilon} \to A_\epsilon$ and $A_\epsilon \to A$, which implies A is measurable.

This leads us to the construction of continuous Brownian motions.

Definition 4.2. Let $I \subset [0, \infty)$. A process $(X_t)_{t\in I}$ is said to be continuous in probability if for all $t \in I$ and $t_n \to t$, $X_{t_n} \to X_t$ in probability.

Theorem 4.3. *Let I be an interval in $[0, \infty)$. Assume that a process $(X_t)_{t\in I}$ is continuous in probability and there exists a countable set T dense on I such that*

$$\mathbb{P}(\omega : t \mapsto X_t(\omega) \text{ is uniformly continuous on } T \cap J) = 1$$

for any finite subinterval $J \subset I$. Then, there is a process $(Y_t)_{t\in I}$ on the same probability space such that the map $t \mapsto Y_t(\omega)$ is continuous on I for all ω and $X_t = Y_t$ a.s. for all $t \in I$.

Remark 4.3. Both processes $(X_t)_{t\in I}$ and $(Y_t)_{t\in I}$ have the same distribution.

Proof of Theorem 4.3. Write $T_J = T \cap J$ and set

$$C = \{\omega : t \mapsto X_t(\omega) \text{ is uniformly continuous on } T_J\}.$$

For $t \ge 0$, define

$$Y_t(\omega) := \lim_{j\to\infty} X_{t_j}(\omega), \quad \forall \omega \in C,\ t \in J,\ t_j \in T_J,\ t_j \to t.$$

and $Y_t(\omega) \equiv 0$ for all $\omega \in C^c$ and $t \in J$. Clearly, Y_t is a measurable function (since $X_{t_j} \to Y_t$ on C) and is continuous in $t \in J$ for all ω. Since X_{t_j} converges in probability to X_t and X_{t_j} converges to Y_t a.s., we have $Y_t = X_t$ a.s. for all $t \in J$. The desired construction is then given by applying the above conclusion to $J_n = [0, n] \cap I$. □

Theorem 4.4. *For any Brownian motion $(X_t)_{t\geq 0}$, there exists a countable dense subset T of $[0,\infty)$ such that, for almost all ω, the map $t \mapsto X_t(\omega)$ is uniformly continuous on $T\cap[0,a]$ for all $a<\infty$.*

To prove this theorem, we need the following lemma.

Lemma 4.5. *Let X_t be a Brownian motion. For $t_0<t_1<\cdots<t_n$ and $x>0$,*

$$\mathbb{P}\left(\max_{1\leq k\leq n} X_{t_k}>x\right)\leq 2\mathbb{P}(X_{t_n}>x),\quad \mathbb{P}\left(\max_{1\leq k\leq n} |X_{t_k}|>x\right)\leq 2\mathbb{P}(|X_{t_n}|>x).$$

Proof. Let $N=\inf\{k\geq 1: X_{t_k}>x\}$, where $\inf\emptyset:=\infty$. Since the distribution of $X_{t_n}-X_{t_k}$ is continuous and symmetric about 0, one has

$$\mathbb{P}\left(\max_{1\leq k\leq n} X_{t_k}>x\right)=\sum_{k=1}^n \mathbb{P}(N=k)=2\sum_{k=1}^n \mathbb{P}(N=k)\mathbb{P}(X_{t_n}-X_{t_k}>0).$$

Note that $\{N=k\}\in\mathcal{F}(X_{t_1},...,X_{t_k})$. By the independency of increments, this implies

$$\begin{aligned}\mathbb{P}\left(\max_{1\leq k\leq n} X_{t_k}>x\right)&=2\sum_{k=1}^n \mathbb{P}(N=k,X_{t_n}-X_{t_k}>0)\\&\leq 2\sum_{k=1}^n \mathbb{P}(N=k,X_{t_n}>x)\leq 2\mathbb{P}(X_{t_n}>x).\end{aligned}$$

Since X_t is normalized, $-X_t$ has the same distribution as X_t and thus

$$\mathbb{P}\left(\min_{1\leq k\leq n} X_{t_k}<-x\right)\leq 2\mathbb{P}(X_{t_n}<-x).$$

The last inequality is given by the following fact.

$$\left\{\max_{1\leq k\leq n}|X_{t_k}|>x\right\}\subset\left\{\max_{1\leq k\leq n} X_{t_k}>x\right\}\cup\left\{\min_{1\leq k\leq n} X_{t_k}<-x\right\}.$$

□

Proof of Theorem 4.4. By Exercise 4.1, it suffices to consider normalized Brownian motions. Let $I=\{k/2^n: k\geq 0,\, n\geq 0\}$, $a\in\mathbb{N}$ and $T=\{ak/2^n: k=0,1,...,2^n, n\geq 1\}$. Clearly, $T=I\cap[0,a]$. Define

$$U_n=\sup\{|X_s-X_t|: s,t\in T,\,|s-t|\leq a2^{-n}\},\quad U=\lim_{n\to\infty}U_n.$$

By Theorem 4.3, it suffices to show that $U=0$ a.s.. As U_n is non-increasing, it is equivalent to prove that $U_n\to 0$ in probability.

Set $I_{n,k}=[ak/2^n, a(k+1)/2^n]$ and $Y_{n,k}=\sup\{|X_t-X_{ak/2^n}|: t\in I_{n,k}\cap T\}$ for $k=0,1,...,2^n-1$. By the following inequality

$$\frac{1}{3}U_n\leq\max_{0\leq k<2^n} Y_{n,k},$$

it remains to show that the right hand side converges to 0 in probability. Following assumptions (A1) and (A2), one has

$$\mathbb{P}\left(\max_{0\leq k<2^n} Y_{n,k}>\delta\right)\leq\sum_{k=0}^{2^n-1}\mathbb{P}(Y_{n,k}>\delta)=2^n\mathbb{P}(Y_{n,0}>\delta).$$

Let $T_m = \{ak/2^m : k = 0, 1, ..., 2^m\}$. Obviously, $\max\{|X_t| : t \in I_{n,0} \cap T_m\}$ is non-decreasing in m and converges to $Y_{n,0}$, which implies

$$\mathbb{P}(Y_{n,0} > \delta) = \lim_{m\to\infty} \mathbb{P}\left(\max_{t\in I_{n,0}\cap T_m} |X_t| > \delta\right).$$

By Lemma 4.5, we have

$$\mathbb{P}\left(\max_{t\in I_{n,0}\cap T_m} |X_t| > \delta\right) \le 2\mathbb{P}(|X_{a2^{-n}}| > \delta), \quad \forall m \ge n.$$

Consequently, one may apply (A3) to get

$$\lim_{n\to\infty} \mathbb{P}\left(\max_{0\le k<2^n} Y_{n,k} > \delta\right) \le \lim_{n\to\infty} \frac{\mathbb{P}(|X_{a2^{-n}}| > \delta)}{2^{-n-1}} = 0.$$

□

Thereafter, a Brownian motion refers to a stochastic process $(X_t)_{t\ge 0}$ with continuous sample paths and satisfying $X_0 = 0$ and

(1) X_{t_0}, $X_{t_1} - X_{t_0}$,...,$X_{t_{n-1}} - X_{t_n}$ are independent for $t_0 < t_1 < \cdots < t_n$ and $n \ge 1$.
(2) For $s < t$, $X_t - X_s$ has the normal distribution with mean $\mu(t - s)$ and variance $\sigma^2(t - s)$.

When mentioning normalized Brownian motions, we mean $\mu = 0$ and $\sigma = 1$ and write $(B_t)_{t\ge 0}$ instead. In the following exercises, we use I to denote an interval in $[0, \infty)$.

Exercise 4.3. Let $(X_t)_{t\in I}$ be continuous in probability. Show that, for any countable set $\{t_i : i \ge 1\}$ dense in I,

$$\mathcal{F}(X_t, t \in I) \subset \overline{\mathcal{F}}(X_{t_i}, i \ge 1)$$

where $\overline{\mathcal{F}}(X_{t_i}, i \ge 1)$ is the σ-field containing $\mathcal{F}(X_{t_i}, i \ge 1)$ and all subsets of measure zero set of $\mathbb{P}$.

Exercise 4.4. Let $(X_t)_{t\in I}$ be a stochastic process continuous in probability and T_n be a finite subset of I satisfying $T_n \subset T_{n+1}$ and $T_n \to T$, where T is dense in I. Use Exercise 4.3 to show that, for $A \in \mathcal{F}(X_t, t \in I)$,

$$\mathbb{P}(A|X_t, t \in T_n) \stackrel{a.s.}{\to} \mathbb{P}(A|X_t, t \in I) \quad \text{as } n \to \infty.$$

Exercise 4.5. Let $(X_t)_{t\in I}$ and $(Y_t)_{t\in I}$ be two stochastic processes satisfying $\mathbb{P}(X_t = Y_t) = 1$ for all $t \in I$. Show that if $(X_t)_{t\in I}$ and $(Y_t)_{t\in I}$ almost surely right-continuous (resp. left-continuous), then $\mathbb{P}(X_t = Y_t, \forall t \in I) = 1$.

Exercise 4.6. Let $(X_t)_{t\in I}$ be a stochastic process with right-continuous (resp. left-continuous) sample paths. Show that the map $(t, \omega) \mapsto X_t(\omega)$ is $\mathcal{B}(I) \times \mathcal{F}(X_i, i \in I)$-measurable.

4.2. **Variation and differentiability of Brownian motions.**

Theorem 4.6. *Let $(X_t)_{t\ge 0}$ be a Brownian motion. Then,*

$$\mathbb{P}(\{\omega \in \Omega : t \mapsto B_t(\omega) \text{ is Lipschitz continuous at some } t > 0\}) = 0.$$

Corollary 4.7. *Almost all Brownian paths are nowhere differentiable.*

Proof. This is an immediate result of the fact that if a function is differentiable at some point, say x, then the function is Lipschitz continuous at x. □

Corollary 4.8. *Almost all Brownian paths have infinite variation on any finite interval.*

Proof. This comes from the fact that if a function is of bounded variation on I, then it is differentiable a.s. (in Lebesgue measure) on I. □

Proof of Theorem 4.6. We prove this theorem by following Breiman's book. It suffices to consider the normalized Brownian motion starting from 0. Fix $N \in \mathbb{N}$, let $M > 0$ and set

$$A_n = \{\omega : \text{for some } s \in [0, N],\ |X_t(\omega) - X_s(\omega)| \le M|t - s|,\ \forall |t - s| < 2N/n\}$$

and

$$y_{n,k}(\omega) = \max\left\{\left|X_{N(k+2-i)/n}(\omega) - X_{N(k+1-i)/n}(\omega)\right| i = 0, 1, 2\right\}.$$

For $\omega \in A_n$ with $Nk/n \le s \le N(k + 1)/n$, $y_{n,k} \le 3NM/n$ and this implies $A_n \subset B_n$, where

$$B_n = \{\omega : y_{n,k}(\omega) \le \tfrac{3NM}{n}, \text{ for some } 1 \le k \le n - 2.\}$$

By the independency and stationarity of increments, one has, for $n > 2$,

$$\begin{aligned}\mathbb{P}(B_n) &\le n\mathbb{P}\left(\max_{i=0,1,2}\left|X_{N(i+1)/n} - X_{Ni/n}\right| \le \frac{3NM}{n}\right) = n\mathbb{P}\left(|X_{N/n}| \le \frac{3NM}{n}\right)^3\\ &\le n\left(\sqrt{\frac{n}{2\pi N}}\int_{-3MN/n}^{3MN/n} e^{-nx^2/(2N)}dx\right)^3 = n\left(\sqrt{\frac{N}{2\pi n}}\int_{-3M}^{3M} e^{-Ny^2/(2n)}dy\right)^3 \to 0,\end{aligned}$$

as $n \to \infty$. Let A be the limit of A_n. Clearly, A contains all Brownian paths $t \in [0, N] \mapsto X_t$ which is Lipschitz continuous at some point with Lipschitz constant M. Following the above computations, we obtain $\mathbb{P}(A) = 0$ for all $M > 0$, which says that almost all Brownian paths are nowhere Lipschitz continuous on $[0, N]$. As N is arbitrary, this proves the desired property. □

Let B_t be a normalized Brownian motion. Let $\mathcal{P}_n = \{t_{n,0} < \cdots < t_{n,m_n}\}$ be a partition of $[0, t]$ and set $\|\mathcal{P}_n\| = \max\{t_{n,k+1} - t_{n,k} : 0 \le k < m_n\}$. Consider the following summation.

$$S_n = \sum_{k=0}^{m_n-1} |B_{t_{n,k+1}} - B_{t_{n,k}}|^2.$$

Clearly, one has

$$S_n - t = \sum_{k=0}^{m_n-1} [(B_{t_{n,k+1}} - B_{t_{n,k}})^2 - \mathbb{E}|B_{t_{n,k+1}} - B_{t_{n,k}}|^2].$$

By the independency of increments, this implies

$$\begin{aligned}\mathbb{E}(S_n - t)^2 &= \sum_{k=0}^{m_n-1} \mathbb{E}[(B_{t_{n,k+1}} - B_{t_{n,k}})^2 - (t_{n,k+1} - t_{n,k})]^2\\ &= \mathbb{E}(B_1^2 - 1)^2 \sum_{k=0}^{m_n-1} (t_{n,k+1} - t_{n,k})^2 \le \mathbb{E}(B_1^2 - 1)^2 t\|\mathcal{P}_n\|\end{aligned}$$

and then

$$\sum_{n=1}^{\infty} \mathbb{P}(|S_n - t| > \epsilon) \le \frac{\mathbb{E}(B_1^2 - 1)^2 t}{\epsilon} \sum_{n=1}^{\infty} \|\mathcal{P}_n\|, \quad \forall \epsilon > 0.$$

Theorem 4.9. *Let $\mathcal{P}_n = \{t_{n,0}, ..., t_{n,m_n}\}$ be a partition of $[0, t]$ and set $\|\mathcal{P}_n\| = \max_{0 \le k < m_n} |t_{n,k+1} - t_{n,k}|$. If $\|\mathcal{P}_n\| \to 0$, then*

$$S_n = \sum_{k=0}^{m_n-1} |B_{t_{n,k+1}} - B_{t_{n,k}}|^2 \to t \quad \text{in } L^2(\mathbb{P}).$$

Moreover, if $\sum_{n=1}^{\infty} \|\mathcal{P}_n\| < \infty$, then $S_n \to t$ a.s.

4.3. **Behavior of Brownian Motions at time 0 and ∞.**

Theorem 4.10 (Law of the iterated logarithm)**.** *Let B_t be a normalized Brownian motion. Then,*

$$\limsup_{t\downarrow 0} \frac{B_t}{\sqrt{2t\log\log 1/t}} = 1 \quad a.s.$$

Proof. We start by proving the following two claims. Let $q \in (0,1)$, $t_n = q^n$ and $\varphi(t) = \sqrt{2t\log\log 1/t}$.

Claim 1: For $\delta \in (0,1)$ and $q = 1-\delta/2$, there are constants $C > 0$ and $\lambda > 1$ such that

$$\mathbb{P}(B_{t_n} > (1+\delta)\varphi(t_{n+1})) \leq Cn^{-\lambda} \quad \forall n.$$

Claim 2: Let $Z_n = B_{t_n} - B_{t_{n+1}}$. For $\epsilon \in (0,1)$ and $q = \epsilon$, there are constants $C' > 0$ and $\beta \in (0,1)$ (depending on ϵ) such that

$$\mathbb{P}(Z_n > (1-\epsilon)\varphi(t_n)) \geq C'\frac{1}{n^\beta \log n} \quad \forall n \text{ large enough.}$$

For the first claim, note that

$$\mathbb{P}(B_{t_n} > x\sqrt{t_n}) = \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-s^2/2}ds \sim \frac{e^{-x^2/2}}{\sqrt{2\pi}x} \quad \text{as } x \to \infty,$$

and

$$(1+\delta)\varphi(t_{n+1})/\sqrt{t_n} = \sqrt{2\lambda\log(c(n+1))}$$

where $\lambda = q(1+\delta)^2$ and $c = \log 1/q$. By choosing $q = (1-\delta/2)$, we get $\lambda > 1$ and

$$\mathbb{P}(B_{t_n} > (1+\delta)\varphi(t_{n+1})) \sim \frac{1}{2c^\lambda\sqrt{\pi\lambda}n^\lambda\sqrt{\log n}} \quad \text{as } n \to \infty.$$

For the second claim, note that

$$\mathbb{P}(Z_n > z\sqrt{t_n - t_{n+1}}) \sim \frac{e^{-z^2/2}}{\sqrt{2\pi}z} \quad \text{as } z \to \infty$$

and

$$(1-\epsilon)\varphi(t_n)/\sqrt{t_n - t_{n+1}} = \sqrt{2\beta\log(cn)}$$

where $\beta = (1-\epsilon)^2/(1-q)$. Setting $q = \epsilon$ implies $\beta \in (0,1)$ and

$$\mathbb{P}(Z_n > (1-\epsilon)\varphi(t_n)) \sim \frac{1}{2c^\beta\sqrt{\pi\beta}n^\beta\sqrt{\log n}} \quad \text{as } n \to \infty.$$

Back to the proof of the theorem. By Lemma 4.5, Borel-Cantelli lemma and the above claims, we have

$$\mathbb{P}\left(\max_{t\in[t_{n+1},t_n]} B_t > (1+\delta)\varphi(t_{n+1}) \ i.o.\right) = 0, \quad \forall\delta \in (0,1) \tag{4.1}$$

and

$$\mathbb{P}(Z_n > (1-\epsilon)\varphi(t_n) \ i.o.) = 1, \quad \forall\epsilon \in (0,1). \tag{4.2}$$

Note that $\varphi'(t) > 0$ for $t \leq e^{-e}$. By (4.1), this implies that, for all $\delta \in (0,1)$,

$$\mathbb{P}\left(\limsup_{t\downarrow 0} B_t/\varphi(t) > 1+\delta\right) \leq \mathbb{P}\left(\max_{t\in[t_{n+1},t_n]} B_t > (1+\delta)\varphi(t_{n+1}) \ i.o.\right) = 0,$$

which implies

$$\limsup_{t\downarrow 0}\frac{B_t}{\varphi(t)} \leq 1 \quad \text{a.s..}$$

For the lower bound, note that both $(B_t)_{t\geq 0}$ and $(-B_t)_{t\geq 0}$ have the same distribution. This implies

$$\mathbb{P}\left(\liminf_{t\downarrow 0} \frac{B_t}{\varphi(t)} \geq -1\right) = 1.$$

Equivalently, for $\epsilon >$ and $q \in (0,1)$,

$$\mathbb{P}(B_{t_{n+1}} \geq -(1+\epsilon)\varphi(t_{n+1}) \text{ for } n \text{ large enough}) = 1.$$

Recall that (4.2) says, for $\epsilon \in (0,1)$ and $q = \epsilon$,

$$\mathbb{P}(B_{t_n} - B_{t_{n+1}} > (1-\epsilon)\varphi(t_n) \ i.o.) = 1.$$

As a result, for $\epsilon \in (0,1)$ and $q = 1-\epsilon$, the following event holds for infinitely many n with probability 1.

$$\begin{aligned} B_{t_n} > (1-\epsilon)\varphi(t_n) - (1+\epsilon)\varphi(t_n+1) &= \varphi(t_n)\left[1-\epsilon-(1+\epsilon)\frac{\varphi(t_{n+1})}{\varphi(t_n)}\right] \\ &\sim \varphi(t_n)\left[1-\epsilon-(1+\epsilon)\sqrt{\epsilon}\right] \end{aligned}$$

Letting $\epsilon \to 0$ gives the desired lower bound. □

To see the behavior of Brownian motions as $t \to \infty$, we introduce the following notations.

Definition 4.3. Let $Y_1, ..., Y_n$ be random variables. $Y = (Y_1, ..., Y_n)$ has a joint (multivariate) normal distribution with mean 0 if there exist i.i.d. standard normal random variables $X_1, ..., X_k$ and a $k \times n$ matrix A such that $Y \stackrel{d}{=} XA$, where $X = (X_1, ..., X_k)$.

Proposition 4.11. *Let $Y_1, ..., Y_n$ be random variables with finite second moments and Γ be the covariance matrix, i.e. $\Gamma_{i,j} = \mathrm{Cov}(Y_i, Y_j)$. Then, Γ is symmetric and non-negative definite. Furthermore, if $Y_1, ..., Y_n$ are jointly normally distributed with mean 0, then $\Gamma = A^t A$, where $(Y_1, ..., Y_n) \stackrel{d}{=} (X_1, ..., X_k)A$, $X_1, ..., X_k$ are standard normal and A is an $k \times n$ matrix.*

Let $Y_1, ..., Y_n$ be jointly normal distributed with mean 0 and $Y = (Y_1, ..., Y_n) \stackrel{d}{=} XA$. For $u = (u_1, ..., u_n)$

$$\begin{aligned} \mathbb{E}e^{iYu^t} = \mathbb{E}e^{iX(Au^t)} = \mathbb{E}\left(\prod_{j=1}^{k} e^{iX_j \sum_k A_{jk}u_k}\right) &= \prod_{j=1}^{k} \exp\left\{-\frac{1}{2}\left(\sum_{k=1}^{n} A_{jk}u_k\right)^2\right\} \\ &= \exp\{-\tfrac{1}{2}uA^tAu^t\} = \exp\{-\tfrac{1}{2}u\Gamma u^t\} \end{aligned}$$

This implies that the characteristic function of Y is specified by its covariance matrix. Conversely, if Y is a random vector (n-vector) whose characteristic function is given by $e^{-\frac{1}{2}u\Gamma u^t}$ where Γ is a symmetric and non-negative $n \times n$ matrix, then there exists A (by choosing A such that $\Gamma = A^t A$) and independent standard normal random variables $X_1, ..., X_n$ such that $Y \stackrel{d}{=} XA$.

Exercise 4.7. Assume that $Y_1, ..., Y_n$ have a joint normal distribution with mean 0. Show that $Y_1, ..., Y_n$ are independent if and only if their covariance matrix is a diagonal matrix. *Hint:* Use the fact that if $X_1, .., X_n$ are random variables with characteristic function $f_1, ..., f_n$, then $X_1, ..., X_n$ are independent if and only if

$$\mathbb{E}e^{i(u_1X_1+\cdots+u_nX_n)} = f_1(u_1) \times \cdots \times f_n(u_n) \quad \forall (u_1, ..., u_n) \in \mathbb{R}^n.$$

Definition 4.4. $Y_1, ..., Y_n$ have a joint normal distribution $\mathcal{N}(m, \Gamma)$ if $m = (\mathbb{E}Y_1, ..., \mathbb{E}Y_n)$ and $(Y_1, ..., Y_n) - m$ is jointly normal distributed with mean 0 and covariance matrix Γ.

Definition 4.5. Let $Y_1, ..., Y_n$ be random variable and $Y = (Y_1, ..., Y_n)$.

(1) Y is linearly independent if $c_1Y_1 + \cdots + c_nY_n = 0$ a.s. implies that $c_i = 0$ for $1 \le i \le n$.
(2) Suppose Y has distribution $\mathcal{N}(0, \Gamma)$. Y is *non-degenerate* if Y is linearly independent.

Exercise 4.8. Let $Y = (Y_1, ..., Y_n)$ has a joint normal distribution $\mathcal{N}(0, \Gamma)$. Show that
(1) Y is non-degenerate if and only if $\det \Gamma \neq 0$.
(2) If Y is non-degenerate, then the joint density satisfies

$$f(y) = \frac{1}{(2\pi)^{n/2}\sqrt{\det \Gamma}} \exp\{-\tfrac{1}{2} y \Gamma^{-1} y^t\}.$$

Exercise 4.9. Suppose $Y = (Y_1, ..., Y_n)$ has a joint normal distribution $\mathcal{N}(0, \Gamma)$. Let K be the smallest integer k such that $Y = XA$, where $X = (X_1, ..., X_k)$, $X_1, ..., X_k$ are i.i.d. standard normal random variables and A is a $k \times n$ matrix. Set

$$L = \max\{\ell : Y_{i_1}, ..., Y_{i_\ell} \text{ are linearly independent for } i_1 < \cdots < i_\ell\}.$$

Show that $K = L = Rank(\Gamma)$.

Theorem 4.12. *A stochastic process $(X_t)_{t \ge 0}$ is a normalized Brownian motion if and only if, for any $0 < t_1 < t_2 < \cdots < t_n$, $X_{t_1}, ..., X_{t_n}$ have joint normal distribution with mean 0 and covariance matrix Γ given by $\Gamma_{i,j} = \min\{t_i, t_j\}$.*

Theorem 4.13. *Let B_t be a normalized Brownian motion and set*

$$X_t = \begin{cases} tX_{1/t} & \text{if } t \in (0, \infty) \\ 0 & \text{if } t = 0 \end{cases}$$

Then, X_t is a Brownian motion of which paths are continuous on $[0, \infty)$ with probability 1.

The following is an immediate corollary of the above theorem concerning the behavior of a Brownian motion as time tends to infinity.

Corollary 4.14. *For any normalized Brownian motion B_t, one has*

$$\limsup_{t \to \infty} \frac{B_t}{\sqrt{2t \log \log t}} = 1 \quad a.s.$$

To prove this theorem, we need the following proposition.

Proposition 4.15. *Let B_t be a normalized Brownian motion. Then,*

$$\lim_{t \to \infty} \frac{B_t}{t} = 0 \quad a.s.$$

Proof of Theorem 4.13. It is clear that for $0 < t_1 < \cdots < t_n$, $X_{t_1}, ..., X_{t_n}$ have a joint normal distribution. Note that, for $s > 0$ and $t > 0$,

$$\mathbb{E}(X_t X_s) = st\mathbb{E}(B_{1/t} B_{1/s}) = st \min\{1/t, 1/s\} = \min\{t, s\}.$$

This implies that $(X_t)_{t \ge 0}$ and $(B_t)_{t \ge 0}$ have the same distribution. Note that X_t is continuous on $(0, \infty)$. By Proposition 4.15, one has

$$\lim_{t \downarrow 0} X_t = \lim_{t \downarrow 0} tB_{1/t} = \lim_{s \uparrow \infty} \frac{B_s}{s} = 0 \quad a.s.$$

□

Proof of Proposition 4.15. Let $Y_k = B_k - B_{k-1}$. By the independency and stationarity of increments, $Y_1, Y_2, ...$ are i.i.d. and, by the law of large numbers,

$$\frac{B_k}{k} = \frac{Y_1 + \cdots + Y_k}{k} \xrightarrow{a.s.} \mathbb{E}Y_1 = 0.$$

For $t \geq 0$, let k be an integer such that $t \in [k, k+1)$ and set

$$Z_k = \max_{0 \leq t \leq 1} |B_{k+t} - B_k|.$$

Observe that

$$\left| \frac{B_t}{t} - \frac{B_k}{k} \right| \leq \frac{|B_t - B_k|}{t} + \left(\frac{1}{k} - \frac{1}{t} \right) |B_k| \leq \frac{Z_k}{k} + \frac{|B_k|}{k^2}.$$

The second term has been shown to converge to 0 a.s. For the first term, we recall a fact generalized from Lemma 4.5 using the continuity of Brownian paths.
Fact 1: Let B_t be a normalized Brownian motion. For $t > 0$,

$$\mathbb{P}\left(\max_{0 \leq s \leq t} B_s > x \right) \leq 2\mathbb{P}(B_t > x), \quad \mathbb{P}\left(\max_{0 \leq s \leq t} |B_s| > x \right) \leq 2\mathbb{P}(|B_t| > x).$$

Using this fact, one can show that $\mathbb{E}Z_k < \infty$. Recall another fact in the following.
Fact 2: Let $X_1, X_2, ...$ be i.i.d. random variables. Then, $\mathbb{E}|X_1| < \infty$ if and only if $\mathbb{P}(|X_n| > n \ i.o.) = 0$.

Note that, for any $L > 0$, $LZ_1, LZ_2, ...$ are i.i.d. with $\mathbb{E}(LZ_1) < \infty$. By Fact 2, one has

$$\mathbb{P}\left(\limsup_{k \to \infty} \frac{Z_k}{k} \leq \frac{1}{L} \right) \geq \mathbb{P}(LZ_k \leq k \text{ for } k \text{ large enough}) = 1.$$

This implies $Z_k/k \to 0$ a.s. □

4.4. Strong Markov property.

Definition 4.6. For any process $(X_t)_{t \geq 0}$, set $\mathcal{F}_t := \mathcal{F}(X_s, s \leq t)$.

1. A nonnegative random variable T is called a stopping time for $\mathcal{F}_t$ if $\{T \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$.
2. For any non-negative random variable T, define

$$\mathcal{F}_T := \{B \in \mathcal{F} : B \cap \{T \leq t\} \in \mathcal{F}_t, \forall t \geq 0\}.$$

Remark 4.4. For any stopping times $S \leq T$, $\mathcal{F}_S \subset \mathcal{F}_T$.

Proposition 4.16. *Let X_t be a continuous process and T be a stopping time for $\mathcal{F}_t = \mathcal{F}(X_s; 0 \leq s \leq t)$. If T is real-valued, then X_T is $\mathcal{F}_T$-measurable.*

Proof. For $n \geq 1$, set

$$A_{n,k} = \left\{ \frac{k}{n} \leq T < \frac{k+1}{n} \right\}, \quad X_{n,T} = \sum_{k=0}^{\infty} X_{k/n} \mathbf{1}_{A_{n,k}}.$$

Then, $X_{n,T}$ is $\mathcal{F}$-measurable and, on $\{T \leq N\}$,

$$|X_T - X_{n,T}| \leq \max_{\substack{0 \leq t \leq N \\ 0 \leq h \leq 1/n}} |X_{t+h} - X_t|.$$

By the continuity of X_t, the right hand side converges to 0. This implies $X_{n,T} \to X_T$ and, hence, X_T is $\mathcal{F}_T$-measurable. □

Theorem 4.17 (The strong Markov property)**.** *Let B_t be a normalized Brownian motion and T be a stopping time for $\mathcal{F}_t = \mathcal{F}(B_s; 0 \leq s \leq t)$. Assume $\mathbb{P}(T < \infty) = 1$ and set $Y_t = B_{T+t} - B_T$ for $t \geq 0$. Then, Y_t is a normalized Brownian motion and independent of $\mathcal{F}_T$.*

Proof. First, consider the case that T takes values on a countable set, say $\{\tau_1, \tau_2, ...\}$. Let $0 < t_1 < t_2 < \cdots < t_k$, $A_1, ..., A_k \in \mathcal{B}(\mathbb{R})$ and $B \in \mathcal{F}_T$. Note that $\{T = \tau_j\} \in \mathcal{F}_{\tau_j}$ and

$$\{T = \tau_j\} \cap B = \{T = \tau_j\} \cap (\{T \le \tau_j\} \cap B) \in \mathcal{F}_{\tau_j}.$$

This implies

$$\begin{aligned}
&\mathbb{P}(Y_{t_1} \in A_1, ..., Y_{t_k} \in A_k, B) \\
=&\sum_{j=1}^{\infty} \mathbb{P}(Y_{t_1} \in A_1, ..., Y_{t_k} \in A_k, T = \tau_j, B) \\
=&\sum_{j=1}^{\infty} \mathbb{P}(B_{\tau_j+t_1} - B_{\tau_j} \in A_1, ..., B_{\tau_j+t_k} - B_{\tau_j} \in A_k, T = \tau_j, B) \\
=&\sum_{j=1}^{\infty} \mathbb{P}(B_{\tau_j+t_1} - B_{\tau_j} \in A_1, ..., B_{\tau_j+t_k} - B_{\tau_j} \in A_k)\mathbb{P}(T = \tau_j, B) \\
=&\mathbb{P}(B_{t_1} - B_0 \in A_1, ..., B_{t_k} - B_0 \in A_k)\mathbb{P}(B).
\end{aligned}$$

This case is then proved by the $\pi - \lambda$ lemma.

Next, let T be a stopping time satisfying $\mathbb{P}(T < \infty) = 1$ and set, for $n \ge 1$,

$$T_n = \frac{1}{n}\mathbf{1}_{\{0 \le T \le 1/n\}} + \sum_{k=1}^{\infty} \frac{k+1}{n}\mathbf{1}_{\{k/n < T \le (k+1)/n\}}.$$

Note that, for $t \in [k/n, (k+1)/n)$,

$$\{T_n \le t\} = \{T \le k/n\} \in \mathcal{F}_{k/n} \subset \mathcal{F}_t$$

and, for $B \in \mathcal{F}_T$,

$$B \cap \{T_n \le t\} = B \cap \{T_n \le k/n\} = B \cap \{T \le k/n\} \in \mathcal{F}_{k/n} \subset \mathcal{F}_t.$$

The implies that T_n is a stopping time and $\mathcal{F}_T \subset \mathcal{F}_{T_n}$. For $n \ge 1$ and $t \ge 0$, define $Z_{n,t} = B_{T_n+t} - B_{T_n}$. Since T_n takes values on a countable set, one has, for $0 < t_1 < \cdots < t_k$ and $x_1, ..., x_k \in \mathbb{R}$,

$$\mathbb{P}(Z_{n,t_1} \le x_1, ..., Z_{n,t_k} \le x_k, B) = \mathbb{P}(B_{t_1} - B_0 \le x_1, ..., B_{t_k} - B_0 \le x_k)\mathbb{P}(B).$$

By the continuity of Brownian paths, $Z_{n,t} \to Y_t$ since $T_n \to T$. As a result, if $(x_1, ..., x_k)$ is a continuous point of the joint distribution of $Y_{t_1}, ..., Y_{t_k}$, which is dense on $\mathbb{R}^k$, then

$$\mathbb{P}(Y_{t_1} \le x_1, ..., Y_{t_k} \le x_k, B) = \mathbb{P}(B_{t_1} - B_0 \le x_1, ..., B_{t_k} - B_0 \le x_k)\mathbb{P}(B).$$

The desired conclusion is then given by the $\pi - \lambda$ lemma. $\square$