2. Probability spaces

2.1. **Probabilities.**

Definition 2.1. Given a set Ω, a non-empty collection $\mathcal{F}$ of subsets of Ω is a **field** over Ω if

(1) $A \cap B \in \mathcal{F}$ and $A \cup B \in \mathcal{F}$ for $A, B \in \mathcal{F}$.
(2) $A^c \in \mathcal{F}$ for $A \in \mathcal{F}$.

The elements of $\mathcal{F}$ are called **events**.

Remark 2.1. A field is a collection of subsets which is closed under finite union, finite intersection and complement. In particular, $\emptyset$ and Ω are contained in $\mathcal{F}$.

Remark 2.2. The requirement of $A \cap B \in \mathcal{F}$ in (1) can be removed using the identity $A \cap B = (A^c \cup B^c)^c$.

Remark 2.3. The requirement of closedness under the union can be replaced by the following.

$$A \cup B \in \mathcal{F}, \quad \forall A, B \in \mathcal{F},\ A \cap B = \emptyset.$$

Exercise 2.1. Could Definition 2.1(1) be replaced by the following?

$$A, B \in \mathcal{F},\ A \cap B = \emptyset \quad \Rightarrow \quad A \cup B \in \mathcal{F}.$$

Definition 2.2. Let $\mathcal{F}$ be a field over Ω. A non-negative set function $\mathbb{P}$ defined on $\mathcal{F}$ is called a **finite probability** if

(1) (Normalization) $\mathbb{P}(\Omega) = 1$.
(2) (Finite additivity) $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$ for all $A, B \in \mathcal{F}$ satisfying $A \cap B = \emptyset$.

The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a **finite probability space**.

Remark 2.4. Note that $\mathbb{P}(\emptyset) = 0$ and, for $A, B \in \mathcal{F}$, $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$.

Example 2.1. For $n \geq 1$, let $\Omega_n = \{\omega = (\omega_1, ..., \omega_n) | \omega_i \in \{0, 1\}, \forall i\}$, $\mathcal{F}_n$ be the power set of Ω_n and $\mathbb{P}_n(A) = |A|/2^n$. Then, $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$ forms a finite probability. Set $\Omega = \{\omega = (\omega_1, \omega_2, ...) | \omega_i \in \{0, 1\}, \forall i\}$ and $\mathcal{F} = \bigcup_n (\mathcal{F}_n \times \Omega)$, where $\mathcal{F}_n \times \Omega := \{A \times \Omega | A \in \mathcal{F}_n\}$ and $A \times \Omega := \{\omega = (\omega_1, ...) | (\omega_1, ..., \omega_n) \in A\}$. One may check that $\mathcal{F}$ is a field.

Observe that, for $B \in \mathcal{F}$, there are $n \geq 1$ and $A \in \mathcal{F}_n$ such that $B = A \times \Omega$. In the above setting, define $\mathbb{P}(B) = \mathbb{P}_n(A)$. Note that the definition is independent of the choice of n and A, and $\mathbb{P}$ becomes a finite probability.

Exercise 2.2. Let $A_1, ..., A_n \in \mathcal{F}$. Show that $\mathbb{P}(\bigcup_{i=1}^n A_i) \leq \sum_{i=1}^n \mathbb{P}(A_i)$.

Definition 2.3. A field $\mathcal{F}$ over a set Ω is a σ-field if the countable union and countable intersection of events in $\mathcal{F}$ are closed. That is, if $A_1, A_2, ... \in \mathcal{F}$, then $\bigcup_n A_n \in \mathcal{F}$ and $\bigcap_n A_n \in \mathcal{F}$.

Remark 2.5. The filed $\mathcal{F}$ in Example 2.1 is not a σ-field.

Remark 2.6. Similar to Remarks 2.2 and 2.3, the requirement of $\bigcap_n A_n \in \mathcal{F}$ can be removed or the requirement $\bigcup_n A_n \in \mathcal{F}$ can be restricted to sequences of mutually disjoint sets.

Exercise 2.3. Prove Remark 2.6.

Exercise 2.4. Let $\mathcal{F}$ be a collection of subsets of Ω. Prove that $\mathcal{F}$ is a σ-field if and only if

(1) $\Omega \in \mathcal{F}$.
(2) $A \cap B \in \mathcal{F}$ for all $A, B \in \mathcal{F}$.
(3) $A^c \in \mathcal{F}$ for all $A \in \mathcal{F}$.
(4) $\bigcup_n A_n \in \mathcal{F}$ for any mutually disjoint sequence $A_n \in \mathcal{F}$.

Definition 2.4. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a finite probability space. $\mathbb{P}$ is called **σ-additive** or **countably additive** if $\mathbb{P}(\bigcup_{n=1}^\infty A_n) = \sum_{n=1}^\infty \mathbb{P}(A_n)$ for any sequence of mutually disjoint events $A_n \in \mathcal{F}$ satisfying $\bigcup_{n=1}^\infty A_n \in \mathcal{F}$.

Exercise 2.5. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a finite probability and $A_n \in \mathcal{F}$ for $n \geq 1$. Show that if $\mathbb{P}$ is σ-additive and $\bigcup_{n=1}^\infty A_n \in \mathcal{F}$, then $\mathbb{P}(\bigcup_{n=1}^\infty A_n) \leq \sum_{n=1}^\infty \mathbb{P}(A_n)$.

Theorem 2.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a finite probability. Then, the following are equivalent.*

(1) $\mathbb{P}$ *is σ-additive.*
(2) $\mathbb{P}$ *is continuous from above. That is, $\mathbb{P}(\bigcap_{n=1}^\infty A_n) = \lim_n \mathbb{P}(A_n)$ for $A_n \in \mathcal{F}$ satisfying $A_n \supset A_{n+1}$ and $\bigcap_{n=1}^\infty A_n \in \mathcal{F}$.*
(3) $\mathbb{P}$ *is continuous from below. That is, $\mathbb{P}(\bigcup_{n=1}^\infty A_n) = \lim_n \mathbb{P}(A_n)$ for $A_n \in \mathcal{F}$ satisfying $A_n \subset A_{n+1}$ and $\bigcup_{n=1}^\infty A_n \in \mathcal{F}$.*
(4) $\mathbb{P}$ *is continuous at $\emptyset$. That is, $\lim_n \mathbb{P}(A_n) = 0$ for $A_n \in \mathcal{F}$ satisfying $A_n \supset A_{n+1}$ and $\cap_{n=1}^\infty A_n = \emptyset$.*

Proof. Obviously, (2) and (3) are equivalent. Assume that (1) holds and let $A_n \subset A_{n+1}$. Set $B_1 = A_n$ and $B_{n+1} = A_{n+1} \setminus A_n$ for $n \geq 1$. It is clear that $B_n \in \mathcal{F}$, $\bigcup_{m=1}^n B_m = A_n$ and $\bigcup_{n=1}^\infty A_n = \bigcup_{n=1}^\infty B_n$. Since B_n are mutually disjoint, the σ-additivity implies

$$\mathbb{P}\left(\bigcup_{n=1}^\infty A_n\right) = \mathbb{P}\left(\bigcup_{n=1}^\infty B_n\right) = \sum_{n=1}^\infty \mathbb{P}(B_n) = \lim_{n\to\infty} \sum_{m=1}^n \mathbb{P}(B_m) = \lim_{n\to\infty} \mathbb{P}(A_n).$$

This implies (3). Note that (4) is a special case of (2), it remains to show that (4)⇒(1). Assume (4) holds and let $A_n \in \mathcal{F}$ be mutually disjoint and satisfy $\bigcup_{n=1}^\infty A_n \in \mathcal{F}$. Set $C_n = A \setminus \bigcup_{m=1}^n A_m$. Clearly, $C_n \in \mathcal{F}$, $C_n \supset C_{n+1}$ and $\bigcap_{n=1}^\infty C_n = \emptyset$. Observe that

$$\mathbb{P}(A) = \mathbb{P}(C_n) + \mathbb{P}(A \setminus C_n) = \mathbb{P}(C_n) + \sum_{m=1}^n \mathbb{P}(A_m).$$

Letting n tend to infinity gives the desired identity. □

Exercise 2.6. Prove that, in Example 2.1, $\mathbb{P}$ is σ-additive. *Hint: Cantor's intersection theorem.*

Definition 2.5. A triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a **probability space** if $\mathcal{F}$ is a σ-field and

(1) (Normalization) $\mathbb{P}(\Omega) = 1$.
(2) (σ-additivity) $\mathbb{P}(\bigcup_{n=1}^\infty A_n) = \sum_{n=1}^\infty \mathbb{P}(A_n)$ for any mutually disjoint sequence $A_n \in \mathcal{F}$.

$\mathbb{P}$ is called a **probability**.

Definition 2.6. Let $\mathcal{F}$ be a σ-field and $A_n \in \mathcal{F}$.

(1) $\limsup_n A_n$ is the event that contains infinitely many A_n, which is exactly the set $\bigcap_{n=1}^\infty (\bigcup_{m=n}^\infty A_m)$. We write it $\{A_n \text{ i.o.}\}$ for short.
(2) $\liminf_n A_n$ is the event that contains all except finitely many A_n, which is equal to $\bigcup_{n=1}^\infty (\bigcap_{m=n}^\infty A_m)$.

Write $\lim_n A_n$ for either case if $\limsup_n A_n = \liminf_n A_n$.

Exercise 2.7. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $A_n \in \mathcal{F}$.

(1) Show that $\mathbb{P}(\limsup_n A_n) \geq \limsup_n \mathbb{P}(A_n)$ and $\mathbb{P}(\liminf_n A_n) \leq \liminf_n \mathbb{P}(A_n)$.
(2) Give examples such the equalities in (1) fail.
(3) Use (1) to conclude that if $\lim_n A_n$ exists, then $\mathbb{P}(\lim_n A_n) = \lim_n \mathbb{P}(A_n)$.

2.2. **Fields and σ-fields.** Given a set Ω, there are two trivial σ-fields, $\{A|A \subset \Omega\}$ and $\{\emptyset, \Omega\}$. Note that if, for $\lambda \in L$, $\mathcal{F}_\lambda$ is a field (resp. σ-field), then $\bigcap_{\lambda \in L} \mathcal{F}_\lambda$ is a field (resp. σ-field).

Remark 2.7. Let Ω be a set and $\mathcal{E}$ be a collection of subsets of Ω. There exist smallest field and σ-field containing $\mathcal{E}$, which are denoted by $\mathcal{F}(\mathcal{E})$ and $\sigma(\mathcal{E})$ in this notes.

Remark 2.8. Let $\mathcal{F}$ be a collection of subsets of Ω. Note that, when there is $B \subset \Omega$ such that $A \subset B$ for all $A \in \mathcal{F}$, one may consider the smallest σ-field over B that contains $\mathcal{F}$ and we write it as $\sigma_B(\mathcal{F})$.

Definition 2.7. Let Ω be a set and $\mathcal{E}$ be a collection subsets of Ω. $\mathcal{E}$ is said to be a **monotone class** if $\lim_n A_n \in \mathcal{E}$ for $A_n \in \mathcal{E}$ satisfying $A_n \subset A_{n+1}$ for all n or $A_n \supset A_{n+1}$ for all n.

Remark 2.9. As before, monotone classes are closed under the intersection and this implies that if $\mathcal{E}$ is a collection of subsets of Ω, then there is a smallest monotone class containing $\mathcal{E}$. We write it as $\mu(\mathcal{E})$.

Theorem 2.2 (Monotone class theorem)**.** *Let $\mathcal{F}$ be a field over Ω. Then, $\mu(\mathcal{F}) = \sigma(\mathcal{F})$.*

Proof. It is obvious from the definition that $\mu(\mathcal{F}) \subset \sigma(\mathcal{F})$. For $\mu(\mathcal{F}) \supset \sigma(\mathcal{F})$, it remains to show that $\mu(\mathcal{F})$ is closed under complement and finite intersection. We prove the closedness of complement and leave the closedness of finite intersection for the reader. Set

$$\mathcal{M} = \{A \in \mu(\mathcal{F}) | A^c \in \mu(\mathcal{F})\}.$$

Clearly, $\mathcal{F} \subset \mathcal{M} \subset \mu(\mathcal{F})$ and it suffices to prove that $\mathcal{M}$ is a monotone class, which leads to $\mathcal{M} = \mu(\mathcal{F})$. Let $A_n \in \mathcal{M}$ be a monotone sequence. Clearly, $A_n^c \in \mu(\mathcal{F})$. Since $\mu(\mathcal{F})$ is a monotone class, $\lim_n A_n \in \mu(\mathcal{F})$ and $(\lim_n A_n)^c = \lim_n A_n^c \in \mu(\mathcal{F})$. This implies $\lim_n A_n \in \mathcal{M}$, as desired. The proof is known as the **principle of appropriate sets**. ☐

Definition 2.8. Let $\mathcal{C}$ be a class of subsets of Ω. Then, $\mathcal{C}$ is called

- a *π-system* if it is closed under finite intersection. That is, $A \cap B \in \mathcal{C}$ for $A, B \in \mathcal{C}$.
- a *λ-system* if it contains Ω and is closed under the complements and the countable disjoint unions. That is,

$$\Omega \in \mathcal{C}, \quad A^c \in \mathcal{C}, \ \forall A \in \mathcal{C}$$

  and

$$\bigcup_{n=1}^{\infty} A_n \in \mathcal{C}$$

  for any sequence $(A_n)_{n=1}^{\infty}$ of mutually disjoint sets in $\mathcal{C}$.

Remark 2.10. If a π-system is also a λ-system, then it is a σ-field.

Remark 2.11. Assuming $\Omega \in \mathcal{C}$ and the closedness of finite disjoint unions in $\mathcal{C}$, the closedness of complement is equivalent to the closedness of proper difference, that is, $B \setminus A \in \mathcal{C}$ for all $A, B \in \mathcal{C}$ satisfying $A \subset B$.

Remark 2.12. For any two probabilities $\mathbb{P}_1, \mathbb{P}_2$ on $(\Omega, \mathcal{F})$, the class of events in $\mathcal{F}$ such that $\mathbb{P}_1 = \mathbb{P}_2$ forms a λ-system.

Lemma 2.3 (The $\pi - \lambda$ lemma)**.** *If $\mathcal{P}$ is a π-system and $\mathcal{L}$ is a λ-system that contains $\mathcal{P}$, then $\sigma(\mathcal{P}) \subset \mathcal{L}$.*

Proof. Let $\mathcal{L}_0$ be the smallest λ-system containing $\mathcal{P}$. By Remark 2.10, it suffices to show that $\mathcal{L}_0$ is a π-system. For $A \subset \Omega$, let $\mathcal{L}_A$ be the class of all subsets B of Ω such that $A \cap B \in \mathcal{L}_0$. It is obvious that

$$A \in \mathcal{L}_B \quad \Leftrightarrow \quad B \in \mathcal{L}_A.$$

We prove in the following that $\mathcal{L}_A$ is a λ-system for $A \in \mathcal{L}_0$. Let $A \in \mathcal{L}_0$. Observe that $A \cap \Omega = A \in \mathcal{L}_0$. This implies $\Omega \in \mathcal{L}_A$. Let $(B_n)_{n=1}^\infty$ be a sequence of mutually disjoint sets in $\mathcal{L}_A$. Then,

$$A \cap \left(\bigcup_{n=1}^\infty B_n \right) = \bigcup_{n=1}^\infty (A \cap B_n) \in \mathcal{L}_0,$$

which yields $\bigcup_{n=1}^\infty B_n \in \mathcal{L}_A$. For $B \in \mathcal{L}_A$, one has $B \cap A \in \mathcal{L}_0$. By Remark 2.11, $B^c \cap A = A \setminus (B \cap A) \in \mathcal{L}_0$. This implies $B^c \in \mathcal{L}_A$ as desired.

Let $A \in \mathcal{P}$. Note that if $B \in \mathcal{P}$, then $A \cap B \in \mathcal{P} \subset \mathcal{L}_0$. This implies $\mathcal{P} \subset \mathcal{L}_A$ for $A \in \mathcal{P}$ and, then, $\mathcal{L}_0 \subset \mathcal{L}_A$. Next, for $A \in \mathcal{P}$ and $B \in \mathcal{L}_0$, one has $B \in \mathcal{L}_A$ or equivalently $A \in \mathcal{L}_B$. This implies that, for $B \in \mathcal{L}_0$, $\mathcal{P} \subset \mathcal{L}_B$ and hence $\mathcal{L}_0 \subset \mathcal{L}_B$. Consequently, we may conclude that, for $A, B \in \mathcal{L}_0$, $A \cap B \in \mathcal{L}_0$ and this proves that $\mathcal{L}_0$ is a π-system. □

Exercise 2.8. Let $\mathcal{F}$ be a collection of subsets of Ω. Define $\mathcal{F} \cap B := \{A \cap B | A \in \mathcal{F}\}$ for $B \subset \Omega$ and $\mathcal{F} \times B = \{A \times B | A \in \mathcal{F}\}$ for any set B. Show that, $\sigma_B(\mathcal{F} \cap B) = \sigma_\Omega(\mathcal{F}) \cap B$ for $B \subset \Omega$ and $\sigma_{\Omega \times B}(\mathcal{F} \times B) = \sigma_\Omega(\mathcal{F}) \times B$ for any set B.

Remark 2.13. In the case $\Omega = \mathbb{R}$ and $\mathcal{F} = \{(a, b] : a, b \in \mathbb{R}, a < b\}$, $\sigma(\mathcal{F})$ is call the Borel σ-field on $\mathbb{R}$ and is denoted by $\mathcal{B}(\mathbb{R})$.

Exercise 2.9. Let $\mathcal{F}$ be a class of subsets of $\mathbb{R}$. Prove that if, for any $a < b$, $\mathcal{F}$ contains at least one of (a, b), $(a, b]$, $[a, b)$ and $[a, b]$, then $\sigma(\mathcal{F}) \supset \mathcal{B}(\mathbb{R})$.

In the following, a collection of subsets of any set Ω is mostly assumed to contain Ω.

Example 2.2. Given two sets Ω_1, Ω_2, let $\mathcal{F}_1$ and $\mathcal{F}_2$ be collections of subsets in Ω_1 and Ω_2 containing Ω_1, Ω_2. We use the notation $\mathcal{F}_1 \otimes \mathcal{F}_2$ to denote the smallest σ-filed generated by $\mathcal{F}_1 \times \mathcal{F}_2 := \{A_1 \times A_2 : A_i \in \mathcal{F}_i, i = 1, 2\}$, i.e. $\sigma(\mathcal{F}_1 \times \mathcal{F}_2)$. Then, $\mathcal{F}_1 \otimes \mathcal{F}_2 = \sigma(\mathcal{F}_1) \otimes \sigma(\mathcal{F}_2)$. To prove this identity, it suffices to show that

$$\mathcal{F}_1 \otimes \mathcal{F}_2 = \sigma(\mathcal{F}_1) \otimes \mathcal{F}_2, \quad \mathcal{F}_1 \otimes \mathcal{F}_2 = \mathcal{F}_1 \otimes \sigma(\mathcal{F}_2).$$

We consider the former, while the latter can be proved in a similar way. Clearly, one has $\mathcal{F}_1 \otimes \mathcal{F}_2 \subset \sigma(\mathcal{F}_1) \otimes \mathcal{F}_2$. To see the inverse direction, let $B \in \mathcal{F}_2$. Note that $\Omega_1 \times B \in \mathcal{F}_1 \otimes \mathcal{F}_2$. By Exercise 2.8, this implies

$$\begin{aligned} \mathcal{F}_1 \otimes \mathcal{F}_2 &\supset \sigma(\mathcal{F}_1 \times \mathcal{F}_2) \cap (\Omega_1 \times B) = \sigma_{\Omega_1 \times B}((\mathcal{F}_1 \times \mathcal{F}_2) \cap (\Omega_1 \times B)) \\ &\supset \sigma_{\Omega_1 \times B}(\mathcal{F}_1 \times B) = \sigma(\mathcal{F}_1) \times B. \end{aligned}$$

Hence, we have $\mathcal{F}_1 \otimes \mathcal{F}_2 \supset \sigma(\mathcal{F}_1) \times \mathcal{F}_2$, which leads to $\mathcal{F}_1 \otimes \mathcal{F}_2 \supset \sigma(\mathcal{F}_1) \otimes \mathcal{F}_2$.

Exercise 2.10. For $i = 1, 2$, let $\mathcal{F}_i$ be a collection of subsets of Ω_i. Find an example such that $\mathcal{F}_1 \otimes \mathcal{F}_2 \neq \sigma(\mathcal{F}_1) \otimes \sigma(\mathcal{F}_2)$.

Example 2.3. Let $n \geq 1$. For $1 \leq i \leq n$, let $\mathcal{F}_i$ be a collection of subsets of Ω_i containing Ω_i. We write $\bigotimes_{i=1}^n \mathcal{F}_i$ or $\mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_n$ for the smallest σ-field generated by $\prod_{i=1}^n \mathcal{F}_i = \mathcal{F}_1 \times \cdots \times \mathcal{F}_n$. By Example 2.2, one has

$$\bigotimes_{i=1}^n \mathcal{F}_i = \left(\prod_{i=1}^k \mathcal{F}_i \right) \otimes \left(\prod_{i=k+1}^n \mathcal{F}_i \right) = \left(\bigotimes_{i=1}^k \mathcal{F}_i \right) \otimes \left(\bigotimes_{i=k+1}^n \mathcal{F}_i \right).$$

Inductively, we obtain $\bigotimes_{i=1}^n \mathcal{F}_i = \bigotimes_{i=1}^n \sigma(\mathcal{F}_i)$. This leads to the fact that if $I_1, ..., I_n$ are collections of intervals of $\mathbb{R}$ in Remark 2.13, then $\sigma(I_1 \times \cdots \times I_n) = \mathcal{B}(\mathbb{R}) \otimes \cdots \otimes \mathcal{B}(\mathbb{R})$, which is also written as $\mathcal{B}(\mathbb{R}^n)$. (Why $\mathbb{R} \in I_i$ for all $1 \leq i \leq n$ is not requested?)

Example 2.4. Let $\Omega_1, \Omega_2, ...$ be sets and $\Omega = \{\omega = (\omega_1, \omega_2, ...)|\omega_i \in \Omega_i, \forall i \geq 1\}$. For $n \geq 1$ and $A \subset \prod_{i=1}^n \Omega_i$, define $\mathcal{C}(A) = \{\omega \in \Omega|(\omega_1, ..., \omega_n) \in A\}$. For $n \geq 1$, let $\mathcal{F}_n$ be a collection of subsets of Ω_n including Ω_n. Consider the following classes.

$$\mathcal{C}_1 = \bigcup_{n=1}^{\infty} \{\mathcal{C}(A_1 \times \cdots \times A_n)|A_i \in \mathcal{F}_i, \forall 1 \leq i \leq n\},$$

and

$$\mathcal{C}_2 = \bigcup_{n=1}^{\infty} \{\mathcal{C}(A_1 \times \cdots \times A_n)|A_i \in \sigma(\mathcal{F}_i), \forall 1 \leq i \leq n\},$$

and

$$\mathcal{C}_3 = \bigcup_{n=1}^{\infty} \{\mathcal{C}(A)|A \in \mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_n\}.$$

Obviously, $\mathcal{C}_2$ is a π-system and $\mathcal{C}_3$ is a field. It follows immediately from Exercise 2.8 that $\sigma(\mathcal{C}_1) = \sigma(\mathcal{C}_2) = \sigma(\mathcal{C}_3)$. We write $\sigma(\mathcal{C}_i)$ as $\bigotimes_{n=1}^{\infty} \mathcal{F}_n$.

In the case that $\Omega_n = \mathbb{R}$ and $\mathcal{F}_n$ is the class of intervals as in Remark 2.13, we write $\mathbb{R}^\infty$ and $\mathcal{B}(\mathbb{R}^\infty)$ for Ω and $\sigma(\mathcal{C}_1)$.

For $n \geq 1$, let $\mathcal{F}_n$ be a collection of subsets of Ω_n. Let ρ be a permutation of $\{1, 2, ...\}$. For $B \subset \prod_{n=1}^{\infty} \Omega_n$, define $\rho(B) = \{(\omega_{\rho(1)}, \omega_{\rho(2)}, ...)|(\omega_1, \omega_2, ...) \in B\}$ and, for any collection $\mathcal{C}$ of subsets of $\prod_{n=1}^{\infty} \Omega_n$, define $\rho(\mathcal{C}) = \{\rho(B)|B \in \mathcal{C}\}$. In the case $\Omega_n \in \mathcal{F}_n$ for all n, one can show that $\rho(\prod_{n=1}^{\infty} \mathcal{F}_n) = \prod_{n=1}^{\infty} \mathcal{F}_{\rho(n)}$, which leads to

$$\rho\left(\bigotimes_{n=1}^{\infty} \mathcal{F}_n\right) = \bigotimes_{n=1}^{\infty} \mathcal{F}_{\rho(n)}. \tag{2.1}$$

Further, for any increasing sequence of positive integer, $(s_n)_{n=1}^{\infty}$, and $B \in \bigotimes_{n=1}^{\infty} \mathcal{F}_{s_n}$, one has

$$\{\omega = (\omega_1, \omega_2, ...)|(\omega_{s_n})_{n=1}^{\infty} \in B\} \in \bigotimes_{n=1}^{\infty} \mathcal{F}_n. \tag{2.2}$$

Example 2.5. Let T be a subset of $\mathbb{R}$ and, for $t \in T$, let Ω_t be a set and $\Omega = \{\omega = (\omega_t)_{t \in T}|\omega_t \in \Omega_t, \forall t \in T\}$. For $t \in T$, let $\mathcal{F}_t$ be a collection of subsets of Ω_t and assume that $\Omega_t \in \mathcal{F}_t$. For $n \geq 1$, $t_1, t_2, ..., t_n \in T$ and $A \subset \prod_{i=1}^n \Omega_{t_i}$, set $\mathcal{D}(t_1, ..., t_n, A) = \{\omega \in \Omega|(\omega_{t_1}, ..., \omega_{t_n}) \in A\}$. For any sequence $t_n \in T$ and $A \subset \prod_{n=1}^{\infty} \Omega_{t_n}$, define $\mathcal{D}(t_1, t_2, ..., A) = \{\omega \in \Omega|(\omega_{t_1}, \omega_{t_2}, ...) \in A\}$. Consider the following classes.

$$\mathcal{D}_1 = \bigcup_{n=1}^{\infty} \{\mathcal{D}(t_1, ..., t_n, A_1 \times \cdots \times A_n)|t_1, ..., t_n \in T, A_i \in \mathcal{F}_{t_i}, \forall 1 \leq i \leq n\} \tag{2.3}$$

and

$$\mathcal{D}_2 = \bigcup_{n=1}^{\infty} \{\mathcal{D}(t_1, ..., t_n, A_1 \times \cdots \times A_n)|t_1, ..., t_n \in T, A_i \in \sigma(\mathcal{F}_{t_i}), \forall 1 \leq i \leq n\}$$

and

$$\mathcal{D}_3 = \bigcup_{n=1}^{\infty} \{\mathcal{D}(t_1, ..., t_n, A)|t_1, ..., t_n \in T, A \in \mathcal{F}_{t_1} \otimes \cdots \otimes \mathcal{F}_{t_n}\}. \tag{2.4}$$

As before, one can show that $\sigma(\mathcal{D}_1) = \sigma(\mathcal{D}_2) = \sigma(\mathcal{D}_3)$ and we write $\bigotimes_{t \in T} \mathcal{F}_t$ for them.

Theorem 2.4. *For $A \in \bigotimes_{t \in T} \mathcal{F}_t$, there are a sequence $(t_n)_{n=1}^{\infty}$ in T and a set $B \in \bigotimes_{n=1}^{\infty} \mathcal{F}_{t_n}$ such that $A = \mathcal{D}(t_1, t_2, ..., B)$.*

Proof. Let $\mathcal{D}$ be the following class

$$\mathcal{D} = \left\{ \mathcal{D}(t_1, t_2, ..., B) \middle| t_n \in T, B \in \bigotimes_{n=1}^{\infty} \mathcal{F}_{t_n} \right\}$$

and $\mathcal{D}_1$ be the class in (2.3). Clearly, $\mathcal{D}_1 \subset \mathcal{D}$. To prove this theorem, it remains to show that $\mathcal{D} \subset \sigma(\mathcal{D}_1)$ and $\mathcal{D}$ is a σ-field. First, we fix a sequence $t_n \in T$ and let $\mathcal{C}_1$ be the class relative to the sequence $\mathcal{F}_{t_n}$ in Example 2.4. Clearly, $\sigma(\mathcal{C}_1) = \bigotimes_{n=1}^{\infty} \mathcal{F}_{t_n}$. By setting $\overline{\Omega} = \prod_{n=1}^{\infty} \Omega_{t_n}$ and $\widetilde{\Omega} = \prod_{t \in T \setminus \{t_1, t_2, ...\}} \Omega_t$, one has $\Omega = \overline{\Omega} \times \widetilde{\Omega}$ and $\mathcal{D}(t_1, t_2, ..., B) = B \times \widetilde{\Omega}$ for $B \in \sigma(\mathcal{C}_1)$. By Exercise 2.8, this implies

$$\sigma(\{\mathcal{D}(t_1, t_2, ..., B) | B \in \mathcal{C}_1\}) = \sigma(\mathcal{C}_1 \times \widetilde{\Omega}) = \sigma(\mathcal{C}_1) \times \widetilde{\Omega} = \{\mathcal{D}(t_1, t_2, ..., B) | B \in \sigma(\mathcal{C}_1)\}.$$

Since $\{\mathcal{D}(t_1, t_2, ..., B) | B \in \mathcal{C}_1\} \subset \mathcal{D}_1$, we obtain $\mathcal{D} \subset \sigma(\mathcal{D}_1)$.

Next, we prove that $\mathcal{D}$ is a σ-field. Clearly, $\Omega \in \mathcal{D}$. Note that, for $A = \mathcal{D}(t_1, t_2, ..., B) \in \mathcal{D}$, $A^c = \mathcal{D}(t_1, t_2, ..., B^c) \in \mathcal{D}$. For $n \geq 1$, let $A_n = \mathcal{D}(t_{n,1}, t_{n,2}, ..., B_n)$ for some $B_n \in \bigotimes_{i=1}^{\infty} \mathcal{F}_{t_{n,i}}$. We write $\{s_1, s_2, ...\} = \{t_{n,i} | i, n \geq 1\}$ and set

$$B'_n = \left\{ v \in \prod_{m=1}^{\infty} \Omega_{s_m} \middle| (v_{t_{n,1}}, v_{t_{n,2}}, ...) \in B_n \right\}.$$

Clearly, $A_n = \mathcal{D}(s_1, s_2, ..., B'_n)$. Let $(i_j)_{j=1}^{\infty}$ be a subsequence of $\mathbb{N}$ such that $\{t_{n,j} : j = 1, 2, ...\} = \{s_{i_j} | j = 1, 2, ...\}$ and ρ be a permutation of $\mathbb{N}$ such that $s_{i_j} = t_{n,\rho(j)}$ for $j \geq 1$. By defining $\rho\left((\omega_{t_{n,j}})_{j=1}^{\infty}\right) = \left(\omega_{t_{n,\rho(j)}}\right)_{j=1}^{\infty}$, one has $B'_n = \{v \in \prod_{m=1}^{\infty} \Omega_{s_m} | (s_{i_j})_{j=1}^{\infty} \in \rho(B_n)\}$ and, by (2.1) and (2.2), $B'_n \in \bigotimes_{m=1}^{\infty} \mathcal{F}_{s_m}$. This implies $\bigcup_n A_n = \mathcal{D}(s_1, s_2, ..., \bigcup_n B'_n) \in \mathcal{D}$. The closedness of finite intersection can be shown in a similar way. This proves that $\mathcal{D}$ is a σ-field. $\square$

Remark 2.14. For the special case of $\Omega_t = \mathbb{R}$ and $\mathcal{F}_t = \{(a, b] : a < b\}$, we write $\mathbb{R}^T$ and $\mathcal{B}(\mathbb{R}^T)$ for Ω and $\bigotimes_{t \in T} \mathcal{F}_t$. Let $S \subset T$ be an uncountable set and consider

$$\{\omega \in \Omega | \omega_t < C, \forall t \in S\}, \quad \{\omega \in \Omega | \omega_t = 0, \text{ for some } t \in S\}.$$

In general, these sets are not in $\mathcal{B}(\mathbb{R}^T)$.