4. Random variables and concepts of convergence

4.1. **Random variables.**

Definition 4.1. Let $(\Omega, \mathcal{F})$ and $(R, \mathcal{B})$ be σ-fields. A function $X : \Omega \to R$ is called a $(\mathcal{F}/\mathcal{B})$ **random element** if X is $\mathcal{F}$-measurable, i.e. $\{X \in B\} \in \mathcal{F}$ for all $B \in \mathcal{B}$. If $R = \mathbb{R}$ and $\mathcal{B} = \mathcal{B}(\mathbb{R})$, then X is called a **random variable**. If $R = \mathbb{R}^n$ and $\mathcal{B} = \mathcal{B}(\mathbb{R}^n)$, then X is called an n**-dimensional random vector**.

Remark 4.1. Let X be a map from Ω to R and set $\mathcal{F}(X) = \{X^{-1}(B)|B \in \mathcal{B}\}$ and $\mathcal{B}(X) = \{B \subset R|X^{-1}(B) \in \mathcal{F}\}$. Then, $\mathcal{F}(X), \mathcal{B}(X)$ are σ-fields over Ω, R, and X is $\mathcal{F}$-measurable iff $\mathcal{F}(X) \subset \mathcal{F}$ iff $\mathcal{B}(X) \supset \mathcal{B}$. In particular, if X is a random element, then X is $\mathcal{F}(X)$-measurable.

Lemma 4.1. *Let $\mathcal{F}, \mathcal{B}$ be σ-fields over Ω, R and X be a mapping from Ω to R. Set $\mathcal{B}(X) = \{B \subset R|X^{-1}(B) \in \mathcal{F}\}$. Let $\mathcal{C} \subset \mathcal{B}$ be such that $\sigma(\mathcal{C}) = \mathcal{B}$. Then, X is $\mathcal{F}$-measurable if and only if $\mathcal{C} \subset \mathcal{B}(X)$.*

Corollary 4.2. *Let $(\Omega, \mathcal{F})$ be a σ-field and X be a real-valued function defined on Ω. Then, X is a random variable if and only if $\{X \le c\} \in \mathcal{F}$ for all $c \in \mathbb{R}$.*

Remark 4.2. Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ and $\mathcal{B}(\overline{\mathbb{R}}) = \sigma(\mathcal{B}(\mathbb{R}) \cup \{\{\infty\}, \{-\infty\}\})$. Clearly, $\mathcal{B}(\overline{\mathbb{R}}) = \sigma(\{[-\infty, c]|c \in \mathbb{R}\})$. A function X taking values on the extended real field $\overline{\mathbb{R}}$ is called an extended random variable if $\{X \in B\} \in \mathcal{F}$ for $B \in \mathcal{B}(\overline{\mathbb{R}})$. By Lemma 4.1, Corollary 4.2 also applies for extended random variables.

Proposition 4.3. *Let X, Y be random variables and $a, b \in \mathbb{R}$. Then, $aX + bY, XY, \max\{X, Y\}$ are random variables. Furthermore, if X_n is an extended random variable for all $n \in \mathbb{N}$, then $\sup_n X_n$, $\inf_n X_n$, $\limsup_n X_n$ and $\liminf_n X_n$ are extended random variables. In particular, if X_n converges to X, then X is a random variable.*

Proposition 4.4. *For any extended random variable X defined on Ω, there is a sequence of random variables, say X_n, satisfying $|X_n| \le |X_{n+1}|$, $X_n(\Omega)$ is a finite set and $X_n \to X$, where, in the case $X(\omega) = \infty$, $X_n(\omega) \to X(\omega)$ means that $X_n(\omega)$ diverges to infinity.*

Exercise 4.1. Let Ω be a metric space and $\mathcal{F}$ be the σ-field generated by the open sets in Ω. Prove that if $X : \Omega \to \mathbb{R}$ is upper semicontinuous, i.e. $\limsup_{x \to y} X(x) \le X(y)$ for all $y \in \Omega$, then X is a random variable.

Proposition 4.5. *Let $X : (\Omega, \mathcal{F}) \to (R, \mathcal{B})$ be a random element and $f : (R, \mathcal{B}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be a random variable. Then, $f(X)$ is a random variable.*

Exercise 4.2. Let $X : (\Omega, \mathcal{F}) \to (R, \mathcal{B})$ be a random element and g be a $\mathcal{F}(X)$-measurable random variable. Prove that there is a $\mathcal{B}$-measurable random variable f such that $g = f(X)$.

Lemma 4.6. *For $1 \le i \le n$, let $\mathcal{F}_i$ be a σ-field over Ω_i and set $\Omega = \prod_{i=1}^n \Omega_i$ and $\mathcal{F} = \bigotimes_{i=1}^n \mathcal{F}_i$. Consider the projection mapping $P_i : \Omega \to \Omega_i$ defined by $P_i(\omega_1, ..., \omega_n) = \omega_i$. Then, P_i is a $\mathcal{F}/\mathcal{F}_i$ random element for $1 \le i \le n$.*

Proposition 4.7. *Let $X : \Omega \to \mathbb{R}^n$ and write $X = (X_1, ..., X_n)$. Then, X is an n-dimensional random vector if and only if $X_1, ..., X_n$ are random variables.*

Definition 4.2. Let T be a subset of $\mathbb{R}$ and $(\Omega, \mathcal{F})$ be a σ-field. A **stochastic process** $X : \Omega \to \mathbb{R}^T$ is a $(\mathcal{F}/\mathcal{B}(\mathbb{R}^T))$ random element. The process is of **discrete time** if T is a countable set, e.g. $\{0, 1, ...\}$ and is of **continuous time** if T is an interval, e.g. $[0, 1]$, $[0, \infty)$, $\mathbb{R}$. For any $\omega \in \Omega$, $(X_t(\omega))_{t \in T}$ is called a **realization** corresponding to ω.

Proposition 4.8. *Let $(\Omega, \mathcal{F})$ be a σ-field, $T \subset \mathbb{R}$ and $X = (X_t)_{t \in T} : \Omega \to \mathbb{R}^T$. Then, X is a stochastic process if and only if X_t is a random variable for all $t \in T$.*

Proposition 4.9. *Let T be a set and, for $t_1,..,t_n \in T$, $F_{t_1,...,t_n}$ be a n-dimensional distribution function satisfying the consistency property. Then, there is a stochastic process $(X_t)_{t\in T}$ such that $\mathbb{P}(X_{t_i} \le x_i, \forall 1 \le i \le n) = F_{t_1,...,t_n}(x_1,...,x_n)$ for $t_1,...,t_n \in T$ and $n \in \mathbb{N}$.*

Proof. By Theorem 3.7, there is a probability $\mathbb{P}$ on $(\mathbb{R}^T, \mathcal{B}(\mathbb{R}^T))$ such that

$$\mathbb{P}(\mathcal{D}(t_1,...,t_n, \prod_{i=1}^{n}(-\infty, x_i]) = F_{t_1,...,t_n}(x_1,...,x_n).$$

The desired process is then given by $(X_t)_{t\in T} : \mathbb{R}^T \to \mathbb{R}$, where $X_t(\omega) = \omega_t$ for all $\omega = (\omega_t)_{t\in T} \in \mathbb{R}^T$. □

Example 4.1. Let $E_1, E_2, ...$ be mutually disjoint non-empty sets satisfying $E_1 \cup E_2 \cup \cdots = \Omega$ and $\mathcal{F}$ be the σ-field generated by $(E_n)_{n=1}^{\infty}$. Note that $\mathcal{F}$ consists of $\emptyset$, Ω, finite unions of $(E_n)_{n=1}^{\infty}$ and complements of finite unions of $(E_n)_{n=1}^{\infty}$. This implies that any $\mathcal{F}$-measurable random variable X is constant on E_n for all n. Hence, there is a sequence $(a_n)_{n=1}^{\infty}$ such that $X = \sum_{n=1}^{\infty} a_n \mathbf{1}_{E_n}$.

Exercise 4.3. Let X, Y be $\mathcal{F}$-measurable random variables. Suppose that $X(\Omega) = Y(\Omega) = \{1, 2, ..., N\}$ and $\mathcal{F}(X) = \mathcal{F}(Y)$. Show that there is a permutation σ of $\{1, 2, ..., N\}$ such that $\{X = i\} = \{Y = \sigma(i)\}$ for $1 \le i \le N$.

In the end of this subsection, we introduce a useful technique. For any family Q of real-valued functions defined on Ω, let $\mathcal{F}(Q)$ be the σ-field over Ω generated by Q, which is defined to be the smallest σ-field over Ω such that each function in Q is measurable. Then, $\mathcal{F}(Q) = \sigma(\{f^{-1}(B) : f \in Q, B \in \mathcal{B}(\mathbb{R})\})$. A sequence of functions X_n is said to **converge boundedly** to X if X_n converges to X and X_n is uniformly bounded. A family of real-valued functions on Ω is called a **multiplicative system** if it is closed under multiplication.

Theorem 4.10 (Multiplicative system theorem)**.** *Let H be a linear space of bounded functions on Ω which contains $\mathbf{1}_\Omega$ and is closed under bounded convergence. If H contains a multiplicative system Q, then it contains all bounded $\mathcal{F}(Q)$-measurable functions.*

Proof. Let $\mathcal{L}$ be the class containing subsets $A \subset \Omega$ for which $\mathbf{1}_A \in H$. Let $\mathcal{P}$ be the class of subsets of the following form

$$\{\omega \in \Omega : X_1(\omega) \in I_1, ..., X_n(\omega) \in I_n\} \tag{4.1}$$

where $X_1, ..., X_n \in Q$, $I_1, ..., I_n$ are open intervals and $n \in \mathbb{N}$. It is obvious that $\mathcal{P}$ is a π-system and, by the assumption that H is linear and closed under bounded convergence, $\mathcal{L}$ is a λ-system.

Next, we prove that $\mathcal{P} \subset \mathcal{L}$. Note that the indicator function of the set in (4.1) is equal to

$$\mathbf{1}_{I_1}(X_1)\mathbf{1}_{I_2}(X_2) \cdots \mathbf{1}_{I_n}(X_n).$$

For $1 \le k \le n$, one may choose a sequence of continuous functions $(f_m^{(k)})_{m=1}^{\infty}$ that converges boundedly to $\mathbf{1}_{I_k}$. Set $c = \sup\{|X_k(\omega)| : \omega \in \Omega, 1 \le k \le n\}$. By the Stone-Weierstrass theorem, we may choose, for each k, m, a polynomial $g_m^{(k)}$ such that

$$|g_m^{(k)}(t) - f_m^{(k)}(t)| \le \frac{1}{m}, \quad \text{for } |t| \le c.$$

This implies that, for $1 \le k \le n$, $g_m^{(k)}(X_k) \to \mathbf{1}_{I_k}(X_k)$ boundedly as $m \to \infty$. Since Q is a multiplicative system and H is a linear space, we have

$$g_m^{(1)}(X_1)g_m^{(2)}(X_2) \cdots g_m^{(n)}(X_n) \in H, \quad \forall m \ge 1.$$

By the bounded convergence of H, the indicating function of the set in (4.1) belongs to H, which implies $\mathcal{P} \subset \mathcal{L}$. As a result of the $\pi - \lambda$ lemma, $\mathcal{L} \supset \sigma(\mathcal{P}) = \mathcal{F}(Q)$. Now let X be a bounded $\sigma(\mathcal{P})$-measurable function and set, for $n \geq 1$,

$$X_n = \sum_k \frac{k}{n} \mathbf{1}_{\{\frac{k}{n} < X \leq \frac{k+1}{n}\}}.$$

It is easy to see that $X_n \in H$ for $n \geq 1$ and X_n converges boundedly to X and, hence, $X \in H$. □

4.2. **Expectations.**

Definition 4.3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and X be a random variable on $(\Omega, \mathcal{F})$. The expectation of X is defined by

$$\mathbb{E}X := \int_\Omega X(\omega) d\mathbb{P}(\omega),$$

provided the right side is well-defined.

Remark 4.3. In the case $\mathbb{P}(X \geq c) = 1$ for some $c \in \mathbb{R}$, if $\int_\Omega X(\omega) d\mathbb{P}(\omega)$ does not exist, we write $\mathbb{E}X = \infty$. Similarly, when $\mathbb{P}(X \leq c) = 1$ and the expectation of X does not exist, we write $\mathbb{E}X = -\infty$.

Definition 4.4. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Two random variables X, Y are said to be equal a.s. (almost surely) if $\mathbb{P}(X \neq Y) = 0$. A sequence of random variables $(X_n)_{n=1}^\infty$ is said to converge to a function X a.s. if $\mathbb{P}(X_n \to X) = 1$.

Remark 4.4. Note that $\{\omega \in \Omega | X_n(\omega) \text{ converges}\} \in \mathcal{F}$. This implies that if $\mathbb{P}$ is complete and X_n converges almost surely to X, then X is a random variable.

Proposition 4.11. *Let X, Y be random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose that $\mathbb{E}|X| < \infty$ and $\mathbb{E}|Y| < \infty$. Then,*

(1) *If $X \geq Y$ a.s, then $\mathbb{E}X \geq \mathbb{E}Y$. In particular, if $X \geq c$ a.s., then $\mathbb{E}X \geq c$.*
(2) *If $X = Y$ a.s., then $\mathbb{E}X = \mathbb{E}Y$.*
(3) *For $a, b \in \mathbb{R}$, $\mathbb{E}|aX + bY| < \infty$ and $\mathbb{E}(aX + bY) = a\mathbb{E}X + b\mathbb{E}Y$.*

Remark 4.5. Referring to the setting in Proposition 4.11, one has that $X \geq Y$ a.s. (resp. $X = Y$ a.s.) if and only if $\mathbb{E}(X\mathbf{1}_A) \geq \mathbb{E}(Y\mathbf{1}_A)$ (resp. $\mathbb{E}(X\mathbf{1}_A) = \mathbb{E}(Y\mathbf{1}_A)$) for all $A \in \mathcal{F}$.

Theorem 4.12 (Monotone convergence theorem)**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and X, X_n be nonnegative random variables defined on Ω. Suppose that $X_n \uparrow X$ a.s.. Then, $\mathbb{E}X_n \uparrow \mathbb{E}X$.*

Theorem 4.13 (Lebesgue's dominated convergence theorem)**.** *Let X_n, X, Y be random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose that $|X_n| \leq Y$ a.s., $\mathbb{E}Y < \infty$ and $X_n \to X$ a.s.. Then, $\mathbb{E}X_n \to \mathbb{E}X$.*

Theorem 4.14 (Fatou's lemma)**.** *For $n \geq 1$, let X_n be a non-negative random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then, $\mathbb{E}(\liminf_n X_n) \leq \liminf_n \mathbb{E}X_n$.*

Remark 4.6. The monotone convergence theorem, dominated convergence theorem and Fatou's lemma are equivalent.

Definition 4.5. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $X : (\Omega, \mathcal{F}) \to (R, \mathcal{B})$ be a random element. The **distribution** of X is the probability $\mathbb{P}_X$ on $(R, \mathcal{B})$ induced by X, i.e. $\mathbb{P}_X(B) = \mathbb{P}(X \in B)$ for $B \in \mathcal{B}$. In the case of $(R, \mathcal{B}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, the distribution function of X is defined by $F_X(a) := \mathbb{P}_X((-\infty, a])$ for $a \in \mathbb{R}$.

Remark 4.7. If $R=\mathbb{R}^n$, then $\mathbb{P}_X$ is characterized by its n-dimensional distribution. If $R=\mathbb{R}^T$, then $\mathbb{P}_X$ is characterized by its finite dimensional distributions. There is no confusion to called the n-dimensional and finite dimensional distribution functions as the distribution functions of X.

Theorem 4.15. *Let $X=(X_t)_{t\in T}$ be a stochastic process on $(\Omega,\mathcal{F},\mathbb{P})$ and $\mathbb{P}_X$ be the distribution of X. Let $f:\mathbb{R}^T\to\mathbb{R}$ be a $\mathcal{B}(\mathbb{R}^T)$-measurable random variable. Then,*

$$\int_\Omega f(X(\omega))d\mathbb{P}(\omega)=\int_{\mathbb{R}^T} f(x)d\mathbb{P}_X(x) \tag{4.2}$$

in the sense that if either side is well-defined.

Proof. Let H be the class of bounded $\mathcal{B}(\mathbb{R}^T)$-measurable functions f such that (4.2) holds. Then, by Proposition 4.11, H is a linear space containing constant function $\mathbf{1}$ and, by Lebesgue's dominated convergent theorem, H is closed under bounded convergence. Set $Q:=\{\mathbf{1}_B|B\in\mathcal{B}(\mathbb{R}^T)\}$. It is obvious that Q is a multiplicative system contained in H. Note that $\mathcal{F}(Q)=\mathcal{B}(\mathbb{R}^T)$. By the multiplicative systems theorem, (4.2) holds for all bounded $\mathcal{B}(\mathbb{R}^T)$-measurable functions.

Next, let f be any $\mathcal{B}(\mathbb{R}^T)$-measurable function and set $f_n=f\cdot\mathbf{1}_{\{x:|f(x)|\le n\}}$. Clearly, $|f_n|\le|f|$ and $f_n\to f$. Since f_n is bounded and $\mathcal{B}(\mathbb{R}^T)$-measurable,

$$\int_\Omega f_n(X(\omega))d\mathbb{P}(\omega)=\int_{\mathbb{R}^T} f_n(x)d\mathbb{P}_X(x).$$

By the Lebesgue dominated convergence theorem, (4.2) holds when f is integrable under $\mathbb{P}_X$. □

Corollary 4.16. *Let $(\Omega,\mathcal{F},\mathbb{P})$ and $(\Omega',\mathcal{F}',\mathbb{P}')$ be probability spaces and $X:\Omega\to\mathbb{R}^T$ and $X':\Omega'\to\mathbb{R}^T$ be stochastic processes. If $\mathbb{P}_X=\mathbb{P}_{X'}$, then*

$$\int_\Omega f(X)d\mathbb{P}=\int_{\Omega'} f(X')d\mathbb{P}',$$

for any $\mathcal{B}(\mathbb{R}^T)$-measurable random variable f satisfying $\int|f|d\mathbb{P}_X<\infty$.

Exercise 4.4. Let f be a real-valued function defined on an interval $I\subset\mathbb{R}$ and X be a random variable taking values on I. Prove that if f is convex, $\mathbb{E}|X|<\infty$ and $\mathbb{E}|f(X)|<\infty$, then $\mathbb{E}f(X)\ge f(\mathbb{E}X)$.

4.3. **Convergence concepts.**

Definition 4.6. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space and X,X_n be random variables on Ω. A sequence $(X_n)_{n=1}^\infty$ converges to X

1. almost surely or with probability 1 if $\mathbb{P}(X_n\to X)=1$.
2. in probability if $\mathbb{P}(|X_n-X|>\epsilon)\to 0$ for all $\epsilon>0$.
3. in the rth norm if $\mathbb{E}|X_n-X|^r\to 0$ for $r\in[1,\infty)$ and $\|X_n-X\|_\infty\to 0$ for $r=\infty$, where $\|Y\|_\infty:=\inf\{c|\mathbb{P}(|Y|\le c)=1\}$.
4. in distribution if the distribution functions of X_n and X, say F_n and F, satisfy
$$\lim_{n\to\infty}F_n(x)=F(x),\quad \forall x\in\mathcal{C},$$
where $\mathcal{C}$ is the set of all continuous points of F.
5. almost uniformly if, for any $\epsilon>0$, there is $E_\epsilon\in\mathcal{F}$ such that $\mathbb{P}(E_\epsilon^c)<\epsilon$ and X_n converges uniformly to X on E_ϵ.

Remark 4.8. It is obvious that both the a.s. convergence and the convergence in the rth mean with $r\ge 1$ imply the convergence in probability.

Remark 4.9. For $1 \le r < s \le \infty$, if $X_n \to X$ in the sth norm, then $X_n \to X$ in the rth norm.

Remark 4.10. Note that $X_n \to X$ a.s. if and only if $X_n \to X$ a.u.. (The direction a.s. $\Rightarrow$ a.u. is a special case of Egorov's theorem.)

Exercise 4.5. Prove that X_n converges to X in probability if and only if, any subsequence of $(X_n)_{n=1}^{\infty}$ has a further subsequence that converges to X a.s. Give an example that X_n converges to X in probability but not a.s.

Definition 4.7. A sequence of random variables $(X_n)_{n=1}^{\infty}$ is Cauchy a.s., Cauchy in probability and Cauchy in the rth mean if $X_n - X_m \to 0$ as $n, m \to \infty$ in the sense of Definition 4.6(1)-(3).

Proposition 4.17. *If $(X_n)_{n=1}^{\infty}$ is a sequence of random variables that are Cauchy a.s. (resp. in probability and in the rth mean), then there exists a random variable X such that X_n converges to X a.s (resp. in probability and in the rth mean).*

Proof. The proof for a.s. convergence is given by Exercise 4.6, while the others are left as exercises. □

Exercise 4.6. Let X_n be a sequence of random variables on $(\Omega, \mathcal{F})$ and set

$$E_1 = \{\omega \in \Omega : X_n(\omega) \text{ is a Cauchy sequence}\}$$

and

$$E_2 = \{\omega \in \Omega : X_n(\omega) \to \infty\}, \quad E_3 = \{\omega \in \Omega : X_n(\omega) \to -\infty\}.$$

Prove that E_1, E_2, E_3 are contained in $\mathcal{F}$.

Remark 4.11. It follows immediately from Exercise 4.6 that

$$\left\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) \text{ does not exist}\right\} \in \mathcal{F}.$$

Exercise 4.7. Let X_n, Y_n be random variables on $(\Omega, \mathcal{F}, \mathbb{P}), (\Omega', \mathcal{F}', \mathbb{P}')$. Suppose $\mathbb{P}_X = \mathbb{P}_Y$, where $X = (X_n)_{n=1}^{\infty}$ and $Y = (Y_n)_{n=1}^{\infty}$. Prove that if X_n converges a.s. (resp. in probability and in the rth mean) to $\widetilde{X}$, then there exists another random variable $\widetilde{Y}$ defined on Ω' such that Y_n converges to $\widetilde{Y}$ a.s (resp. in probability and in the rth mean).

We end this section by introducing the concept of uniform integrability.

Definition 4.8. A sequence of random variables $(X_n)_{n=1}^{\infty}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is **uniformly integrable** if

$$\sup_{n\ge1} \int_{\{|X_n|>c\}} |X_n| d\mathbb{P} \to 0, \quad \text{as } c \to \infty.$$

Remark 4.12. If $(X_n)_{n=1}^{\infty}$ is uniformly integrable, then $\sup_n \mathbb{E}|X_n| < \infty$.

Remark 4.13. Note that if $|X_n| \le X$ and $\mathbb{E}|X| < \infty$, then $(X_n)_{n=1}^{\infty}$ is uniformly integrable.

Lemma 4.18. *If $(X_n)_{n=1}^{\infty}$ and $(Y_n)_{n=1}^{\infty}$ are uniformly integrable, then $(aX_n)_{n=1}^{\infty}$ for any $a \in \mathbb{R}$, $(\max\{X_n, 0\})_{n=1}^{\infty}$ and $(X_n + Y_n)_{n=1}^{\infty}$ are uniformly integrable.*

Proof. The uniform integrability of the first two sequences are clear. For the last one, note that

$$\int_{\{|X_n+Y_n|>c, |X_n|\ge|Y_n|\}} |X_n + Y_n| d\mathbb{P} \le 2 \int_{\{|X_n|>c/2\}} |X_n| d\mathbb{P}$$

and

$$\int_{\{|X_n+Y_n|>c, |X_n|<|Y_n|\}} |X_n + Y_n| d\mathbb{P} \le 2 \int_{\{|Y_n|>c/2\}} |Y_n| d\mathbb{P}.$$

This implies that, when $c \to \infty$,

$$\sup_{n\geq 1}\int_{\{|X_n+Y_n|>c\}}|X_n+Y_n|d\mathbb{P}\leq 2\left\{\sup_{n\geq 1}\int_{\{|X_n|>c/2\}}|X_n|d\mathbb{P}+\sup_{n\geq 1}\int_{\{|Y_n|>c/2\}}|Y_n|d\mathbb{P}\right\}\to 0.$$

□

Theorem 4.19. *Let $(X_n)_{n=1}^\infty$ be a sequence of random variables defined on $(\Omega, \mathcal{F})$. Then, $(X_n)_{n=1}^\infty$ is uniformly integrable if and only if the following holds.*

(1) $\sup_n \mathbb{E}|X_n| < \infty$.
(2) *For any sequence of events $(A_m)_{m=1}^\infty$ in $\mathcal{F}$ satisfying $\mathbb{P}(A_m) \to 0$, $\sup_n \mathbb{E}(|X_n|\mathbf{1}_{A_m}) \to 0$ as $m \to \infty$.*

Remark 4.14. An equivalent statement of (2) says that, for any $\epsilon > 0$, there is $\delta > 0$ such that $\sup_n \mathbb{E}(|X_n|\mathbf{1}_A) < \epsilon$ for all $A \in \mathcal{F}$ satisfying $\mathbb{P}(A) < \delta$.

Proof of Theorem 4.19. It loses no generality to assume that $X_n \geq 0$. First, assume that $(X_n)_{n=1}^\infty$ is uniformly integrable. Note that, for $A \in \mathcal{F}$ and $c > 0$,

$$\sup_{n\geq 1}\mathbb{E}(X_n\mathbf{1}_A)\leq \sup_{n\geq 1}\mathbb{E}(X_n\mathbf{1}_{\{X_n\geq c\}})+c\mathbb{P}(A).$$

Setting $A = \Omega$ and choosing c large enough gives (1), while (2) is provided by replacing A with A_m and passing m to the infinity and then c to the infinity.

Next, assume that (1) and (2) hold. Note that, for $c > 0$, $\mathbb{E}X_n \geq c\mathbb{P}(X_n \geq c)$. By (1), this implies $\sup_n \mathbb{P}(X_n \geq c) \leq c^{-1} \sup_n \mathbb{E}X_n \to 0$ as $c \to \infty$ or, equivalently, for any $\delta > 0$, there is $c > 0$ such that $\sup_n \mathbb{P}(X_n \geq c) < \delta$. By Remark 4.14 and the assumption of (2), this implies that, for any $\epsilon > 0$, there exists $c > 0$ such that $\sup_n \mathbb{E}(X_n\mathbf{1}_{\{X_n\geq c\}}) < \epsilon$, which proves the uniform integrability of $(X_n)_{n=1}^\infty$. □

Exercise 4.8. Let $(X_n)_{n=1}^\infty$ be a sequence of random variables. Prove that $(X_n)_{n=1}^\infty$ is uniformly integrable if and only if $\mathbb{E}|X_n| < \infty$ and

$$\limsup_{n\to\infty}\int_{\{|X_n|>c\}}|X_n|d\mathbb{P}\to 0,\quad \text{as } c\to\infty.$$

Exercise 4.9. Let $(X_n)_{n=1}^\infty$ be a sequence of random variable and G be a non-negative increasing function defined on $[0, \infty)$. Suppose $\mathbb{E}|X_n| < \infty$ and

$$\lim_{t\to\infty}\frac{G(t)}{t}=\infty,\quad \sup_{n\geq 1}\mathbb{E}G(|X_n|)<\infty.$$

Show that $(X_n)_{n=1}^\infty$ is uniformly integrable.

Exercise 4.10. Let $(X_n)_{n=1}^\infty$ be a sequence of random variables. Show that if $\mathbb{E}|X_n| < \infty$, $\mathbb{E}|X| < \infty$ and $\mathbb{E}|X_n - X| \to 0$, then $(X_n)_{n=1}^\infty$ is uniformly integrable.

Theorem 4.20. *Let $(X_n)_{n=1}^\infty$ be uniformly integrable. Then,*

(1) $\mathbb{E}\liminf_n X_n \leq \liminf_n \mathbb{E}X_n$ *and* $\limsup_n \mathbb{E}X_n \leq \mathbb{E}\limsup_n X_n$.
(2) *If $X_n \to X$ a.s., then X is integrable and $\mathbb{E}|X_n - X| \to 0$.*

Proof. For (1), let $\epsilon > 0$. Since $(X_n)_{n=1}^\infty$ is uniformly integrable, we may choose $c > 0$ such that $\mathbb{E}(|X_n|\mathbf{1}_{\{|X_n|>c\}}) < \epsilon$ for all $n \in \mathbb{N}$. This implies $\mathbb{E}(|X_n|\mathbf{1}_{\{X_n<-c\}}) < \epsilon$. By Fatou's lemma, one has

$$\mathbb{E}\liminf_{n\to\infty}(X_n\mathbf{1}_{\{X_n\geq -c\}})\leq \liminf_{n\to\infty}\mathbb{E}(X_n\mathbf{1}_{\{X_n\geq -c\}}).$$

Since $X_n \leq X_n\mathbf{1}_{\{X_n\geq -c\}}$, the above inequality implies

$$\mathbb{E}\liminf_{n\to\infty}X_n\leq \mathbb{E}\liminf_{n\to\infty}(X_n\mathbf{1}_{\{X_n\geq -c\}})\leq \liminf_{n\to\infty}\mathbb{E}(X_n\mathbf{1}_{\{X_n\geq -c\}})\leq \liminf_{n\to\infty}\mathbb{E}X_n+\epsilon.$$

Letting $\epsilon \to 0$ gives the desired result. The other inequality can be proved using the uniform integrability of $(-X_n)_{n=1}^\infty$.

For (2), as a result of (1) and Lemma 4.18, it remains to show that X is integrable. Note that, for $c > 0$, $\{|X| > c\} \subset \liminf_n \{|X_n| > c\}$ almost surely. By Fatou's lemma, this implies

$$\int_{\{|X|>c\}} |X| d\mathbb{P} \leq \liminf_{n\to\infty} \int_{\{|X_n|>c\}} |X_n| d\mathbb{P} \leq \sup_{n\geq 1} \mathbb{E}|X_n| < \infty.$$

□

Theorem 4.21. *Let X_n, X be random variables. Suppose $X_n \geq 0$, $\mathbb{E}X_n < \infty$ and $X_n \to X$ almost surely. Then, the following are equivalent.*

(1) *$\mathbb{E}X_n \to \mathbb{E}X$ with $\mathbb{E}X < \infty$.*
(2) *$(X_n)_{n=1}^\infty$ is uniformly integrable.*

Proof. (2)⇒(1) is immediate from Theorem 4.20. Assume that (1) holds and let $\mathcal{C}$ be set of continuous points of F_X, the distribution function of X. It is clear that $\mathbb{R} \setminus \mathcal{C}$ is finite or countable. Note that, for $c \in \mathcal{C}$, $X_n \mathbf{1}_{\{X_n < c\}} \to X\mathbf{1}_{\{X<c\}}$ almost surely. By the Lebesgue dominated convergence theorem, this implies $\mathbb{E}(X_n \mathbf{1}_{\{X_n<c\}}) \to \mathbb{E}(X\mathbf{1}_{\{X<c\}})$. Since $\mathbb{P}(X = c) = 0$ for $x \in \mathcal{C}$, we have $\mathbb{E}(X_n\mathbf{1}_{\{X_n \geq c\}}) \to \mathbb{E}(X\mathbf{1}_{\{X>c\}})$ for all $c \in \mathcal{C}$.

For $\epsilon > 0$, choose $c_0 \in \mathcal{C}$ such that $\mathbb{E}(X\mathbf{1}_{\{X>c_0\}}) < \epsilon/2$. Next, we may select $N \in \mathbb{N}$ such that $\mathbb{E}(X_n\mathbf{1}_{\{X_n\geq c_0\}}) < \mathbb{E}(X\mathbf{1}_{\{X>c_0\}}) + \epsilon/2$ for $n \geq N$. This implies, for $c \geq c_0$ and $n \geq N$,

$$\mathbb{E}(X_n\mathbf{1}_{\{X_n>c\}}) \leq \mathbb{E}(X_n\mathbf{1}_{\{X_n\geq c_0\}}) < \epsilon.$$

Since $\mathbb{E}X_n < \infty$, we may select $c_1 > c_0$ such that $\mathbb{E}(X_n\mathbf{1}_{\{X_n>c_1\}}) < \epsilon$ for $1 \leq n \leq N$. Consequently, this leads to $\sup_n \mathbb{E}(X_n\mathbf{1}_{\{X_n>c\}}) \leq \epsilon$ for $c \geq c_1$, as desired. □

As a consequence of Theorems 4.20 and 4.21, we obtain the following corollary.

Corollary 4.22. *Assume that $X_n \to X$ almost surely. Then, the following are equivalent.*

(1) *$(X_n)_{n=1}^\infty$ is uniformly integrable.*
(2) *$\mathbb{E}|X| < \infty$ and $\mathbb{E}|X_n - X| \to 0$.*
(3) *$\mathbb{E}|X| < \infty$ and $\mathbb{E}|X_n| \to \mathbb{E}|X|$.*

Exercise 4.11. Show that the assumption in Corollary 4.22 can be replaced by convergence in probability.