8.1. The central limit theorem for i.i.d. sequences. Recall that $C_0^\infty(\mathbb{R})$ is $\mathcal{N}$-separating.

Theorem 8.1. *Let $X_1, X_2, ...$ be i.i.d. random variables with $\mathbb{E}X_1 = 0$ and $\mathbb{E}X_1^2 = \sigma^2 \in (0,\infty)$. Suppose that there is a random variable X such that*

$$\frac{X_1 + \cdots + X_n}{\sqrt{n}\sigma} \to X \quad \textit{in distribution.}$$

Then, for any i.i.d. random variables $Y_1, Y_2, ...$ with $\mathbb{E}Y_1 = 0$ and $\mathbb{E}Y_1^2 = \delta^2 \in (0,\infty)$,

$$\frac{Y_1 + \cdots + Y_n}{\sqrt{n}\delta} \to X \quad \textit{in distribution.}$$

Proof. It loses no generality to assume that $\sigma = \delta = 1$ and that $(X_n)_{n=1}^\infty$ and $(Y_n)_{n=1}^\infty$ are independent. Set

$$S_n = \frac{X_1 + \cdots + X_n}{\sqrt{n}}, \quad T_n = \frac{Y_1 + \cdots + Y_n}{\sqrt{n}}.$$

Clearly, $\mathbb{E}(S_n^2) = \mathbb{E}(T_n^2) = 1$. By Exercise 7.3, $(S_n)_{n=1}^\infty$ and $(T_n)_{n=1}^\infty$ are mass-preserving. Note that $\mathbb{E}f(S_n) \to \mathbb{E}f(X)$ for any bounded continuous function f. Since $C_0^\infty(\mathbb{R})$ is $\mathcal{N}$-separating, to prove this theorem, it remains to show that $\mathbb{E}f(S_n) - \mathbb{E}f(T_n) \to 0$ for all $f \in C_0^\infty(\mathbb{R})$.

Let $f \in C_0^\infty(\mathbb{R})$. For $0 \le k \le n$, set

$$U_{n,k} = \frac{X_1 + \cdots + X_{k-1} + Y_{k+1} + \cdots + Y_n}{\sqrt{n}}$$

and

$$V_{n,k} = f\left(U_{n,k} + \frac{X_k}{\sqrt{n}}\right) - f\left(U_{n,k} + \frac{Y_k}{\sqrt{n}}\right).$$

Clearly, $f(S_n) - f(T_n) = \sum_{k=1}^n V_{n,k}$. By Taylor's theorem, there are $\theta, \widetilde{\theta} \in [0,1]$ such that

$$V_{n,k} = \frac{X_k - Y_k}{\sqrt{n}} f'(U_{n,k}) + \frac{X_k^2}{2n} f''\left(U_{n,k} + \theta \frac{X_k}{\sqrt{n}}\right) - \frac{Y_k^2}{2n} f''\left(U_{n,k} + \widetilde{\theta}\frac{Y_k}{\sqrt{n}}\right). \tag{8.1}$$

Set $\Delta_f(h) = \max\{|f''(x) - f''(y)| : |x-y| \le h\}$. Since f'' is uniformly continuous, $\Delta_f(h) \to 0$ as $h \to 0$. Note that $\Delta_f(h) \le \Delta_f(h+\delta) \le \Delta_f(h) + \Delta_f(\delta)$. This implies that Δ_f is continuous and, thus, Borel measurable. By (8.1), we obtain

$$\left|\mathbb{E}V_{n,k} - \frac{1}{2n}\mathbb{E}\left[(X_k^2 - Y_k^2) f''(U_{n,k})\right]\right| \le \frac{1}{2n}\mathbb{E}\left[X_k^2 \Delta_f\left(\frac{|X_k|}{\sqrt{n}}\right) + Y_k^2 \Delta_f\left(\frac{|Y_k|}{\sqrt{n}}\right)\right]$$

or, equivalently,

$$|\mathbb{E}V_{n,k}| \le \frac{1}{2n}\mathbb{E}\left[X_1^2 \Delta_f\left(\frac{|X_1|}{\sqrt{n}}\right) + Y_1^2 \Delta_f\left(\frac{|Y_1|}{\sqrt{n}}\right)\right].$$

Consequently, this yields

$$|\mathbb{E}f(S_n) - \mathbb{E}f(T_n)| \le \frac{1}{2}\mathbb{E}\left[X_1^2 \Delta_f\left(\frac{|X_1|}{\sqrt{n}}\right) + Y_1^2 \Delta_f\left(\frac{|Y_1|}{\sqrt{n}}\right)\right].$$

By the Lebesgue dominated convergence theorem, $\mathbb{E}f(S_n) - \mathbb{E}f(T_n) \to 0$ as $n \to \infty$. □

The next theorem is a simple corollary of Theorems 1.4 and 8.1.

Theorem 8.2 (The central limit theorem)**.** *Let $X_1, X_2, ...$ be i.i.d. random variables with mean μ and variance $\sigma^2 > 0$. Then,*

$$\frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \to N(0,1) \quad \textit{in distribution.}$$

Remark 8.1. Let $X_1, X_2, ...$ be i.i.d. random variables with $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = 1/2$ and X be a standard normal random variable. Let f_n, f be the characteristic functions of X_n, X. Note that

$$f_n(u) = \frac{e^{iu} + e^{-iu}}{2} = \cos u, \quad \forall n = 1, 2, ...$$

By the central limit theorem, we have

$$f_n\left(\frac{u}{\sqrt{n}}\right)^n = \left(\cos\left(\frac{u}{\sqrt{n}}\right)\right)^n \to f(u) \quad \forall u \in \mathbb{R}.$$

Using Taylor's theorem, one has, for fixed $u \in \mathbb{R}$,

$$\cos(u/\sqrt{n}) = \left(1 - \frac{u^2}{2n}\right)(1 + O(n^{-2})), \quad \text{as } n \to \infty.$$

This leads to

$$f(u) = \lim_{n\to\infty} \left(1 - \frac{u^2}{2n}\right)^n (1 + O(n^{-2}))^n = e^{-u^2/2}.$$

8.2. **The central limit theorem for non-identical distributions.** In this section, we introduce two famous results related to the central limit theorem given by J. Lindeburg and W. Feller respectively in 1922 and 1935.

For $n \geq 1$, let r_n be a positive integers and $X_{n,1}, ..., X_{n,r_n}$ be independent random variables with mean 0 and variance $\mathrm{Var}(X_{n,k}) = \sigma_{n,k}^2$. Set

$$S_n = \sum_{k=1}^{r_n} X_{n,k}, \quad s_n^2 = \mathrm{Var}(S_n) = \sum_{k=1}^{r_n} \sigma_{n,k}^2.$$

The triangular array $\{X_{n,k} : 1 \leq k \leq r_n, n \geq 1\}$ is said to possess

(1) the **Lindeberg condition** (LC) if, for all $\epsilon > 0$,

$$\frac{1}{s_n^2} \sum_{k=1}^{r_n} \int_{\{|X_{n,k}| \geq \epsilon s_n\}} X_{n,k}^2 d\mathbb{P} \to 0 \quad \text{as } n \to \infty.$$

(2) the **uniform asymptotic negligibility** (UAN) if

$$\max_{1 \leq k \leq r_n} \frac{\sigma_{n,k}^2}{s_n^2} \to 0 \quad \text{as } n \to \infty.$$

(3) the **central limit theorem** (CLT) if

$$\frac{S_n}{s_n} \xrightarrow{\mathcal{D}} N(0,1), \quad \text{as } n \to \infty.$$

Theorem 8.3 (Lindeberg(1922))**.** *$LC \Rightarrow UAN + CLT$.*

Theorem 8.4 (Feller(1935))**.** *$UAN + CLT \Rightarrow LC$.*

Remark 8.2. Under UAN, $CLT \Leftrightarrow LC$.

To prove the above theorems, we need the following setting. For $n \geq 1$ and $\epsilon > 0$, set

$$A_n(\epsilon) = \frac{1}{s_n^2} \sum_{k=1}^{r_n} \int_{\{|X_{n,k}| \geq \epsilon s_n\}} X_{n,k}^2 d\mathbb{P} \quad (LC)$$

$$B_n(\epsilon) = \frac{1}{s_n^3} \sum_{k=1}^{r_n} \int_{\{|X_{n,k}| < \epsilon s_n\}} |X_{n,k}|^3 d\mathbb{P}$$

and

$$C_n = \frac{1}{s_n^4}\sum_{k=1}^{r_n} \sigma_{n,k}^4, \quad D_n = \max_{1\le k\le r_n} \frac{\sigma_{n,k}^2}{s_n^2} \quad (UAN).$$

For $n \geq 1$ and $1 \leq k \leq r_n$, let $f_{n,k}$ be the characteristic function of $X_{n,k}$ and define

$$\Sigma_n(u) = \sum_{k=1}^{r_n} \left| f_{n,k}\left(\frac{u}{s_n}\right) - e^{-u^2\sigma_{n,k}^2/(2s_n^2)} \right|$$

Lemma 8.5. *In the above setting, we have*

(1) $B_n(\epsilon) \leq \epsilon$.
(2) $C_n \leq D_n$.
(3) $D_n \leq A_n(\epsilon) + \epsilon^2$.
(4) $\Sigma_n(u) \leq u^2 A_n(\epsilon) + |u|^3 B_n(\epsilon) + u^4 C_n$.

Proof. (1) and (2) are obvious. (3) follows immediately from the following inequality.

$$\frac{\sigma_{n,k}^2}{s_n^2} = \frac{1}{s_n^2}\int_{\{|X_{n,k}|\geq \epsilon s_n\}} X_{n,k}^2 d\mathbb{P} + \frac{1}{s_n^2}\int_{\{|X_{n,k}|<\epsilon s_n\}} X_{n,k}^2 d\mathbb{P} \leq A_n(\epsilon) + \epsilon^2.$$

To see (4), we need the fact that, for any random variable Y with mean 0 and variance $b > 0$,

$$|\mathbb{E}e^{iY} - e^{-b/2}| \leq \mathbb{E}|R(Y)| + b^2,$$

where $R(y) = e^{iy} - (1 + iy - \frac{1}{2}y^2)$. Observe that $e^{-a} - 1 + a \in [0, a^2/2]$ for $a \geq 0$ and

$$R(Y) - (e^{iY} - e^{-b/2}) = e^{-b/2} - 1 - iY + \frac{Y^2}{2}.$$

Replacing a with $b/2$ and taking the expectation on both sides gives the desired inequality. Next, let $Y = \frac{uX_{n,k}}{s_n}$. This implies

$$|f_{n,k}(u/s_n) - e^{-u^2\sigma_{n,k}^2/(2s_n^2)}| \leq \mathbb{E}\left|R\left(\frac{uX_{n,k}}{s_n}\right)\right| + \frac{u^4\sigma_{n,k}^4}{s_n^4}.$$

We will use the inequality $|R(y)| \leq y^2 \wedge |y|^3$ to bound the first term of the right side. To prove this inequality, it suffices to consider $y > 0$, since $R(-y) = \overline{R(y)}$. Note that

$$R'(y) = ie^{iy} - i + y,\ R''(y) = -e^{iy} + 1,\ R'''(y) = -ie^{iy}.$$

This implies

$$R(0) = R'(0) = R''(0) = 0,\ |R''(y)| \leq 2,\ |R'''(y)| = 1.$$

Since R, R', R'' are continuously differentiable, we have that, for $y \geq 0$,

$$|R''(y)| \leq \int_0^y |R'''(z)|dz = y \quad \Rightarrow \quad |R''(y)| \leq 2 \wedge y$$
$$|R'(y)| \leq \int_0^y |R''(z)|dz \leq (2y) \wedge (y^2/2)$$
$$|R(y)| \leq \int_0^y |R'(z)|dz \leq y^2 \wedge (y^3/6) \leq y^2 \wedge y^3$$

Consequently, we obtain

$$\mathbb{E}\left|R\left(\frac{uX_{n,k}}{s_n}\right)\right| \leq \int_{\{|X_{n,k}|\geq\epsilon s_n\}} \left(\frac{uX_{n,k}}{s_n}\right)^2 d\mathbb{P} + \int_{\{|X_{n,k}|<\epsilon s_n\}} \left|\frac{uX_{n,k}}{s_n}\right|^3 d\mathbb{P}$$

and this proves (4). □

Proof of Theorem 8.3. First, it is clear that LC is exactly the case $A_n(\epsilon) \to 0$ for all $\epsilon > 0$ and UAN is equivalent to $D_n \to 0$. By Lemma 8.5(3), LC implies UAN and then $\Sigma_n(u) \to 0$, as $n \to \infty$, for all $u \in \mathbb{R}$.

To show the CLT, it is equivalent to prove

$$\prod_{k=1}^{r_n} f_{n,k}\left(\frac{u}{s_n}\right) \to e^{-u^2/2},$$

where $f_{n,k}$ is the c.f. of $X_{n,k}$. Let $z_1, ..., z_n$ and $w_1, ..., w_n$ be complex numbers with absolute values at most 1. Note that

$$\left|\prod_{k=1}^{n} z_k - \prod_{k=1}^{n} w_k\right| \le \sum_{k=1}^{n} |z_k - w_k|.$$

Letting $z_k = f_{n,k}(u/s_n)$ and $w_k = \exp\{-\sigma_{n,k}^2 u^2/(2s_n^2)\}$ implies

$$\left|\prod_{k=1}^{r_n} f_{n,k}\left(\frac{u}{s_n}\right) - e^{-u^2/2}\right| \le \Sigma_n(u) \to 0, \quad \text{as } n \to \infty.$$

□

Definition 8.1. Any triangular array $\{X_{n,k} : 1 \le k \le r_n, n \ge 1\}$ with $\mathbb{E}X_{n,k} = 0$ is said to have **Lyapunov's condition** if

$$L_n(\delta) = \frac{1}{s_n^{2+\delta}} \sum_{k=1}^{r_n} \mathbb{E}|X_{n,k}|^{2+\delta} \to 0 \quad \text{for some } \delta > 0.$$

Remark 8.3. Note that, for any random variable X,

$$\int_{\{|X|>\epsilon s\}} X^2 d\mathbb{P} \le \int_{\{|X|>\epsilon s\}} X^2 \left|\frac{X}{\epsilon s}\right|^{\delta} d\mathbb{P} \le \frac{\mathbb{E}|X|^{2+\delta}}{\epsilon^\delta s^\delta}.$$

This implies

$$\text{Lyapunov's condition} \quad \Rightarrow \quad LC.$$

Corollary 8.6. *Lyapunov's condition implies CLT.*

Proof of Theorem 8.4. For $n \ge 1$ and $1 \le k \le r_n$, let $\varphi_{n,k}(u) = f_{n,k}(u) - 1$ and

$$\psi_n(u) = \sum_{k=1}^{r_n} \varphi_{n,k}\left(\frac{u}{s_n}\right) + \frac{u^2}{2} = \sum_{k=1}^{r_n} \mathbb{E}\left[e^{iuX_{n,k}/s_n} - 1 - \frac{iuX_{n,k}}{s_n} + \frac{u^2X_{n,k}^2}{2s_n^2}\right].$$

Step 1: $A_n(\epsilon) \le \frac{\epsilon^2}{6}\Re\psi_n\left(\frac{4}{\epsilon}\right)$. To see this, observe that, for $1 \le k \le r_n$,

$$\Re\left[\varphi_{n,k}\left(\frac{u}{s_n}\right) + \frac{u^2\sigma_{n,k}^2}{2s_n^2}\right] = \Re\mathbb{E}\left[e^{iuX_{n,k}/s_n} - 1 + \frac{u^2X_{n,k}^2}{2s_n^2}\right]$$

$$=\mathbb{E}\left[\cos\left(\frac{uX_{n,k}}{s_n}\right) - 1 + \frac{u^2X_{n,k}^2}{2s_n^2}\right] \ge \int_{\{|X_{n,k}|\ge\epsilon s_n\}} \left[\cos\left(\frac{uX_{n,k}}{s_n}\right) - 1 + \frac{u^2X_{n,k}^2}{2s_n^2}\right] d\mathbb{P}$$

$$\ge\frac{1}{s_n^2}\int_{\{|X_{n,k}|\ge\epsilon s_n\}} X_{n,k}^2\left(\frac{u^2}{2} - \frac{2}{\epsilon^2}\right) d\mathbb{P}.$$

In the above computations, the first inequality uses the fact

$$\cos t - 1 + \frac{t^2}{2} = \int_0^t (s - \sin s)ds \ge 0, \quad \forall t \ge 0,$$

and the second inequality applies $\cos s - 1 \geq -2$. Summing up k yields

$$\Re\psi_n(u) \geq \left(\frac{u^2}{2} - \frac{2}{\epsilon^2}\right) A_n(\epsilon).$$

The desired inequality is given by choosing $u = 4/\epsilon$.

Step 2: $UAN \Rightarrow \max_k |\varphi_{n,k}(u/s_n)| \to 0$ for all $u \in \mathbb{R}$. First, observe that, for $t > 0$,

$$e^{it} - 1 = \int_0^t ie^{is}ds. \tag{8.2}$$

This implies $|e^{it} - 1| \leq |t| \wedge 2$. Using this fact, we have

$$\begin{aligned}\left|\varphi_{n,k}\left(\frac{u}{s_n}\right)\right| &\leq \mathbb{E}|e^{iuX_{n,k}/s_n} - 1| \leq \mathbb{E}\left[2 \wedge \left|\frac{uX_{n,k}}{s_n}\right|\right] \\ &\leq 2\mathbb{P}(|X_{n,k}| \geq \epsilon s_n) + \int_{\{|X_{n,k}|<\epsilon s_n\}} \left|\frac{uX_{n,k}}{s_n}\right| d\mathbb{P} \leq \frac{2\sigma_{n,k}^2}{\epsilon^2 s_n^2} + \epsilon|u|\end{aligned}$$

Thus, for all $\epsilon > 0$,

$$\max_{1\leq k\leq r_n} |\varphi_{n,k}(u/s_n)| \leq 2\epsilon^{-2}D_n + \epsilon|u|,$$

which proves the desired property.

Step 3: $UAN \Rightarrow \prod_{k=1}^{r_n} e^{\varphi_{n,k}(u/s_n)} - \prod_{k=1}^{r_n} f_{n,k}(u/s_n) \to 0$. Since $|f_{n,k}(v)| \leq 1$, we have $\Re\varphi_{n,k}(v) = \Re f_{n,k}(v) - 1 \leq 0$ and then $|e^{\varphi_{n,k}(v)}| = \exp\{\Re\varphi_{n,k}(v)\} \leq 1$. Fix $u \in \mathbb{R}$. By Step 2, there is $N > 0$ such that $|\varphi_{n,k}(u/s_n)| < \epsilon$ for $1 \leq k \leq r_n$ and $n \geq N$. This implies

$$\begin{aligned}\left|\prod_{k=1}^{r_n} e^{\varphi_{n,k}(u/s_n)} - \prod_{k=1}^{r_n} f_{n,k}(u/s_n)\right| &\leq \sum_{k=1}^{r_n} \left|e^{\varphi_{n,k}(u/s_n)} - f_{n,k}(u/s_n)\right| \\ = \sum_{k=1}^{r_n} \left|e^{\varphi_{n,k}(u/s_n)} - 1 - \varphi_{n,k}(u/s_n)\right| &\leq e^\epsilon \sum_{k=1}^{r_n} |\varphi_{n,k}(u/s_n)|^2, \quad \forall n \geq N,\end{aligned}$$

where the last inequality comes from the following fact

$$|e^\alpha - 1 - \alpha| = \left|\alpha^2 \sum_{k=2}^\infty \frac{\alpha^{k-2}}{k!}\right| \leq |\alpha|^2 e^{|\alpha|}, \quad \forall \alpha \in \mathbb{C}.$$

Moreover, by (8.2), one has, for $t \in \mathbb{R}$,

$$|e^{it} - 1 - it| = \left|\int_0^t i(e^{is} - 1)ds\right| \leq \int_0^{|t|} |e^{is} - 1|ds \leq \int_0^{|t|} \int_0^s drds = \frac{t^2}{2}.$$

This implies

$$\begin{aligned}\sum_{k=1}^{r_n} |\varphi_{n,k}(u/s_n)|^2 &\leq \epsilon \sum_{k=1}^{r_n} |\varphi_{n,k}(u/s_n)| \leq \epsilon \sum_{k=1}^{r_n} \mathbb{E}\left|e^{iuX_{n,k}/s_n} - 1 - \frac{iuX_{n,k}}{s_n}\right| \\ &\leq \frac{\epsilon u^2}{2s_n^2} \sum_{k=1}^{r_n} \mathbb{E}X_{n,k}^2 = \frac{\epsilon u^2}{2}, \quad \forall n \geq N.\end{aligned}$$

This part is proved by the result in Step 2.

Finally, if UAN and CLT hold, then by Step 3,

$$\exp\left\{\sum_{k=1}^{r_n} \varphi_{n,k}(u/s_n)\right\} \to e^{-u^2/2}$$

or equivalently $e^{\psi_n(u)} \to 1$. This implies

$$\exp\{\Re\psi_n(u)\} = |e^{\psi_n(u)}| \to 1,$$

which gives $\Re\psi_n(u) \to 0$. By Step 1, LC follows. $\square$

8.3. **$\mathcal{D}$-Convergence under** UAN. Consider the triangular array $\{X_{n,k} | 1 \le k \le r_n, n \ge 1\}$, where $X_{n,1}, ..., X_{n,r_n}$ are independent and

$$\mathbb{E}X_{n,k} = 0, \quad \mathbb{E}X_{n,k}^2 = \sigma_{n,k}^2, \quad \sum_{k=1}^{r_n} \sigma_{n,k}^2 = 1.$$

Let $f_{n,k}$ be the characteristic function of $X_{n,k}$ and set $S_n = \sum_{k=1}^{r_n} X_{n,k}$.
Assumption: $S_n \overset{\mathcal{D}}{\to} L$ under UAN.

Note that

$$S_n \overset{\mathcal{D}}{\to} L \quad \Leftrightarrow \quad f_{S_n} = \prod_{k=1}^{r_n} f_{n,k} \to f_L \quad \Leftrightarrow \quad \exp\left\{\sum_{k=1}^{r_n} \varphi_{n,k}\right\} \to f_L$$

where $\varphi_{n,k} = f_{n,k} - 1$ and the second equivalence comes from step 3 in the proof of Theorem 8.4. Observe that,

$$\sum_{k=1}^{r_n} \varphi_{n,k}(u) = \int_{\mathbb{R}} (e^{iux} - 1 - iux) dF_n(x) \tag{8.3}$$

where $F_n = \sum_{k=1}^{r_n} F_{n,k}$ and $F_{n,k}$ is the distribution function of $X_{n,k}$. For $n \ge 1$, set $d\nu_n(x) = x^2 dF_n(x)$. Clearly, $\nu_n(\{0\}) = 0$ and ν_n is absolutely continuous w.r.t. F_n with Radon derivative x^2. Note that F_n is not a probability, but ν_n is, because

$$\nu_n(\mathbb{R}) = \int_{\mathbb{R}} x^2 dF_n(x) = \sum_{k=1}^{r_n} \int_{\mathbb{R}} x^2 dF_{n,k}(x) = \sum_{k=1}^{r_n} \sigma_{n,k}^2 = 1.$$

For $u \in \mathbb{R}$, set

$$h(u,x) = \begin{cases} (e^{iux} - 1 - iux)/x^2 & \text{if } x \ne 0 \\ -u^2/2 & \text{if } x = 0 \end{cases} \tag{8.4}$$

By (8.3), we have

$$\sum_{k=1}^{r_n} \varphi_{n,k}(u) = \int_{\mathbb{R}} h(u,x) d\nu_n(x)$$

Recall that the class $\mathcal{M}$ is sequentially compact under the $\mathcal{D}$-convergence. One may choose $\nu \in \mathcal{M}$ and a subsequence $(\nu_{n_k})_{k=1}^{\infty}$ which $\mathcal{D}$-converges to ν. Clearly, h is continuous and vanishes at $\pm\infty$. This implies

$$\int_{\mathbb{R}} h(u,x) d\nu_{n_k}(x) \to \int_{\mathbb{R}} h(u,x) d\nu(x).$$

Exercise 8.1. Let $\mu_n \in \mathcal{N}$ and $\mu \in \mathcal{M}$. Assume that $\mu_n(x_0) \to \mu(x_0)$ for some $x_0 \in \mathcal{C}(\mu)$. Prove that $\mu_n \to \mu$ in distribution if and only if $\int_{\mathbb{R}} f d\mu_n \to \int_{\mathbb{R}} f d\mu$ for all continuous functions vanishing at $\pm\infty$.

Next, let $\sigma^2 = \nu(\{0\})$ and Q be a measure satisfying $dQ(x) = \frac{1}{x^2} d\nu(x)$ for $x \ne 0$ and $Q(\{0\}) = 0$. In this setting, we may rewrite

$$\int_{\mathbb{R}} h(u,x) d\nu(x) = -\frac{\sigma^2}{2} u^2 + \int_{\mathbb{R}} (e^{iux} - 1 - iux) dQ(x).$$

This implies, for $u \in \mathbb{R}$,

$$f_L(u) = \exp\left\{-\frac{\sigma^2}{2}u^2 + \int_{\mathbb{R}} (e^{iux} - 1 - iux)dQ(x)\right\}, \tag{8.5}$$

where $Q(\{0\}) = 0$ and $\sigma^2 + \int_{\mathbb{R}} x^2 dQ(x) \leq 1$.

Theorem 8.7. *For $n \geq 1$, let $X_{n,1}, ..., X_{n,r_n}$ be independent random variables with finite variances and set $S_n = \sum_{k=1}^{r_n} X_{n,k}$. Assume that UAN holds and*

$$\mathbb{E}S_n \to a, \quad \mathrm{Var}(S_n) \to b^2 < \infty.$$

If $S_n \xrightarrow{\mathcal{D}} L$, then $f_L(u) = e^{\psi(u)}$ with

$$\psi(u) = iau - \frac{\sigma^2 b^2 u^2}{2} + \int_{\mathbb{R}} (e^{iux} - 1 - iux)dK(x),$$

where σ is a constant, $K(\{0\}) = 0$ and $\int_{\mathbb{R}} x^2 dK(x) < \infty$.

Proof. For $n \geq 1$ and $1 \leq k \leq r_n$, set

$$\widetilde{X}_{n,k} = \frac{X_{n,k} - \mathbb{E}X_{n,k}}{\sqrt{\mathrm{Var}(S_n)}}, \quad \widetilde{S}_n = \sum_{k=1}^{r_n} \widetilde{X}_{n,k} = \frac{S_n - \mathbb{E}S_n}{\sqrt{\mathrm{Var}(S_n)}}.$$

Clearly, $\mathbb{E}\widetilde{X}_{n,k} = 0$ and $\sum_{k=1}^{r_n} \mathbb{E}\widetilde{X}_{n,k}^2 = 1$. Since $S_n \xrightarrow{\mathcal{D}} L$, $\mathbb{E}S_n \to a$ and $\mathrm{Var}(S_n) \to b^2$, we have

$$\widetilde{S}_n \xrightarrow{\mathcal{D}} \widetilde{L} = \frac{L - a}{b}.$$

By (8.5), there is a constant $\sigma > 0$ and a measure Q on $\mathbb{R}$ satisfying $Q(\{0\}) = 0$ and $\sigma^2 + \int_{\mathbb{R}} x^2 dQ(x) \leq 1$ such that

$$f_{\widetilde{L}}(u) = \exp\left\{-\frac{\sigma^2}{2}u^2 + \int_{\mathbb{R}} (e^{iux} - 1 - iux)dQ(x)\right\}.$$

By setting $dK(y) = dQ(y/b)$, we obtain

$$f_L(u) = e^{iau} f_{\widetilde{L}}(bu) = \exp\left\{iau - \frac{\sigma^2 b^2}{2}u^2 + \int_{\mathbb{R}} (e^{iuy} - 1 - iuy)dK(y)\right\}.$$

□

Remark 8.4. Consider some particular cases.

(1) If $K = 0$ and $\sigma = 0$, then $L = a$ a.s..
(2) If $K = 0$, then $L \stackrel{\mathcal{D}}{=} N(a, \sigma^2 b^2)$.
(3) If $a = \sigma = 0$ and $K = c\delta_x$, then $f_L(u) = \exp\{c(e^{iux} - 1 - iux)\}$. Let Z_λ be a Poisson random variable with parameter $\lambda > 0$, i.e. $\mathbb{P}(Z_\lambda = n) = e^{-\lambda}\lambda^n/n!$ for $n = 0, 1, 2, ...$ Then $f_{Z_\lambda}(u) = \exp\{(e^{iu} - 1)\lambda\}$ and $L \stackrel{\mathcal{D}}{=} x(Z_c - c)$.
(4) If $a = \sigma = 0$ and $K = \sum_{k=1}^n c_k \delta_{x_k}$, then $L \stackrel{\mathcal{D}}{=} \sum_{k=1}^n x_k(Z_{c_k} - c_k)$, where $Z_{c_1}, ..., Z_{c_n}$ are independent.

Proposition 8.8. *Let $\psi(u) = iau - \sigma^2 u^2/2 + \int_{\mathbb{R}} (e^{iux} - 1 - iux)dK(x)$, where $K(\{0\}) = 0$ and $\int_{\mathbb{R}} x^2 dK(x) < \infty$. Then, $e^{\psi(u)}$ is a characteristic function of some random variable with finite second moment.*

Proof. It loses no generality to assume that $a = \sigma = 0$. Set $d\mu(x) = x^2 dK(x)$ and let h be the function in (8.4). Clearly, $\mu(\mathbb{R}) < \infty$ and

$$\int_{\mathbb{R}} (e^{iux} - 1 - iux) dK(x) = \int_{\mathbb{R}} h(u,x) d\mu(x). \tag{8.6}$$

Since h is uniformly continuous and bounded on $A \times \mathbb{R}$ for any bounded set $A \subset \mathbb{R}$, the right side of (8.6), as a function of u, is continuous on $\mathbb{R}$.

For $n \geq 1$, let $x_{n,k} = -n + \frac{k}{n}$ for all $k = 0, 1, ..., 2n^2 - 1$ and set

$$\mu_n = \sum_{k=0}^{2n^2-1} \mu([x_{n,k}, x_{n,k} + \tfrac{1}{n}))\delta_{x_{n,k}}, \quad h_n(u,x) = \sum_{k=0}^{2n^2-1} h(u, x_{n,k}) \mathbf{1}_{[x_{n,k}, x_{n,k}+\frac{1}{n})}(x).$$

Immediately, we have

$$\begin{aligned} \int_{\mathbb{R}} h(u,x) d\mu_n(x) &= \sum_{k=0}^{2n^2-1} h(u, x_{n,k}) \mu([x_{n,k}, x_{n,k} + \tfrac{1}{n})) \\ &= \int_{\mathbb{R}} h_n(u,x) d\mu(x) \to \int_{\mathbb{R}} h(u,x) d\mu(x), \end{aligned}$$

where the convergence is given by the Lebesgue dominated convergence theorem. (In fact, μ_n $\mathcal{D}$-converges to μ and, for $u \in \mathbb{R}$, $x \mapsto h(u,x)$ vanishes at infinity. By Exercise 8.1, one has the above convergence.)

Next, set $c_{n,k} = x_{n,k}^{-2} \mu([x_{n,k}, x_{n,k} + \frac{1}{n}))$ if $k \neq n^2$, $c_{n,n^2} = 0$ and

$$K_n = \sum_{k=0}^{2n^2-1} c_{n,k} \delta_{x_{n,k}}, \quad X_n = \sum_{k=0}^{2n^2-1} x_{n,k} (Z_{c_{n,k}} - c_{n,k})$$

where $(Z_\lambda)_{\lambda > 0}$ are independent Poisson random variables with $\mathbb{E} Z_\lambda = \lambda$ and $Z_0 \equiv 0$. Then,

$$\int_{\mathbb{R}} h(u,x) d\mu_n(x) = \int_{\mathbb{R}} (e^{iux} - 1 - iux) dK_n(x) - \frac{u^2}{2} \mu([0, \tfrac{1}{n}))$$

and

$$\exp\left\{ \int_{\mathbb{R}} (e^{iux} - 1 - iux) dK_n(x) \right\} = f_{X_n}(u) = \mathbb{E}[e^{iuX_n}].$$

Letting $n \to \infty$ implies

$$f_{X_n}(u) \to \exp\left\{ \int_{\mathbb{R}} (e^{iux} - 1 - iux) dK(x) \right\}, \quad \forall u \in \mathbb{R}.$$

By the continuity theorem, there is a random variable X such that $X_n \to X$ in distribution and

$$f_X(u) = \exp\left\{ \int_{\mathbb{R}} (e^{iux} - 1 - iux) dK(x) \right\}.$$

Observe that

$$\mathbb{E} X_n^2 = \sum_{k=0}^{2n^2-1} x_{n,k}^2 \mathbb{E}(Z_{c_{n,k}} - c_{n,k})^2 = \mu([-n, n)) - \mu([0, 1/n)).$$

By Corollary 7.6, one has

$$\mathbb{E} X^2 = \liminf_{n \to \infty} \mathbb{E} X_n^2 = \mu(\mathbb{R}) < \infty.$$

□