9. Infinite divisibility

Definition 9.1. A random variable X is **infinitely divisible** if, for any $n \geq 1$, there are i.i.d. random variables $X_{n,1}, ..., X_{n,n}$ such that

$$X \stackrel{\mathcal{D}}{=} X_{n,1} + \cdots + X_{n,n}.$$

Remark 9.1. Equivalently, X is infinitely divisible if and only if, for $n \geq 1$, $f_X(u) = f_n(u)^n$, where f_n is a characteristic function of some random variable.

Example 9.1. Consider the following two examples.

(1) Normal random variables: $N(\mu, \sigma^2) \stackrel{\mathcal{D}}{=} N_1(\frac{\mu}{n}, \frac{\sigma^2}{n}) + \cdots + N_n(\frac{\mu}{n}, \frac{\sigma^2}{n})$.

(2) Poisson random variables: $Z_\lambda \stackrel{\mathcal{D}}{=} Z^1_{\lambda/n} + \cdots + Z^n_{\lambda/n}$ since $f_{Z_\lambda}(u) = \exp\{\lambda(e^{iu} - 1)\}$.

Theorem 9.1. *The following three classes are identical.*

C_1: *The class of all limiting distributions X under UAN, that is,*

$$S_n = X_{n,1} + \cdots + X_{n,r_n} \xrightarrow{\mathcal{D}} X \quad \text{as } n \to \infty,$$

where $X_{n,1}, ..., X_{n,r_n}$ are independent with finite second moments such that $\mathbb{E}S_n$ and $\mathrm{Var}(S_n)$ converges and $\max_k\{\mathrm{Var}(X_{n,k})/\mathrm{Var}(S_n)\} \to 0$.

C_2: *The class of all distributions of which characteristic functions are of the form $e^{\psi(u)}$, where*

$$\psi(u) = iau - \frac{\sigma^2 u^2}{2} + \int_{\mathbb{R}} (e^{iux} - 1 - iux) dK(x)$$

with $K(\{0\}) = 0$ and $\int_{\mathbb{R}} x^2 dK(x) < \infty$.

C_3: *The class of all infinitely divisible distributions with finite second moment.*

Proof. $C_1 \subset C_2$ has been proved in Theorem 8.7. For $C_2 \subset C_3$, let X be a random variable with c.f. $f_X(u) = e^{\psi(u)}$, where

$$\psi(u) = iau - \frac{\sigma^2 u^2}{2} + \int_{\mathbb{R}} (e^{iux} - 1 - iux) dK(x)$$

and $K(\{0\}) = 0$ and $\int_{\mathbb{R}} x^2 dK(x) < \infty$. By Proposition 8.8, let $X_{n,1}, ..., X_{n,n}$ be i.i.d. random variables with c.f.s

$$\exp\left\{\frac{ia}{n}u - \frac{(\sigma/\sqrt{n})^2}{2}u^2 + \int_{\mathbb{R}} (e^{iux} - 1 - iux) dK_n(x)\right\},$$

where $K_n(\{0\}) = 0$ and $dK_n(x) = \frac{1}{n} dK(x)$. This implies $f_X(u) = \prod_{k=1}^n f_{X_k}(u)$ or equivalently $X \stackrel{\mathcal{D}}{=} X_1 + \cdots + X_n$. The finiteness of $\mathbb{E}X^2$ is immediate from Proposition 8.8.

For $C_3 \subset C_1$, let X be an infinitely divisible random variable with $\mathbb{E}X = \mu$ and $\mathrm{Var}(X) = \sigma^2$. For $n \geq 1$, let $X_{n,1}, ..., X_{n,n}$ be i.i.d. random variables such that $X \stackrel{\mathcal{D}}{=} X_{n,1} + \cdots + X_{n,n}$. Clearly, $\mathrm{Var}(X_{n,k}) = \frac{\sigma^2}{n}$ for $1 \leq k \leq n$ and $n \geq 1$ and the UAN holds for the triangular array $\{X_{n,k} | 1 \leq k \leq n, n \geq 1\}$. This implies $X \in C_1$. □

In the following, we consider infinitely divisible random variables without the assumption of finite second moment and generalize Theorem 9.1.

Proposition 9.2. *A random variable X is infinitely divisible if and only if there are i.i.d. random variables $X_{n,1}, ..., X_{n,n}$ such that $S_n = X_{n,1} + \cdots + X_{n,n} \xrightarrow{\mathcal{D}} X$.*

Proof. The necessity for the infinite divisibility is obvious. For the sufficiency, assume that $S_n \stackrel{\mathcal{D}}{\to} X$. Fix $m \geq 1$ and set

$$Y_{n,k} = \sum_{i=1}^{n} X_{mn,(k-1)n+i}, \quad \forall 1 \leq k \leq m.$$

Obviously, $Y_{n,1}, ..., Y_{n,m}$ are i.i.d. and $S_{mn} = Y_{n,1} + \cdots + Y_{n,m}$. This implies

$$\mathbb{P}(Y_{n,1} \leq -y)^m = \mathbb{P}(Y_{n,k} \leq -y, \forall 1 \leq k \leq n) \leq \mathbb{P}(S_{mn} \leq -my).$$

Similarly, one may derive $\mathbb{P}(Y_{n,1} \geq y)^m \leq \mathbb{P}(S_{mn} \geq my)$. Since $S_{mn} \stackrel{\mathcal{D}}{\to} X$, S_{mn} and, thus, $Y_{n,1}$ are mass-preserving. As a consequence, we may choose a subsequence $(n_\ell)_{\ell=1}^\infty$ of $\mathbb{N}$ such that $Y_{n_\ell,k}$ converges in distribution to Y_k for all $1 \leq k \leq m$. Clearly, $Y_1, ..., Y_m$ are identically distributed. By creating the independency of $Y_1, ..., Y_m$, the continuity theorem implies that $X \stackrel{\mathcal{D}}{=} Y_1 + \cdots + Y_m$. □

Theorem 9.3. *Let X be a random variable with characteristic function f. Then, X is infinitely divisible if and only if*

$$f(u) = \exp\left\{ i\beta u - \frac{\sigma^2 u^2}{2} + \int_{\mathbb{R}} \left(e^{iux} - 1 - \frac{iux}{1+x^2} \right) \frac{1+x^2}{x^2} d\nu(x) \right\} \tag{9.1}$$

where $\beta, \sigma \in \mathbb{R}$ and ν is a finite measure with $\nu(\{0\}) = 0$.

To prove this theorem, we need the following proposition.

Proposition 9.4. *Let $X_{n,1}, ..., X_{n,n}$ be i.i.d. random variables and set $S_n = \sum_{i=1}^n X_{n,i}$. If $S_n \stackrel{\mathcal{D}}{\to} X$, then $X_{n,1} \stackrel{\mathcal{D}}{\to} 0$.*

Proof. Let f_n and g_n be characteristic functions of $X_{n,1}$ and S_n and, for $z \in \mathbb{C} \setminus (-\infty, 0]$, let $\mathrm{Arg} z \in (-\pi, \pi)$ be the argument of z. Since S_n $\mathcal{D}$-converges to X, we may choose, by Theorem 7.15, $a > 0$ and $\delta > 0$ such that $\Re g_n \geq \delta > 0$ on $(-a, a)$ for all $n \geq 1$. This implies $|\mathrm{Arg} g_n(u)| < \pi/2$ for $|u| < a$. Note that $g_n = (f_n)^n$. By the continuity of f_n, this implies

$$\mathrm{Arg} f_n(u) = \frac{\mathrm{Arg} g_n(u)}{n}, \quad \forall |u| < a,$$

which leads to $\mathrm{Arg} f_n(u) \to 0$ for $|u| < a$. Moreover, by the fact $|f_n|^n = |g_n| \geq \delta$ on $(-a, a)$ for all $n \geq 1$, we have $|f_n| \to 1$ on $(-a, a)$. As a result, $f_n(u) \to 1$ for $|u| < a$. By Corollary 7.16, $X_{n,1}$ converges to 0 in distribution. □

Corollary 9.5. *For any infinitely divisible random variable X, $\mathbb{E}e^{iuX} \neq 0$ for all $u \in \mathbb{R}$.*

Proof. Let $X_{n,1}, ..., X_{n,n}$ be i.i.d. random variables such that $S_n = X_{n,1} + \cdots + X_{n,n} \stackrel{\mathcal{D}}{=} X$. Clearly, $S_n \to X$ in distribution. By Proposition 9.4, $X_{n,1} \to 0$ in distribution. Let f_n, f be characteristic functions of $X_{n,1}, X$. Then, $f(u) = f_n(u)^n$ and $f_n(u) \to 1$ for all $u \in \mathbb{R}$. If $f(u_0) = 0$ for some $u_0 \in \mathbb{R}$, then $f_n(u_0) = 0$ for all $n \geq 1$, which leads to $f_n(u_0) \to 0$. A contradiction! □

Proof of Theorem 9.3: Necessity for infinite divisibility. Assume that X is infinitely divisible and $X \stackrel{\mathcal{D}}{=} X_{n,1} + \cdots + X_{n,n}$, where $X_{n,1}, ..., X_{n,n}$ are i.i.d. random variables. Let f, f_n be characteristic functions of $X, X_{n,1}$. By Corollary 9.5, $f(u) \neq 0$ for all $u \in \mathbb{R}$ and thus $f_n(u) \neq 0$ for all $u \in \mathbb{R}$. By the continuity of f_n, there is a (unique) complex-valued continuous function ψ_n defined on $\mathbb{R}$ such that $\psi_n(0) = 0$ and $f_n = e^{\psi_n}$. Set $\psi := n\psi_n$. Obviously, ψ is continuous and $f = e^\psi$

By Proposition 9.4 and Theorem 7.15, $f_n \to 1$ and then $\psi_n(u) \to 0$ uniformly on any finite interval. Write

$$\psi(u) = n[f_n(u) - 1] \times [1 + \epsilon_n(u)], \tag{9.2}$$

where

$$\epsilon_n(u) = \begin{cases} \frac{\psi_n(u)}{e^{\psi_n(u)}-1} - 1 & \text{for } \psi_n(u) \neq 0, \\ 0 & \text{for } \psi_n(u) = 0. \end{cases}$$

Clearly, one has that $\epsilon_n(u) \to 0$ uniformly on any finite interval. Let F_n be the distribution of $X_{n,1}$ and $d\mu_n(x) = ndF_n(x)$. Then, (9.2) becomes

$$\psi(u) = (1 + \epsilon_n(u)) \int_{\mathbb{R}} (e^{iux} - 1) d\mu_n(x). \tag{9.3}$$

Consider the following two facts.

Fact 1: There is $A > 0$ such that

$$\limsup_{n\to\infty} \mu_n([-a, a]^c) \leq Aa \int_0^{1/a} |\Re\psi(v)| dv, \quad \forall a > 0.$$

It is easy to see from (9.2) that

$$\lim_{n\to\infty} n[1 - \Re f_n(u)] = -\lim_{n\to\infty} \Re\left(\frac{\psi(u)}{1 + \epsilon_n(u)}\right) = -\Re\psi(u) = |\Re\psi(u)|, \tag{9.4}$$

uniformly on any finite interval. Letting A be the constant in Proposition 7.12, we have

$$\mu_n([-a, a]^c) \leq Aa \int_0^{1/a} n[1 - \Re f_n(v)] dv, \quad \forall a > 0.$$

As $\Re\psi$ is bounded on any finite interval, the desired inequality is then given by the Lebesgue dominated convergence theorem.

Fact 2:

$$\limsup_{n\to\infty} \int_{[-1,1]} x^2 d\mu_n(x) < \infty.$$

Note that

$$n[1 - \Re f_n(1)] = \int_{\mathbb{R}} (1 - \cos x) d\mu_n(x) \geq \int_{[-1,1]} (1 - \cos x) d\mu_n(x).$$

Consider the inequality $\sin t \geq 2t/\pi$ for $t \in [0, 1]$. This implies, for $x \in [0, 1]$,

$$1 - \cos x = \int_0^x \sin t dt \geq \frac{x^2}{\pi},$$

which leads to

$$n[1 - \Re f_n(1)] \geq \frac{1}{\pi} \int_{[-1,1]} x^2 d\mu_n(x).$$

By (9.4), the left hand side converges to $|\Re\psi(1)| < \infty$ and this proves Fact 2.

To see the characteristic function of X, we define

$$\alpha_n = \int_{\mathbb{R}} \frac{y^2}{1 + y^2} d\mu_n(y), \quad G_n(x) = \frac{1}{\alpha_n} \int_{(-\infty,x]} \frac{y^2}{1 + y^2} d\mu_n(y), \quad \beta_n = \int_{\mathbb{R}} \frac{x}{1 + x^2} d\mu_n(x),$$

and

$$\varphi(u, x) = \left(e^{iux} - 1 - \frac{iux}{1 + x^2}\right) \frac{1 + x^2}{x^2}, \quad \forall x \neq 0, \quad \varphi(u, 0) = -\frac{u^2}{2}.$$

Following the above setting, (9.3) can be rewritten as

$$\psi(u) = (1 + \epsilon_n(u))\alpha_n \int_{\mathbb{R}} \varphi(u, x) dG_n(x) + (1 + \epsilon_n(u)) iu\beta_n. \tag{9.5}$$

It is obvious that G_n is a distribution function and $|\varphi(u,x)| \leq |u| + 4$ for $|x| > 1$. By writing

$$\varphi(u, x) = iux + \sum_{k=2}^{\infty} \frac{(iu)^k x^{k-2}(1 + x^2)}{k!},$$

one has $|\varphi(u,x)| \leq 2(e^{|u|} - 1)$ for $|x| \leq 1$. As a result, this implies that, for fixed u, $\varphi(u,x)$ is bounded and continuous on $\mathbb{R}$ and then $\{\int_{\mathbb{R}} \varphi(u,x) dG_n(x) | n \geq 1\}$ is bounded.

Recall that $f(u) \neq 0$ for all $u \in \mathbb{R}$. By Facts 1 and 2, $(\alpha_n)_{n=1}^{\infty}$ is a bounded sequence and, hence, $(\beta_n)_{n=1}^{\infty}$ is bounded. By (9.5), if $\alpha_n \to 0$, then β_n converges, which implies that X is degenerate, i.e. $\sigma^2 = 0$ and $\nu(\mathbb{R}) = 0$ in (9.1). When $\alpha_n \nrightarrow 0$, we select a convergent subsequence α_{n_k} with limit $\alpha > 0$. By Fact 1, one has

$$\limsup_{a \to \infty} \limsup_{n \to \infty} \mu_n([-a, a]^c) = 0.$$

This implies that $(G_{n_k})_{k=1}^{\infty}$ is mass-preserving. As a result, we may choose a further subsequence $(m_k)_{k=1}^{\infty}$ of $(n_k)_{k=1}^{\infty}$ and $\beta \in \mathbb{R}$, $G \in \mathcal{N}$ such that $\beta_{m_k} \to \beta$ and $G_{m_k} \to G$ in distribution. Replacing n with m_k in (9.5) and passing k to infinity yields

$$\begin{aligned} f(u) &= \exp\left\{ \alpha \int_{\mathbb{R}} \left(e^{iux} - 1 - \frac{iux}{1+x^2} \right) \frac{1+x^2}{x^2} dG(x) + i\beta u \right\} \\ &= \exp\left\{ \alpha \int_{\mathbb{R}\setminus\{0\}} \left(e^{iux} - 1 - \frac{iux}{1+x^2} \right) \frac{1+x^2}{x^2} dG(x) + i\beta u - \frac{\alpha u^2}{2} G(\{0\}) \right\}. \end{aligned}$$

□

Proof of Theorem 9.3: Sufficiency for infinite divisibility. As in the Proof of Theorem 9.1, it remains to prove that $f(u)$ is the characteristic function of some random variable. Without loss of generality, we may assume that $\beta = \sigma = 0$. We first consider the case that ν is supported on a finite set, say $\{x_1, ..., x_n\}$. Clearly, $x_i \neq 0$, for $1 \leq i \leq n$, and

$$f(u) = \prod_{j=1}^{n} \exp\{\lambda_j (e^{iux_j} - 1)\} \times \exp\{-iub\},$$

where $\lambda_j = \nu(\{x_j\})(1 + x_j^2)/x_j^2$ and $b = -\sum_{j=1}^{n} \nu(\{x_j\})/x_j$. Clearly, $f(u)$ is the characteristic function of X, where $X = \sum_{i=1}^{n} x_i Y_i - b$ and $Y_1, ..., Y_n$ are independent Poisson random variables with parameters $\lambda_1, ..., \lambda_n$.

For the general case, let $(\nu_n)_{n=1}^{\infty}$ be a sequence of measures supported on finite sets satisfying $\nu_n(\mathbb{R}) \to \nu(\mathbb{R})$ and $\nu_n/\nu_n(\mathbb{R})$ $\mathcal{D}$-converges to $\nu/\nu(\mathbb{R})$. (How to create ν_n?) Let

$$f_n(u) = \exp\left\{ \int_{\mathbb{R}} \varphi(u, x) d\nu_n(x) \right\}, \quad \forall n \in \mathbb{N},$$

and X_n be a random variable with characteristic function f_n. By the Lebesgue dominated convergence theorem, since, for $u \in \mathbb{R}$, $\varphi(u, \cdot)$ is bounded and continuous, $f_n \to f$. As φ is uniformly bounded on $[-a, a] \times \mathbb{R}$ for any $a > 0$, one my apply the LDCT to derive that, for any $u \in \mathbb{R}$ and any sequence $(h_n)_{n=1}^{\infty}$ converging to 0,

$$\lim_{n \to \infty} f(u + h_n) = f(u).$$

This yields the continuity of f. As a consequence of the continuity theorem, f is the characteristic function of some random variable. □

Exercise 9.1. Show that, for any finite measure ν on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, there is a sequence of measures $(\nu_n)_{n=1}^\infty$ supported on finite sets satisfying $\nu_n(\mathbb{R}) \to \nu(\mathbb{R})$ and $\nu_n/\nu_n(\mathbb{R})$ converges to $\nu/\nu(\mathbb{R})$ in distribution.

In the following, we examine the uniqueness of constants β and ν in Theorem 9.3. First, recall the following function

$$\varphi(u,0) = -u^2/2, \quad \varphi(u,x) = \left(e^{iux} - 1 - \frac{iux}{1+x^2}\right)\frac{1+x^2}{x^2} \quad \forall x \neq 0. \tag{9.6}$$

Let γ be the finite measure on $\mathbb{R}$ defined by

$$\gamma(B) = \begin{cases} \nu(B) + \sigma^2 & \text{if } 0 \in B \\ \nu(B) & \text{if } 0 \notin B \end{cases}.$$

Using the above setting, we may rewrite the function f in Theorem 9.3 as

$$f(u) = \exp\left\{i\beta u + \int_{\mathbb{R}} \varphi(u,x) d\gamma(x)\right\}, \tag{9.7}$$

where γ is a finite measure on $\mathbb{R}$.

Proposition 9.6. *β, γ are uniquely determined by* (9.7).

Proof. Let $\psi(u)$ be a continuous function satisfying $f = e^\psi$ and set

$$\theta(u) = \psi(u) - \int_0^1 \frac{\psi(u+h) + \psi(u-h)}{2} dh.$$

Clearly, θ is independent of the choice of ψ and is determined by f. Note that

$$\begin{aligned}\int_0^1 \frac{\varphi(u+h,x) + \varphi(u-h,x)}{2} dh &= \varphi(u,x) - e^{iux}\frac{1+x^2}{x^2}\int_0^1 (1-\cos(hx))dh \\ &= \varphi(u,x) - e^{iux}\left(1 - \frac{\sin x}{x}\right)\frac{1+x^2}{x^2}\end{aligned}$$

This implies

$$\theta(u) = \int_{\mathbb{R}} e^{iux} d\mu(x),$$

where

$$d\mu(x) = g(x)d\gamma(x), \quad g(x) = \left(1 - \frac{\sin x}{x}\right)\frac{1+x^2}{x^2}.$$

Since $0 < \inf_x g(x) \le \sup_x g(x) < \infty$, μ is a finite measure and $d\gamma(x) = (1/g(x))d\mu(x)$.

Assume that $\psi(u) = i\beta' u + \int_{\mathbb{R}} \varphi(x,u) d\gamma'(x)$ and set $d\mu'(x) = g(x)d\gamma'(x)$. As before, one has $\theta(u) = \int_{\mathbb{R}} e^{iux} d\mu'(x)$ and $d\gamma'(x) = (1/g(x))d\mu'(x)$. When $u = 0$, $\theta(0) = \mu(\mathbb{R}) = \mu'(\mathbb{R})$. By Theorem 7.11, this implies $\mu = \mu'$ and, hence, $\gamma = \gamma'$. Putting this fact back to (9.7) yields $e^{i\beta u} = e^{i\beta' u}$ for all $u \in \mathbb{R}$. Again, Theorem 7.11 implies $\beta = \beta'$. $\square$

To state the next theorem, we need the following generalization of mass-preserving measures. A set $\mathcal{A}$ of finite measures on $\mathbb{R}$ is said to be mass-preserving if, for any $\epsilon > 0$, there is a finite interval I such that $\mu(I^c) < \epsilon$ for all $\mu \in \mathcal{A}$.

Remark 9.2. Let μ_n, μ be finite measures on $\mathbb{R}$ satisfying $\mu_n(\mathbb{R}) > 0$ and $\mu(\mathbb{R}) > 0$. Then, $(\mu_n)_{n=1}^\infty$ is mass-preserving and $\mathcal{D}$-converges to μ if and only if $\mu_n(\mathbb{R}) \to \mu(\mathbb{R}) < \infty$ and $\mu_n/\mu_n(\mathbb{R})$ $\mathcal{D}$-converges to $\mu/\mu(\mathbb{R})$.

Remark 9.3. If $(\mu_n)_{n=1}^\infty$ is mass-preserving and $\mathcal{D}$-converges to μ, then $\int_{\mathbb{R}} f(x)d\mu_n(x) \to \int_{\mathbb{R}} f(x)d\mu(x)$ for any bounded measurable function f satisfying $\mu(\mathcal{C}(f)^c) = 0$.

Theorem 9.7. *Let $X_{n,1}, ..., X_{n,n}$ be i.i.d. random variables and $S_n = \sum_{i=1}^n X_{n,i}$. Let F_n be the distribution of $X_{n,1}$, $\beta_n = \int_{\mathbb{R}} nx/(1+x^2)dF_n(x)$ and $\gamma_n(x) = \int_{(-\infty,x]} ny^2/(1+y^2)dF_n(y)$. Then, S_n converges in distribution to some random variable X if and only if β_n converges and γ_n is mass-preserving and converges in distribution. Furthermore, if γ_n is mass-preserving and $\mathcal{D}$-converges to γ and $\beta_n \to \beta$, then the characteristic function f of X is given by*

$$f(u) = \exp\left\{iu\beta + \int_{\mathbb{R}} \varphi(u,x)d\gamma(x)\right\}. \tag{9.8}$$

Proof. Let f_n, h_n be characteristic functions of $X_{n,1}, S_n$ and $I_n \subset \mathbb{R}$ be the largest open interval such that $f_n(u) \neq 0$ (also for $h_n(u)$) for $u \in I_n$. Let $\psi_n(u)$ be the (unique) complex-valued continuous function on I_n satisfying $\psi_n(0) = 0$ and $f_n(u) = e^{\psi_n(u)}$ for $u \in I_n$. Set $\theta_n := n\psi_n$. Clearly, θ_n is continuous and $h_n = e^{\theta_n}$ on I_n. Similar to (9.2) and (9.5), one may write

$$\begin{aligned}\theta_n(u) &= (1+\epsilon_n(u))n[f_n(u)-1] \\ &= (1+\epsilon_n(u))\alpha_n \int_{\mathbb{R}} \varphi(u,x)dG_n(x) + (1+\epsilon_n(u))iu\beta_n, \quad \forall u \in I_n,\end{aligned} \tag{9.9}$$

where $G_n(x) = \gamma_n(x)/\alpha_n$ and

$$\alpha_n = \int_{\mathbb{R}} \frac{nx^2}{1+x^2}dF_n(x), \quad \gamma_n(x) = \int_{(-\infty,x]} \frac{ny^2}{1+y^2}dF_n(y), \quad \beta_n = \int_{\mathbb{R}} \frac{nx}{1+x^2}dF_n(x).$$

First, assume that $S_n \to X$ in distribution. By Propositions 9.2 and 9.4, X is infinitely divisible and $X_{n,1} \to 0$ in distribution. Let f be the characteristic function of X and ψ be the (unique) complex-valed continuous function satisfying $\psi(0) = 0$ and $f = e^{\psi}$. Then, $f_n \to 1$ and $\theta_n \to \psi$ uniformly on any finite interval and thus $\liminf_n I_n = \mathbb{R}$ and $\epsilon_n(u) \to 0$ uniformly on any finite interval. A similar reasoning as in the proof of Theorem 9.3 (Facts 1 and 2) yields

$$\limsup_{n\to\infty} nF_n([-a,a]^c) \le Aa \int_0^{1/a} |\Re\psi(v)|dv < \infty, \quad \forall a > 0,$$

for some $A > 0$ and

$$\limsup_{n\to\infty} \int_{[-1,1]} x^2 ndF_n(x) \le \pi|\Re\psi(1)| < \infty.$$

This implies that $(\alpha_n)_{n=1}^\infty$ is bounded and $(\gamma_n)_{n=1}^\infty$ is mass-preserving. Let $(\alpha_{n_k})_{k=1}^\infty$ be a convergent subsequence with limit α.

Case 1: $\alpha = 0$**.** Clearly, γ_{n_k} converges to 0 in distribution.

Case 2: $\alpha > 0$**.** One can show that $(G_{n_k})_{k=1}^\infty$ is mass-preserving. Let $(G_{m_k})_{k=1}^\infty$ be a further subsequence that $\mathcal{D}$-converges to $G \in \mathcal{N}$. By Remark 9.2, γ_{m_k} $\mathcal{D}$-converges to $\gamma := \alpha G$.

By Proposition 9.6, this implies $(\gamma_n)_{n=1}^\infty$ is mass-preserving and $\mathcal{D}$-converges. By (9.9), $(\beta_n)_{n=1}^\infty$ is convergent. Moreover, if γ, β are limits of γ_n, β_n, then $\psi(u) = \int_{\mathbb{R}} \varphi(u,x)d\gamma(x) + iu\beta$.

Next, suppose γ_n $\mathcal{D}$-converges to γ and β_n converges to β. Note that $\gamma_n \to \gamma$ in distribution implies $F_n \to \mathbf{1}_{[0,\infty)}$ in distribution or equivalently $X_{n,1} \to 0$ in distribution. In (9.9), this implies $\epsilon_n(u) \to 0$ uniformly on any finite interval. By Remark 9.3, we obtain

$$\lim_{n\to\infty} \theta_n(u) = \int_{\mathbb{R}} \varphi(u,x)d\gamma(x) + iu\beta, \quad \forall u \in \mathbb{R}.$$

By the LDCT, since $\varphi(u,x)$ is bounded on $[-a,a] \times \mathbb{R}$ for any $a > 0$ and $\varphi(u,x) \to \varphi(0,x) = 0$ as $u \to 0$, the integral at the right side of the above equality is continuous at 0. As a consequence of the continuity theorem (Theorem 7.13), S_n converges in distribution to some random variable X. □

Corollary 9.8. *Let $X_{n,1}, ..., X_{n,n}$ be i.i.d. random variables and set $S_n = \sum_{i=1}^n X_{n,i}$ and $M_n = \max\{|X_{n,i}| : 1 \le i \le n\}$. Assume that S_n converges in distribution to X. Then, X is normal if and only if M_n converges to 0 in distribution.*

Proof. Consider the expression of (9.8). By Proposition 9.6, X is normal with mean μ and variance σ^2 if and only if $\beta = \mu$ and $\gamma = \sigma^2 \mathbf{1}_{[0,\infty)}$. By Theorem 9.7, $\gamma = \sigma^2 \mathbf{1}_{[0,\infty)}$ for some $\sigma^2 > 0$ if and only if

$$\int_{[-a,a]^c} \frac{y^2}{1+y^2} n dF_n(x) \to 0, \quad \forall a > 0,$$

or equivalently $n\mathbb{P}(|X_{n,1}| > a) = nF_n([-a,a]^c) \to 0$ for al $a > 0$. Note that, for $a > 0$,

$$\mathbb{P}(M_n \le a) = [1 - \mathbb{P}(|X_{n,1}| > a)]^n = e^{n \log[1-\mathbb{P}(|X_{n,1}|>a)]}.$$

Since $\mathbb{P}(|X_{n,1}| > a) \to 0$ for $a > 0$ (by Proposition 9.4), this implies

$$M_n \overset{\mathcal{D}}{\to} 0 \quad \Leftrightarrow \quad n \log[1 - \mathbb{P}(|X_{n,1}| > a)] \to 0, \ \forall a > 0 \quad \Leftrightarrow \quad n\mathbb{P}(|X_{n,1}| > a) \to 0, \ \forall a > 0.$$

□