LECTURE NOTES IN PROBABILITY THEORY

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1. Combinatorial analysis

Counting: A bijection $\phi : A \to B$ is a one-to-one (injective) and onto (surjective) mapping, where ϕ is one-to-one if

$$\phi(x) = \phi(y) \quad \Rightarrow \quad x = y$$

and ϕ is onto if for any $b \in B$, there is $a \in A$ such that $\phi(a) = b$. Given a non-empty finite set S, the number of elements in S is denoted and defined by

$$\begin{aligned} |S| &:= \min\{n \in \mathbb{N} | \text{ There is a one-to-one map } \phi : S \to \{1, 2, ..., n\}\} \\ &:= \max\{n \in \mathbb{N} | \text{ There is a onto map } \phi : S \to \{1, 2, ..., n\}\}. \end{aligned}$$

Conventionally, we set $|\emptyset| = 0$. It follows from the above definition that, for $n \in \mathbb{N}$, $|S| = n$ if and only if there is a bijection $\phi : S \to \{1, 2, ..., n\}$. Note that two finite sets, S and T, are said to have the same number of elements (or the same cardinality) if there is a bijection $\phi : S \to T$.

Example 1.1. If $S = A \cup B$ and $A \cap B = \emptyset$, then $|S| = |A| + |B|$. To see a proof, assume that $|A| = m$, $|B| = n$ and write $A = \{a_1, ..., a_m\}$, $B = \{b_1, ..., b_n\}$. Set $c_i = a_i$ for $1 \le i \le m$ and $c_{m+i} = b_i$ for $1 \le i \le n$. Then, $S = \{c_i | 1 \le i \le n + m\}$ and $|S| = n + m = |A| + |B|$. An inductive argument yields the following proposition.

Proposition 1.1. *Let $n \ge 2$ and $A_1, ..., A_n$ be mutually disjoint finite sets, that is, $A_i \cap A_j = \emptyset$ if $i \ne j$. Then, $|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|$.*

Example 1.2. Let $A = \{(i, j) | 1 \le i \le m, 1 \le j \le n\}$ and $B = \{1, 2, ..., nm\}$. Consider the map $\phi : A \to B$ defined by

$$\phi(i, j) = (i - 1)n + j.$$

Note that if $\phi(i, j) = \phi(k, l)$, that is, $(i - 1)n + j = (k - 1)n + l$, then

$$(i - k)n = l - j.$$

This implies n divides $|l - j|$. Since $1 \le j, l \le n$, it is clear that $0 \le |j - l| < n$. Hence, $j = l$ and $i = k$. This proves that ϕ is one-to-one. To see that ϕ is onto, let $b \in B$ and write $b = (i - 1)n + j$ with $1 \le j \le n$. Since $b \ge 1$, it is clear that $i \ge 1$; since $b \le nm$, $i \le m$. As a result, $\phi(i, j) = b$. Consequently, we obtain $|A| = |B| = nm$. (One may also prove this identity using Proposition 1.1).

Proposition 1.2 (The generalized basic principle of counting)**.** *Let $n \ge 2$. For $1 \le i \le n$, let A_i be a finite set with $|A_i| = m_i$ and set $S_n = A_1 \times A_2 \times \cdots \times A_n = \{(a_1, ..., a_n) | a_i \in A_i, \forall 1 \le i \le n\}$. Then, $|S_n| = m_1 m_2 \cdots m_n$.*

Proof. We prove this theorem by mathematical induction. It has been shown that this proposition holds for $n = 2$. Assume that this principle holds for some integer n with $n \ge 2$. For the case of $n + 1$, let $A = A_1 \times \cdots \times A_n$ and $B = A_{n+1}$. Clearly, $A \times B$ and S_{n+1} have the same cardinality. By induction, $|A| = m_1 m_2 \cdots m_n$ and this implies $|S_{n+1}| = |A| \times |B| = m_1 m_2 \cdots m_{n+1}$. □

Proposition 1.3 (Permutation)**.** *Let S be a set of n distinct objects and S_n be the set of all permutations (arrangements) of elements in S, i.e.*

$$S_n = \{(a_1, ..., a_n) | a_i \in S \text{ for } 1 \le i \le n,\ a_i \neq a_j \text{ for } i \neq j\}.$$

Then, $|S_n| = n!$.

Proposition 1.4 (Combination)**.** *Let S be a set of n distinct objects and, for $1 \le k \le n$, let C_k be the set of all k-subsets (subsets with k elements) of S. Then, $|C_k| = \binom{n}{k} := n!/[k!(n-k)!]$, where $0! := 1$.*

Corollary 1.5. *Let S be a set of n elements and, for $k \ge 1$, let*

$$P_k = \{(a_1, ..., a_k) | a_i \in S \text{ for } 1 \le i \le n,\ a_i \neq a_j \text{ for } i \neq j\}.$$

Then, $|P_k| = n!/(n-k)!$.

Remark 1.1. For $n \in \mathbb{N}$ and $1 \le r < n$, $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$. Such an identity is known as Pascal's identity.

Theorem 1.6 (The binomial theorem)**.** *For $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$, one has*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k},$$

where $0^0 := 1$. In particular, $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

Combinatorial proof. It is clear that the binomial theorem holds if $xy = 0$. Consider the case of $xy \neq 0$. Let $A = \{(a_1, ..., a_n) | a_i \in \{0, 1\}\}$. It is obvious that

$$(x+y)^n = \sum_{(a_1,...,a_n)\in A} \prod_{k=1}^{n} x^{a_k} y^{1-a_k} = \sum_{(a_1,...,a_n)\in A} x^{a_1+\cdots+a_n} y^{n-(a_1+\cdots+a_n)}.$$

Note that A is the disjoint union of $C_0, ...C_n$, where

$$C_k = \{(a_1, ..., a_n) \in A | a_1 + \cdots + a_n = k\}.$$

By the formula of combination, $|C_k| = \binom{n}{k}$ and thus

$$(x+y)^n = \sum_{k=0}^{n} |C_k| x^k y^{n-k} = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

□

Remark 1.2. Note that $|C_k|$ can be also regarded as the number of ways to arrange n elements in a line, where k objects are of type I and $n-k$ objects are of type II.

Remark 1.3. One may prove the binomial theorem by the mathematical induction.

The following theorem is an immediate result of the binomial theorem and should be regarded as a generalization.

Theorem 1.7 (The multinomial theorem)**.** *Let n, k be positive integers with $k \le n$ and $x_1, ..., x_k$ be real numbers. Then,*

$$(x_1 + \cdots + x_k)^n = \sum_{\substack{n_1,...,n_k \in \mathbb{N}_0: \\ n_1+\cdots+n_k=n}} \binom{n}{n_1, n_2, ..., n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k},$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\binom{n}{n_1,n_2,...,n_k} = n!/(n_1!n_2! \cdots n_k!)$.

Proof. We prove the multinomial theorem by induction on k. For $k = 2$, the theorem turns out the binomial theorem. Assume that the theorem holds for some $k \geq 2$. Then, for the case of $k+1$, we first apply the binomial theorem to obtain

$$(x_1 + \cdots + x_{k+1})^n = \sum_{m=0}^{n} \binom{n}{m} (x_1 + \cdots + x_k)^m x_{k+1}^{n-m}.$$

By the inductive assumption, we have

$$(x_1 + \cdots + x_k)^m = \sum_{\substack{n_1,\ldots,n_k \in \mathbb{N}_0: \\ n_1+\cdots+n_k = m}} \binom{m}{n_1, n_2, \ldots, n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}.$$

With the replacement of $m = n - n_{k+1}$, one may derive from the above identities that

$$\begin{aligned}(x_1 + \cdots + x_{k+1})^n &= \sum_{n_{k+1}=0}^{n} \sum_{\substack{n_1,\ldots,n_k \in \mathbb{N}_0: \\ n_1+\cdots+n_k = n-n_{k+1}}} \binom{n}{n_{k+1}} \binom{n - n_{k+1}}{n_1, \ldots, n_k} x_1^{n_1} \cdots x_{k+1}^{n_{k+1}} \\ &= \sum_{\substack{n_1,\ldots,n_{k+1} \in \mathbb{N}_0: \\ n_1+\cdots+n_{k+1} = n}} \binom{n}{n_1, \ldots, n_{k+1}} x_1^{n_1} \cdots x_{k+1}^{n_{k+1}}.\end{aligned}$$

□

Remark 1.4. Note that, for $n_1 + \cdots + n_k = n$, $\binom{n}{n_1,\ldots,n_k}$ also refers to the number of arrangements of n objects with k types, where n_i elements are of type i for $1 \leq i \leq k$.

Example 1.3. A child has 9 blocks, of which 3 are red, 4 are blue and 2 are yellow. There are $\binom{9}{3,4,2} = 1260$ to line those blocks in a linear order.

Example 1.4. Consider a group of 20 people. If everyone shakes hands with everyone else, then $\binom{20}{2}$ handshakes take place in total.

Example 1.5. To simplify the sum $s = \sum_{j=i}^{n} \binom{n}{j}\binom{j}{i}$ with $i \leq n$, let's consider the following combinatorial statement. From a group of n people, a committee of j persons is selected and then a subcommittee of i persons is chosen from this committee. Then, s refers to the total number of ways to select both committees. Equivalently, one may first select the subcommittee and then choose the remainder committee members from the other $n-i$ persons. This implies $s = \binom{n}{i} 2^{n-i}$. In fact, a direct computation using the fact of $\binom{n}{j}\binom{j}{i} = \binom{n}{i}\binom{n-i}{j-i}$ yields

$$s = \sum_{j=i}^{n} \binom{n}{i}\binom{n-i}{j-i} = \binom{n}{i} \sum_{k=0}^{n-i} \binom{n-i}{k} = \binom{n}{i} 2^{n-i}.$$

Remark 1.5. Could one use a similar statement to simplify the sum $\sum_{j=i}^{n} (-1)^j \binom{n}{j}\binom{j}{i}$?