LECTURE NOTES IN CALCULUS II

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11. Infinite sequences and series

11.1. Sequences.

Definition 11.1. A sequence is a list of numbers written in a definite order:

$$a_1,\ a_2,\ a_3,\ \ldots,\ a_n,\ \ldots$$

Briefly, we also write the sequence $\{a_1, a_2, a_3, \ldots\}$ as $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$.

Example 11.1. For the formulas $a_n = n^2 - n$ and $b_n = \sqrt[n]{n}$, the corresponding sequences are

$$\{a_n\} = \{0,\ 2,\ 6,\ \ldots,\ n(n-1),\ \ldots\}, \quad \{b_n\} = \{1,\ \sqrt{2},\ \sqrt[3]{3},\ \ldots,\ \sqrt[n]{n},\ \ldots\}.$$

In the above examples, $n^2 - n$ and $\sqrt[n]{n}$ are called the general formulas of sequences. A sequence of which the first 5 terms are

$$\frac{-2}{1\cdot 2},\ \frac{4}{2\cdot 3},\ \frac{-8}{3\cdot 4},\ \frac{16}{4\cdot 5},\ \frac{-32}{5\cdot 6},$$

has the general formula $a_n = (-2)^n/[n(n+1)]$.

Example 11.2. The Fibonacci sequence is a sequence $\{a_n\}$ satisfying

$$a_1 = 1, \quad a_2 = 1, \quad a_n = a_{n-1} + a_{n-2} \quad \forall n \geq 3.$$

Definition 11.2. A sequence $\{a_n\}$ has the limit L if, for any $\epsilon > 0$, there is a corresponding integer N such that $|a_n - L| < \epsilon$ for $n \geq N$. In this case, we write $\lim_{n\to\infty} a_n = L$ and say that $\{a_n\}$ converges or $\{a_n\}$ is convergent. Otherwise, we say that $\{a_n\}$ diverges or $\{a_n\}$ is divergent.

Remark 11.1. When a_n gets arbitrarily large as n increases, we write $\lim_{n\to\infty} a_n = \infty$ and refer it to the definition that, for any $M > 0$, there is an integer N such that $a_n > M$ for $n \geq N$.

Theorem 11.1. *Let f be a function satisfying $f(n) = a_n$. Then, for $L \in \mathbb{R} \cup \{\pm\infty\}$,*

$$\lim_{x\to\infty} f(x) = L \quad \Rightarrow \quad \lim_{n\to\infty} a_n = L.$$

Remark 11.2. The converse of this theorem can fail! (e.g. $f(x) = \sin(x\pi)$)

Example 11.3. To find $\lim_{n\to\infty} \ln n/n$, we set $f(x) = \ln x/x$. By L'Hôpital's rule, one has $\lim_{x\to\infty} f(x) = 0$, which implies $\ln n/n \to 0$.

Limit laws Assume that $\{a_n\}$ and $\{b_n\}$ converge with limits a and b. Then, for $\alpha, \beta \in \mathbb{R}$,

$$\lim_{n\to\infty}(\alpha a_n \pm \beta b_n) = \alpha a \pm \beta b, \quad \lim_{n\to\infty} a_n b_n = \alpha\beta \quad \lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\alpha}{\beta}, \quad \text{if } \beta \neq 0.$$

Lemma 11.2. *If $\{a_n\}$ is convergent, then $\lim_{n\to\infty}(a_{n+1} - a_n) = 0$.*

The Squeeze Theorem Assume that there is $N > 0$ such that $a_n \leq b_n \leq c_n$ for $n \geq N$. If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Theorem 11.3. $\lim_{n\to\infty} |a_n| = 0$ *if and only if* $\lim_{n\to\infty} a_n = 0$.

Proof. Clear from the identity $|a_n - 0| = ||a_n| - 0|$. □

Example 11.4. Let b_n be the Fibonacci sequence and $a_n = (-1)^{b_n}/n$. Since $|a_n| = 1/n$ converges to 0, $\lim_{n\to\infty}(-1)^{b_n}/n = 0$.

Theorem 11.4. *If* $\lim_{n\to\infty} a_n = L$ *and* f *is continuous at* L, *then* $\lim_{n\to\infty} f(a_n) = f(L)$.

Example 11.5. Since $\lim_{n\to\infty} \ln n/n = 0$ and e^x is continuous at 0, the limit of $n^{1/n} = \exp\{\ln n/n\}$ is $e^0 = 1$.

Definition 11.3. A sequence $\{a_n\}$ is increasing if $a_n < a_{n+1}$ for $n \geq 1$ and decreasing if $a_n > a_{n+1}$ for $n \geq 1$. In either case, $\{a_n\}$ is called monotonic.

Example 11.6. The sequence $a_n = \frac{n}{n^2+1}$ is decreasing because, for $n \geq 1$,

$$a_n - a_{n+1} = \frac{n}{n^2+1} - \frac{n+1}{(n+1)^2+1} = \frac{n^2+n-1}{(n^2+1)[(n+1)^2+1]} > 0.$$

Definition 11.4. A sequence $\{a_n\}$ is bounded above (below) if there is $M \in \mathbb{R}$ such that

$$a_n \leq M \quad \forall n \geq 1. \quad (a_n \geq M \text{ for all } n \geq 1)$$

If $\{a_n\}$ is bounded above and below, then $\{a_n\}$ is called bounded.

Theorem 11.5 (Monotonic Sequence Theorem)**.** *Bounded and monotonic sequences converge.*

Remark 11.3. Bounded and monotonic sequences are either increasing and bounded above or decreasing and bounded below.

Example 11.7. Consider the sequence $\{a_n\}$, where

$$a_1 = 2, \quad a_{n+1} = 2 + \sqrt{a_n} \quad \forall n \geq 1.$$

Note that if $a_n > 0$, then $a_{n+1} = 2 + \sqrt{a_n} > 0$. As $a_1 = 2 > 0$, one has $a_n > 0$ for all $n \geq 1$. Besides, if $a_n > a_{n-1}$, then

$$a_{n+1} - a_n = 2 + \sqrt{a_n} - (2 + \sqrt{a_{n-1}}) = \sqrt{a_n} - \sqrt{a_{n-1}} > 0.$$

If $a_n < 4$, then

$$a_{n+1} = 2 + \sqrt{a_n} < 2 + \sqrt{4} = 4.$$

Since $a_2 = 2 + \sqrt{2} > 2 = a_1$ and $a_n < 4$, the mathematical induction implies that $\{a_n\}$ is increasing and bounded above. By the Monotonic Sequence Theorem, a_n converges. Assume that the limit is L. Since $\sqrt{\cdot}$ is a continuous function, one has

$$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} (2 + \sqrt{a_n}) = 2 + \sqrt{L}.$$

This implies $L = 4$.