11.9. **Taylor and Maclaurin series.**

Theorem 11.21. *If a function has the following power series representation*

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \quad \forall |x-a| < R, \quad R > 0,$$

then f is infinitely differentiable (i.e. $f^{(m)}$ exists for all $m \geq 0$) and $c_m = f^{(m)}(a)/m!$

Proof. The proof follows immediately from the fact that $f^{(m)}(x) = \sum_{n=m}^{\infty} c_n n(n-1)\cdots(n-m+1)(x-a)^{n-m}$ for $|x-a| < R$ and $m \geq 0$. □

Remark 11.21. If f has a power series representation at a, then $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$.

Definition 11.10. Let f be a function which is infinitely differentiable at a. The Taylor series of f at a is the following series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots.$$

A Taylor series of a function at 0 is also called a Maclaurin series.

Example 11.28. Let $f(x) = e^x$ and $a \in \mathbb{R}$. Observe that $f^{(n)}(a) = e^a$. This implies that the Taylor series of e^x at a and the Maclaurin series of e^x are

$$\sum_{n=0}^{\infty} \frac{e^a}{n!}(x-a)^n, \quad \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

By the ratio test, both series have radii of convergence ∞.

Definition 11.11. For $n \geq 0$, the nth degree Taylor polynomial of f at a is defined to be

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

and the remainder of the Taylor series is $R_n(x) = f(x) - T_n(x)$.

Remark 11.22. Note that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \quad \Leftrightarrow \quad \lim_{n\to\infty} T_n(x) = f(x) \quad \Leftrightarrow \quad \lim_{n\to\infty} R_n(x) = 0.$$

Theorem 11.22 (Taylor's inequality)**.** *If $|f^{(n+1)}(x)| \leq M$ for $|x-a| \leq d$, then*

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}, \quad \forall |x-a| \leq d.$$

Proof. Let T_n be the nth degree Taylor polynomial of f at a and $R_n(x) = f(x) - T_n(x)$. By the fundamental theorem of calculus, one has

$$R_n(x) = \int_a^x [f'(t) - T_n'(t)]dt \tag{11.4}$$

We prove this theorem by induction. When $n = 0$, $T_0(x) = f(a)$ and (11.4) implies

$$|R_0(x)| \leq \left| \int_a^x |f'(t)|dt \right| \leq M|x-a|, \quad \forall |x-a| \leq d.$$

Assume that Taylor's inequality holds for any function with $n \le m$. For $n = m+1$, suppose $|f^{(m+2)}(x)| \le M$ for $|x-a| \le d$. Set $g = f'$ and let S_n be the nth degree Taylor polynomial of g at a. Clearly, $|g^{(m+1)}(x)| \le M$ for all $|x-a| < d$ and

$$S_n(x) = \sum_{i=0}^{n} \frac{g^{(i)}(a)}{i!}(x-a)^i = \sum_{i=1}^{n+1} \frac{f^{(i)}(a)}{(i-1)!}(x-a)^{i-1} = T'_{n+1}(x).$$

By induction, we obtain

$$|f'(x) - T'_{m+1}(x)| = |g(x) - S_m(x)| \le \frac{M}{(m+1)!}|x-a|^{m+1}, \quad \forall |x-a| \le d.$$

As a consequence of (11.4), this implies

$$|R_{m+1}(x)| \le \left| \int_a^x |f'(t) - T'_{m+1}(t)| dt \right| \le \left| \int_a^x \frac{M}{(m+1)!}|t-a|^{m+1} dt \right| \le \frac{M}{(m+2)!}|x-a|^{m+2}.$$

□

Example 11.29. Let $f(x) = e^x$, $a \in \mathbb{R}$ and $d > 0$. Note that $|f^{(n+1)}(x)| \le e^{a+d}$ for $|x-a| \le d$. This implies

$$|R_n(x)| \le \frac{e^{a+d}}{(n+1)!}|x-a|^{n+1} \le \frac{e^{a+d}d^{n+1}}{(n+1)!}, \quad \forall |x-a| \le d.$$

By the ratio test, $\sum_{n=1}^{\infty} d^{n+1}/(n+1)!$ is convergent and, hence, $d^{n+1}/(n+1)! \to 0$. As a consequence, $|R_n(x)| \to 0$ for all $|x-a| \le d$, which is equivalent to $e^x = \sum_{n=0}^{\infty} \frac{e^a}{n!}(x-a)^n$ for $|x-a| \le d$. Since d is arbitrary, we obtain $e^x = \sum_{n=0}^{\infty} \frac{e^a}{n!}(x-a)^n$ for $x \in \mathbb{R}$. Particularly, one has

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

Example 11.30. Let $f(x) = \sin x$. Note that

$$f^{(4m)}(x) = \sin x, \quad f^{(4m+1)}(x) = \cos x, \quad f^{(4m+2)}(x) = -\sin x, \quad f^{(4m+3)}(x) = -\cos x.$$

Clearly, the Maclaurin series of $\sin x$ is

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots.$$

By Taylor's inequality, as $|f^{(n)}(x)| \le 1$ for all x, one has $|R_n(x)| \le \frac{|x|^{n+1}}{(n+1)!}$. As a consequence of the Squeeze theorem, this implies $R_n(x) \to 0$ for all $x \in \mathbb{R}$. Hence, we obtain

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots, \quad \forall x \in \mathbb{R},$$

and, for $x \in \mathbb{R}$,

$$\cos x = \frac{d \sin x}{dx} = \sum_{n=0}^{\infty} \frac{d}{dx}(-1)^n \frac{x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots.$$

Example 11.31. Let $f(x) = (1+x)^k$, where k is any real number. Note that $f^{(n)}(x) = k(k-1)(k-2) \times \cdots \times (k-n+1)(1+x)^{k-n}$ and the Maclaurin series of f is $g(x) = \sum_{n=0}^{\infty} \binom{k}{n} x^n$, where

$$\binom{k}{0} := 1, \quad \binom{k}{n} = \frac{k(k-1) \times \cdots \times (k-n+1)}{n!}, \quad \forall n \ge 1.$$

g is named a binomial series. To see the radius of convergence of g, note that

$$\left|\frac{\binom{k}{n+1}x^{n+1}}{\binom{k}{n}x^n}\right| = \frac{|k-n|}{n+1}|x| \to |x| \quad \text{as } n \to \infty.$$

By the ratio test, g has radius of convergence 1. To see that $f = g$ on $(-1, 1)$, we set $h(x) = (1+x)^{-k}g(x)$. Consider the following computation with $|x| < 1$.

$$g'(x) = \sum_{n=1}^{\infty} n\binom{k}{n}x^{n-1} = k\sum_{n=1}^{\infty}\binom{k-1}{n-1}x^{n-1} = k\sum_{n=0}^{\infty}\binom{k-1}{n}x^n.$$

Immediately, this implies

$$(1+x)g'(x) = k\sum_{n=0}^{\infty}\binom{k-1}{n}x^n + k\sum_{n=1}^{\infty}\binom{k-1}{n-1}x^n = k\sum_{n=0}^{\infty}\binom{k}{n}x^n = kg(x)$$

where the last inequality uses the fact

$$\begin{aligned}\binom{k}{n} &= \frac{k(k-1)\cdots(k-n+1)}{n!} \\ &= \frac{(k-1)(k-2)\cdots(k-n+1)}{(n-1)!}\left(1+\frac{k-n}{n}\right) = \binom{k-1}{n-1} + \binom{k-1}{n}.\end{aligned}$$

As a result, the above computation leads to $h'(x) = (-k)(1+x)^{-k-1}g(x) + (1+x)^{-k}g'(x) = 0$ for $|x| < 1$. As $h(0) = g(0) = 1$, we obtain $g(x) = (1+x)^k$ for $|x| < 1$.

Example 11.32. To evaluate $\int e^{-x^2}dx$, we recall the formula $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$ for $x \in \mathbb{R}$. This implies

$$\int e^{-x^2}dx = \int\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{n!}dx = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int x^{2n}dx = C + \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)n!}x^{2n+1}.$$

Example 11.33. Let $f(x) = \frac{e^x-1-x}{x^2}$ for $x \neq 0$ and $g(x) = \sum_{n=0}^{\infty}\frac{x^n}{(n+2)!} = \frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{4!} + \cdots$. Note that $f(x) = g(x)$ for $x \neq 0$ and g has radius of convergence ∞. Since g is differentiable everywhere, it is continuous on $\mathbb{R}$. This implies

$$\lim_{x\to 0}\frac{e^x-1-x}{x^2} = g(0) = \frac{1}{2}.$$

Example 11.34. Consider the function $f(x) = e^{-1/x^2}$ for $x \neq 0$ and $f(0) = 0$. Clearly, f is infinitely differentiable on $\mathbb{R}\setminus\{0\}$. By induction, one can show that $f^{(n)}(x) = f(x)P_n(x^{-1})$ for $x \neq 0$, where P_n is a polynomial of degree at most $3n$. Inductively, one may use this observation to derive $f^{(n)}(0) = 0$ for all $n \geq 1$ and, hence, the Maclaurin series of f equals 0. It's worthwhile to remark that $f(x)$ equals 0 only at 0.