11.2. **Series.**

Definition 11.5. A series is a sum of an infinite sequence $\{a_n\}$ and is written by $\sum_{n=1}^{\infty} a_n$ or $a_1 + a_2 + a_3 + \cdots + a_n + \cdots$.

Remark 11.4. Note that the symbol $\sum_{n=1}^{\infty} a_n$ makes no sense unless a precise definition is set. Consider the following two series.

$$1 + (-1) + 1 + (-1) \cdots, \quad 2^{-1} + 2^{-2} + 2^{-3} + \cdots$$

Intuitively, the first series fluctuates between 0 and 1, while the second series sums up to 1.

Definition 11.6. Given a series $\sum_{n=1}^{\infty} a_n$, let $s_n = a_1 + \cdots + a_n$ be the nth partial sum. Then, the series $\sum_{n=1}^{\infty} a_n$ is convergent if the sequence $\{s_n\}$ is convergent. In this case, we define

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i,$$

and call the limit of $\{s_n\}$ the sum of the series. Otherwise, the series is divergent if $\{s_n\}$ is divergent.

Example 11.8. Consider the following three series.

$$\text{(a)} \sum_{n=1}^{\infty} \frac{n}{(n+1)!}, \quad \text{(b)} \sum_{n=0}^{\infty} ar^n \quad \text{with } a \neq 0, \quad \text{(c)} \sum_{n=1}^{\infty} 1/n.$$

Remark: (b) is called the geometric series and (c) is called the harmonic series.

Let s_n be the nth partial sum. For (a), note that

$$\frac{n}{(n+1)!} = \frac{n+1}{(n+1)!} - \frac{1}{(n+1)!} = \frac{1}{n!} - \frac{1}{(n+1)!} \quad \forall n \geq 1.$$

This implies $s_n = 1 - 1/(n+1)!$. By the squeeze theorem and the inequalities $0 < 1/n! < 1/n$, we may conclude that $\sum_{n=1}^{\infty} \frac{n}{(n+1)!} = 1$. For (b), observe that

$$s_n = \sum_{i=0}^{n-1} ar^i = a + ar + ar^2 + \cdots + ar^{n-1} = \begin{cases} \frac{a(1-r^n)}{1-r} & \text{if } r \neq 1 \\ na & \text{if } r = 1 \end{cases}$$

When $r = 1$ or $r = -1$, it is clear that the series is divergent. For $|r| < 1$, the series is convergent with limit $a/(1-r)$. For $|r| > 1$, the series diverges. Next, we consider (c). For $k \geq 1$, we write

$$s_{2^k} = 1 + \frac{1}{2} + \sum_{\ell=1}^{k-1} \sum_{i=2^\ell+1}^{2^{\ell+1}} \frac{1}{i}.$$

Note that $1/i \geq 1/2^{-\ell-1}$ for $i \leq 2^{\ell+1}$. This yields

$$\sum_{i=2^\ell+1}^{2^{\ell+1}} \frac{1}{i} \geq (2^{\ell+1} - 2^\ell)2^{-\ell-1} = \frac{1}{2} \quad \Rightarrow \quad s_{2^k} \geq 1 + \frac{1}{2} + \sum_{\ell=1}^{k-1} \frac{1}{2} = 1 + \frac{k}{2}.$$

Since S_n is increasing and not bounded above, we may conclude that $s_n \to \infty$.

Theorem 11.6. *If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$. Conversely, if the sequence $\{a_n\}$ diverges or converges with nonzero limit, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.*

Proof. Let s_n be the nth partial sum and set $L = \lim_{n\to\infty} s_n$. By the limit law, one has

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = L - L = 0.$$

□

Remark 11.5. The converse of Theorem 11.6 can be wrong! E.g. the harmonic series.

Limit laws for series If $\sum_{n=1}^{\infty} a_n = a$ and $\sum_{n=1}^{\infty} b_n = b$, then $\sum_{n=1}^{\infty}(\alpha a_n + \beta b_n)$ is convergent for $\alpha, \beta \in \mathbb{R}$ and $\sum_{n=1}^{\infty}(\alpha a_n + \beta b_n) = \alpha a + \beta b$.

Example 11.9. To evaluate the series $\sum_n [3/(n(n+1)) + 2^{-n}]$, one may compute in advance

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = 1 - \frac{1}{n+1} \to 1, \quad \sum_{i=1}^{n} 2^{-i} = \frac{(1/2)(1-2^{-n})}{1-1/2} \to 1.$$

By the limit law, $\sum_n [3/(n(n+1)) + 2^{-n}] = 3 \cdot 1 + 1 = 4$.

Lemma 11.7. *Suppose $\sum_{n=1}^{\infty} a_n$ is convergent. Then, $\sum_{n=m}^{\infty} a_n$ is convergent for any m. Further, if $t_m = \sum_{n=m}^{\infty} a_n$, then $t_m \to 0$.*

Proof. First, fix $m \in \mathbb{N}$. Set $s_n = \sum_{i=1}^{n} a_i$ and $s_n^{(m)} = \sum_{i=m}^{n+m-1} a_i$. Clearly, $s_n^{(m)} = s_{n+m-1} - s_{m-1}$. By the limit law, as $\{s_n\}$ converges, $\{s_n^{(m)}\}$ converges. To see the limit of t_m, let $s = \sum_{n=1}^{\infty} a_n$. Again, the limit law yields $t_m = s - s_{m-1}$, which implies $t_m \to s - s = 0$. □

Example 11.10. Which equations are false in the following computation?

$$\begin{aligned} 0 &= 0 + 0 + 0 + \cdots \\ &= (1-1) + (1-1) + (1-1) + \cdots \\ &= 1 - 1 + 1 - 1 + 1 - 1 + \cdots \\ &= 1 + (-1+1) + (-1+1) + (-1+1) + \cdots \\ &= 1 + 0 + 0 + \cdots = 1 \end{aligned}$$