11.3. **The integral test and estimates of sums.** In this section, we consider the convergency of sums of positive and decreasing sequences.

Theorem 11.8 (The integral test)**.** *Suppose $a_n = f(n)$, where f is a continuous, positive and decreasing function on $[1,\infty)$. Then, the convergency of $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)dx$ are consistent. That is:*

(1) *If $\int_1^{\infty} f(x)dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.*
(2) *If $\int_1^{\infty} f(x)dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.*

Proof. By the monotonicity of f, one can see that

$$\int_1^{n+1} f(x)dx \le s_n = \sum_{i=1}^{n} a_i \le a_1 + \int_1^{n} f(x)dx. \tag{11.1}$$

If $\int_1^{\infty} f(x)dx = L \in \mathbb{R}$, then the second inequality of (11.1) implies that $s_n \le a_1 + L$ for $n \ge 1$. As s_n is increasing and bounded above, $\sum_{n=1}^{\infty} a_n$ converges. If $\int_1^{\infty} f(x)dx = \infty$, then the first inequality of (11.1) implies $s_n \to \infty$. ☐

Example 11.11. Consider the p-series, $\sum_{n=1}^{\infty} n^p$. When $p > 0$, $n^p \to \infty$; when $p = 0$, $n^p \to 1$. This implies that the p-series is divergent. When $p = -1$, the p-series is the harmonic series and, thus, diverges. For $p < 0$ and $p \ne -1$, set $f(x) = x^p$. Clearly, f is continuous, positive and decreasing. Note that

$$\int_1^t x^p dx = \frac{x^{p+1}}{p+1}\bigg|_1^t = \frac{t^{p+1}-1}{p+1}.$$

Letting $t \to \infty$ implies that $\int_1^{\infty} f(x)dx$ is convergent if $p < -1$ and divergent if $-1 < p < 0$. By the integral test, we may conclude that $\sum_{n=1}^{\infty} n^p$ is convergent for $p < -1$ and divergent for $p \ge -1$.

Remark 11.6. The integral test is also valid for continuous and positive functions which are decreasing on $[r,\infty)$, where r is some constant, due to Lemma 11.7, i.e.

$$\sum_{n=1}^{\infty} a_n \text{ converges} \quad \Leftrightarrow \quad \sum_{n=r}^{\infty} a_n \text{ converges.}$$

Example 11.12. Consider the series $\sum_n \ln n/n$. Let $f(x) = \ln x/x$ for $x \ge 1$. Note that $f'(x) = x^{-2}(1-\ln x)$. Clearly, $f'(x) < 0$ if and only if $x > e$. This implies that f is decreasing on $[e,\infty)$. Since

$$\int_1^{\infty} f(x)dx = \lim_{t\to\infty} \frac{(\ln x)^2}{2}\bigg|_1^t = \infty,$$

the series $\sum_{n=1}^{\infty} \ln n/n$ is divergent.

Remainder estimate Let f be continuous, positive and decreasing on $[N,\infty)$. Suppose $a_n = f(n)$ and $\sum_{n=1}^{\infty} a_n$ converges with sum s. Then,

$$\int_{N+1}^{\infty} f(x)dx \le s - s_N \le \int_N^{\infty} f(x)dx.$$