11.4. **The comparison tests.** In this section, we determine the convergency of sums of positive sequences through comparison.

Theorem 11.9 (The comparison test)**.** *Let $a_n > 0$ and $b_n > 0$ for all $n \geq 1$. Assume that $a_n \leq b_n$ for all n.*

(1) *If $\sum_{n=1}^{\infty} b_n$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.*
(2) *If $\sum_{n=1}^{\infty} a_n$ is divergent, then $\sum_{n=1}^{\infty} b_n$ is divergent.*

Proof. As (2) is an equivalent statement of (1), we prove (1) here. Set $s_n = \sum_{i=1}^{n} a_i$ and $t_n = \sum_{i=1}^{n} b_i$. Clearly, $\{s_n\}$ and $\{t_n\}$ are increasing sequences satisfying $s_n \leq t_n$ for all n. If $t_n \to t$, then $s_n \leq t$ for all n, which implies $\sum_{n=1}^{\infty} a_n$ is convergent. □

Example 11.13. Consider two sequences, $a_n = \frac{1}{2^n+3n+1}$ and $b_n = \frac{1}{3n+1}$. Note that $a_n \leq 2^{-n}$ and $b_n \geq \frac{1}{4n}$. By the comparison test, since $\sum_{n=1}^{\infty} 2^{-n}$ is convergent and the harmonic series is divergent, $\sum_{n=1}^{\infty} a_n$ is convergent and $\sum_{n=1}^{\infty} b_n$ is divergent.

Theorem 11.10 (The limit comparison test)**.** *Assume that $a_n > 0$ and $b_n > 0$ and*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0.$$

Then, $\sum_{n=1}^{\infty} a_n$ is convergent if and only if $\sum_{n=1}^{\infty} b_n$ is convergent.

Proof. Let m, M be positive constants such that $m < c < M$. Since a_n/b_n converges to c, we may choose $N > 0$ such that

$$m < \frac{a_n}{b_n} < M \quad \forall n \geq N. \tag{11.2}$$

This implies

$$b_n < \frac{1}{m} a_n, \quad a_n < M b_n \quad \forall n \geq N. \tag{11.3}$$

By the comparison test, $\sum_{n=N}^{\infty} a_n$ converges if and only if $\sum_{n=N}^{\infty} b_n$ converges. The desired equivalence is then given by Lemma 11.7. □

Remark 11.7. In the above theorem, when $c = 0$, the conclusion one can make from the second inequality of (11.2) is

$$\sum_{n=1}^{\infty} b_n \text{ converges} \quad \Rightarrow \quad \sum_{n=1}^{\infty} a_n \text{ converges.}$$

The converse can fail. See e.g. $a_n = 1/n^2$ and $b_n = 1/n$.

Example 11.14. Consider the series $\sum_{n=1}^{\infty} 1/(2n^2 + 3n + (-1)^n)$. Note that

$$\lim_{n\to\infty} \frac{1/(2n^2 + 3n + (-1)^n)}{1/n^2} = 1/2.$$

As $\sum_{n=1}^{\infty} 1/n^2$ converges, $\sum_{n=1}^{\infty} 1/(2n^2 + 3n + (-1)^n)$ converges.

Example 11.15. Consider the series $\sum_{n=1}^{\infty} 1/(n^3+1)$. As $n^3+1 > n^3$ and $\sum_{n=1}^{\infty} 1/n^3$ converges, $\sum_{n=1}^{\infty} 1/(n^3+1)$ converges. Let s_n and s be the nth partial sum and the sum. Then, for $n \geq 1$,

$$0 < s - s_n \leq \int_n^{\infty} \frac{1}{x^3+1} dx \leq \int_n^{\infty} \frac{1}{x^3} dx = \frac{1}{2n^2}.$$

When $n = 100$, $0 < s - s_{100} \leq 5 \cdot 10^{-5}$.