11.5. **Alternating series.**

Definition 11.7. An alternating series is a series of which terms are alternately positive and negative.

Example 11.16. The following series are alternating series.

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n},\qquad -\frac{3}{2}+\frac{4}{3}-\frac{5}{4}+\frac{6}{5}-\cdots=\sum_{n=1}^{\infty}\frac{(-1)^n(n+2)}{n+1}.$$

Theorem 11.11 (The alternating series test)**.** *Assume that $b_{n+1}\le b_n$ for $n\ge 1$ and $b_n\to 0$. Then, the series $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$ converges.*

Proof. Let s_n be the nth partial sum of the series. Then, $s_{2n}=\sum_{i=1}^{n}(b_{2i-1}-b_{2i})\ge 0$ and $s_{2n}\le s_{2n+2}$ for all $n\ge 1$. Note that $s_{2n}=b_1+\sum_{i=1}^{n-1}(-b_{2i}+b_{2i+1})-b_{2n}\le b_1$. By the monotonic sequence theorem, $\{s_{2n}\}$ converges. Let s be the limit of $\{s_{2n}\}$. Since $\lim_{n\to\infty}b_n=0$, one has

$$\lim_{n\to\infty}s_{2n+1}=\lim_{n\to\infty}(s_{2n}+b_{2n+1})=\lim_{n\to\infty}s_{2n}+\lim_{n\to\infty}b_{2n+1}=s.$$

This proves that $\{s_n\}$ converges to s. □

Example 11.17. Consider the following series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^3+1},\qquad \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}.$$

Note that $n^2/(n^3+1)\to 0$, $1/n!\to 0$ and

$$\frac{n^2}{n^3+1}-\frac{(n+1)^2}{(n+1)^3+1}=\frac{n^4+(2n+1)(n^2-1)}{(n^3+1)[(n+1)^3+1]}\ge 0,\qquad \frac{1}{n!}>\frac{1}{(n+1)!}.$$

By the alternating series test, both series are convergent.

Remark 11.8. Note that the assumption $b_{n+1}\le b_n$ in the alternating series test cannot be removed. Consider the series $\sum_{n=1}^{\infty}a_n$, where $a_{2k-1}=1/k$ and $a_{2k}=\ln k/k$. Let $s_n=\sum_{i=1}^{n}(-1)^{i+1}a_i$. Then, $s_{2k}=\sum_{i=1}^{k}1/i-\sum_{i=1}^{k}\ln i/i$. Through a comparison, one has

$$\int_1^{k+1}\frac{dx}{x}\le\sum_{i=1}^{k}\frac{1}{i}\le 1+\int_1^k\frac{dx}{x},\qquad \frac{\ln 2}{2}+\int_3^{k+1}\frac{\ln x}{x}dx\le\sum_{i=1}^{k}\frac{\ln i}{i}\le\frac{\ln 2}{2}+\frac{\ln 3}{3}+\int_3^k\frac{\ln x}{x}dx.$$

By the formulas, $\int\frac{1}{x}dx=\ln x+C$ and $\int\frac{\ln x}{x}dx=\frac{1}{2}(\ln x)^2+C$, we obtain

$$s_{2k}\le 1-\frac{\ln 2}{2}+\ln k-\frac{1}{2}[(\ln k)^2-(\ln 3)^2]\to-\infty,\quad \text{as } k\to\infty.$$

Remark 11.9. Note that the assumption of $b_{n+1}\le b_n$ is not necessary for the convergence of an alternating series. See e.g. the example of $\sum_{n=1}^{\infty}(-1)^{n+1}(a_n+b_n)$ with $a_1=1$, $a_{2n+1}=a_{2n}=1/(n+1)$, $b_{2n}=0$ and $b_{2n-1}=2^{-n}$.

Theorem 11.12 (Alternating series estimation theorem)**.** *Assume that $b_n\ge b_{n+1}$ and $b_n\to 0$. Set $s_n=\sum_{i=1}^{n}(-1)^{i+1}b_i$ and $s=\sum_{i=1}^{\infty}(-1)^{i+1}b_i$. Then, $|s-s_n|\le b_{n+1}$.*

Proof. The proof follows immediately from the fact $s_{2n}\le s\le s_{2n+1}=s_{2n}+b_{n+1}$ for $n\ge 1$. □