11.6. **Absolute convergence and the ratio and root tests.**

Definition 11.8. A series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ is convergent. A series $\sum_{n=1}^{\infty} a_n$ is conditionally convergent if it is convergent but not absolutely convergent.

Remark 11.10. The harmonic series is divergent. The alternating harmonic series is conditionally convergent. The geometric series $\sum_{n=1}^{\infty} r^n$ with $|r| < 1$ is absolutely convergent.

Theorem 11.13. *If a series is absolutely convergent, then it is convergent.*

Proof. The proof follows immediately from the fact of $0 \le a_n + |a_n| \le 2|a_n|$, while the details are omitted. □

Remark 11.11. If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_{\sigma(n)} = \sum_{n=1}^{\infty} a_n$ for any permutation σ of $\mathbb{N}$.

Remark 11.12. If $\sum_{n=1}^{\infty} a_n$ converges conditionally, then, for any $x \in \mathbb{R}$, there is a permutation σ of $\mathbb{N}$ such that $\sum_{n=1}^{\infty} a_{\sigma(n)} = x$.

Example 11.18. Consider the series $\sum_{n=1}^{\infty} \sin n/n^2$. Note that $|\sin n/n^2| \le 1/n^2$. By the comparison test, as $\sum_{n=1}^{\infty} n^{-2}$ is convergent, $\sum_{n=1}^{\infty} \sin n/n^2$ is absolutely convergent and, thus, convergent.

Theorem 11.14 (The ratio test)**.** *Assume that $\{|a_{n+1}/a_n|\}$ converges with limit L.*

(1) *When $L < 1$, $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.*
(2) *When $L > 1$, $\sum_{n=1}^{\infty} a_n$ is divergent.*
(3) *When $L = 1$, no conclusion is available.*

Proof. Since $|a_{n+1}/a_n| \to L$, we may choose $N > 0$ such that $||a_{n+1}/a_n| - L| < |L-1|/2$ for $n \ge N$. By the triangle inequality, one has

$$s := L - \frac{|L-1|}{2} \le \left|\frac{a_{n+1}}{a_n}\right| \le t := L + \frac{|L-1|}{2}, \quad \forall n \ge N.$$

When $L < 1$, $t = (L+1)/2 < 1$. This implies $|a_{n+N}| \le |a_N| t^n$ for $n \ge 0$ and

$$\sum_{n=1}^{\infty} |a_n| \le \sum_{n=1}^{N-1} |a_n| + |a_N| \sum_{n=0}^{\infty} t^n = \sum_{n=1}^{N-1} |a_n| + \frac{|a_N|}{1-t} < \infty.$$

When $L > 1$, $s = (L+1)/2 > 1$. This implies $|a_{n+N}| \ge |a_N| s^n$ for $n \ge 0$ and, thus, $a_n \nrightarrow 0$. For the case of $L = 1$, two typical examples are $a_n = 1/n^2$ and $b_n = 1/n$. □

Example 11.19. Consider the sequences, $a_n = n^3/3^n$ and $b_n = n^n/n!$. Note that $|a_{n+1}/a_n| \to 1/3$ and $|b_{n+1}/b_n| \to e$. By the ratio test, $\sum_{n=1}^{\infty} a_n$ is absolutely convergent but $\sum_{n=1}^{\infty} b_n$ is divergent.

Theorem 11.15 (The root test)**.** *Assume that $\{|a_n|^{1/n}\}$ converges with limit L. Then, the conclusion is the same as the ratio test.*

The proof is exactly the same as that of the ratio test and we omit it here.

Example 11.20. For $a_n = \left(\frac{2n+3}{3n+2}\right)^n$, since $|a_n|^{1/n} \to 2/3$, $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

Remark 11.13. It is remarkable that $|a_{n+1}/a_n| \to L$ implies $|a_n|^{1/n} \to L$. Thus, the ratio test and the root test would fail simultaneously.