11.7. **Power series.**

Definition 11.9. A power series is a series of the following form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots,$$

where x is a variable and a, c_n are constants. More precisely, we call it a power series centered at a and call c_n's the coefficients.

Remark 11.14. The domain of a power series is the set of all x such that the power series is convergent. For instance, the geometric series, $1 + x + x^2 + x^3 + \cdots$, has domain $(-1, 1)$. Note that any power series centered at a must be convergent at $x = a$.

Example 11.21. Consider the following power series.

$$\sum_{n=0}^{\infty} n!x^n, \quad \sum_{n=0}^{\infty} \frac{(x-3)^n}{n+1}, \quad \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}.$$

To see their domains, we set

$$a_n = n!x^n, \quad b_n = \frac{(x-3)^n}{n+1}, \quad c_n = \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}.$$

For the first series, if $x \neq 0$, then

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} n|x| = \infty.$$

By the ratio test, $\sum_{n=0}^{\infty} a_n$ is divergent for $x \neq 0$.

For the second series, note that

$$\lim_{n\to\infty} |b_n|^{1/n} = \lim_{n\to\infty} \frac{|x-3|}{(n+1)^{1/n}} = |x-3|.$$

By the root test, the series is absolutely convergent if $|x-3| < 1$ but divergent if $|x-3| > 1$. Note that $|x-3| = 1$ implies $x = 2$ or $x = 4$. When $x = 4$, $b_n = 1/(n+1)$ and $\sum_{n=0}^{\infty} b_n$ is the harmonic series, which is divergent. When $x = 2$, $b_n = (-1)^n/(n+1)$ and $\sum_{n=0}^{\infty} b_n$ is the alternating harmonic series, which is convergent. Thus, the domain of the second series is $[2, 4)$.

The third series is introduced by the German astronomer Friedrich Bessel and widely used in mathematics, physics and chemistry. It is named after Friedrich Bessel as a Bessel function. Observe that

$$\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = \lim_{n\to\infty} \frac{x^2}{2^2(n+1)^2} = 0, \quad \forall x \in \mathbb{R}.$$

This implies that the Bessel function has domain $\mathbb{R}$.

Lemma 11.16. *Consider the power series* $\sum_{n=0}^{\infty} c_n(x-a)^n$.

(1) *If the power series converges for some* $x_0 \neq a$, *then it converges absolutely for all* $|x-a| < |x_0 - a|$.
(2) *If the power series converges conditionally or diverges for some* $y_0 \neq a$, *then it diverges for all* $|x-a| > |y_0 - a|$.

Proof. We prove (1) in the following, while (2) is an immediate result of (1). Let $r_0 = |x_0 - a|$. Fix $x \in (a - r_0, a + r_0)$, set $b_n = c_n(x_0 - a)^n$, $r = (x-a)/(x_0-a)$ and write $c_n(x-a)^n = b_n r^n$. Note that

$$|x-a| < r_0 = |x_0 - a| \quad \Rightarrow \quad |r| < 1.$$

Since $\sum_{n=0}^{\infty} b_n$ converges, $b_n \to 0$ and, hence, we may choose $M > 0$ such that $|b_n| < M$ for all n. As a consequence, this implies

$$\sum_{n=0}^{\infty} |c_n(x-a)^n| = \sum_{n=0}^{\infty} |b_n| \cdot |r|^n \leq \frac{M}{1-|r|} < \infty.$$

□

Theorem 11.17. *For any power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, exactly one of the following conditions holds.*

(1) *The series converges only at a.*
(2) *The series converges for all $x \in \mathbb{R}$.*
(3) *There is $R > 0$ such that the series converges for $|x-a| < R$ and diverges for $|x-a| > R$.*

Remark 11.15. By Lemma 11.16, the convergence in Theorem 11.17(2)-(3) are in fact the absolute convergence.

Remark 11.16. In Theorem 11.17(3), R is called the radius of convergence of the power series. Generally, we say that the radius of convergence of the power series in (1) and (2) are respectively 0 and ∞.

Remark 11.17. In Theorem 11.17(3), the domain of the power series must be one of the following four cases,

$$(a-R, a+R), \quad [a-R, a+R), \quad (a-R, a+R], \quad [a-R, a+R],$$

and is called the interval of convergence of the power series.

Corollary 11.18. *For the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, assume that at least one of the following limit exists,*

$$L := \lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|}, \quad L := \lim_{n\to\infty} |c_n|^{1/n}.$$

Then, the radius of convergence of this power series is $1/L$, where $1/0 := \infty$ and $1/\infty := 0$.

Example 11.22. Let $k \in \mathbb{N}$. Consider the power series $\sum_{n=0}^{\infty} \frac{(n!)^k}{(kn)!} x^n$. By the ratio test, we have

$$\begin{aligned}\frac{[(n+1)!]^k |x|^{n+1}/[k(n+1)]!}{(n!)^k |x|^n/(kn)!} &= \frac{(n+1)^k |x|}{(kn+1)(kn+2)\cdots(kn+k)} \\ &= \frac{(1+1/n)^k |x|}{k^k(1+1/kn)(1+2/kn)\cdots(1+k/kn)} \to \frac{|x|}{k^k}\end{aligned}$$

as $n \to \infty$. This implies that the radius of the convergence of this power series is k^k.

Remark 11.18. Is it possible that a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ is convergent if and only if $x \in [0, \infty)$?