11.8. **Representations of functions as power series.** In this section, we will learn how to represent certain type of functions as sums of power series.

Example 11.23. Note that the geometric series $\sum_{n=0}^{\infty} x^n$ has interval of convergence $(-1,1)$ and is equal to $1/(1-x)$ for $|x|<1$. An immediate application of this observation yields that, for nonzero constants a,b,

$$\frac{1}{ax+b}=\frac{1}{b}\sum_{n=0}^{\infty}\left(\frac{-ax}{b}\right)^n=\sum_{n=0}^{\infty}\frac{(-1)^n a^n}{b^{n+1}}x^n,\quad \forall |x|<\left|\frac{b}{a}\right|.$$

Theorem 11.19. *Assume that the power series* $\sum_{n=0}^{\infty} c_n(x-a)^n$ *has radius of convergence* $R>0$. *Set* $f(x)=\sum_{n=0}^{\infty} c_n(x-a)^n$. *Then,* f *is differentiable on* $(a-R,a+R)$ *and*

$$f'(x)=\sum_{n=1}^{\infty} nc_n(x-a)^{n-1}=c_1+2c_2(x-a)+3c_3(x-a)^2+\cdots$$

and

$$\int f(x)dx=C+\sum_{n=0}^{\infty}\frac{c_n}{n+1}(x-a)^{n+1}=C+c_0(x-a)+\frac{c_1}{2}(x-a)^2+\frac{c_2}{3}(x-a)^3+\cdots.$$

Moreover, the radii of convergence of these two power series equal R.

Remark 11.19. In a word, the above theorem says

$$\frac{d}{dx}\sum_{n=0}^{\infty}c_n(x-a)^n=\sum_{n=0}^{\infty}\frac{d}{dx}[c_n(x-a)^n],\qquad \int\left[\sum_{n=0}^{\infty}c_n(x-a)^n\right]dx=\sum_{n=0}^{\infty}\int[c_n(x-a)^n]dx,$$

and either of them preserves the radius of convergence.

Example 11.24. To express $1/(1-x)^2$ as a power series, note that $1/(1-x)^2=f'(x)$, where $f(x)=1/(1-x)=\sum_{n=0}^{\infty}x^n$ for $|x|<1$. By Theorem 11.19, this implies

$$\frac{1}{(1-x)^2}=\sum_{n=1}^{\infty}nx^{n-1}=\sum_{n=0}^{\infty}(n+1)x^n,\quad \forall |x|<1.$$

Example 11.25. By Theorem 11.19, as $\ln(1-x)=-\int(1-x)^{-1}dx$, one has

$$\ln(1-x)=-\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}+C,\quad \forall |x|<1.$$

Letting $x=0$ in the above identity yields $C=0$. Consequently, this leads to

$$\ln(1-x)=-\sum_{n=1}^{\infty}\frac{x^n}{n}\quad \forall |x|<1,\quad \ln(1+y)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{y^n}{n}\quad \forall |y|<1.$$

To see the identity for $y=1$, set $s=\sum_{n=1}^{\infty}(-1)^{n+1}/n$ and $s_m(y)=\sum_{n=1}^{m}(-1)^{n+1}y^n/n$. Note that

$$|\ln 2-s|\le|\ln 2-\ln(1+y)|+|\ln(1+y)-s_m(y)|+|s_m(y)-s_m(1)|+|s_m(1)-s|.$$

By the alternating series estimation theorem, one has $|\ln(1+y)-s_m(y)|\le|y|^{m+1}/(m+1)\le 1/(m+1)$ for $|y|<1$ and $|s_m(1)-s|\le 1/(m+1)$. Let $\epsilon>0$ and select $M\in\mathbb{N}$ such that $1/M<\epsilon/4$. Since $\ln(1+y)$ and $s_M(y)$ are continuous, we may choose $|y_0|<1$ such that $|\ln(1+y_0)-\ln 2|<\epsilon/4$ and $|s_M(y_0)-s_M(1)|<\epsilon/4$. Combining all above together yields $|\ln 2-s|<\epsilon$, as desired.

Example 11.26. Consider the power series $1/(1+x^2) = \sum_{n=0}^{\infty}(-1)^n x^{2n}$ for $|x| < 1$. By Theorem 11.19, we have

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} + C \quad \forall |x| < 1.$$

Letting $x = 0$ implies $C = 0$ and, hence,

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

A similar reasoning as that for $\ln 2$ yields $\pi/4 = \sum_{n=0}^{\infty}(-1)^n/(2n+1)$.

Remark 11.20. The radius of convergence for power series is preserved under integration and differentiation. But, this does not mean that they share the same interval of convergence. See e.g. the following example.

Example 11.27. Let $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$. Note that

$$\lim_{n\to\infty} \frac{|x|^{n+1}/(n+1)^2}{|x|^n/n^2} = |x|.$$

By the ratio test, f has radius of convergence 1. By the integral test and the comparison test, $\sum_{n=1}^{\infty} n^{-2}$ and $\sum_{n=1}^{\infty}(-1)^n n^{-2}$ are convergent. This implies that f has interval of convergence $[-1, 1]$.

Following Theorem 11.19, one may derive

$$f'(x) = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n}, \quad f''(x) = \sum_{n=2}^{\infty} \frac{n-1}{n} x^{n-2}$$

and both of their radii of convergence are 1. By the alternating series test, f' has interval of convergence $[-1, 1)$, while f'' has interval of convergence $(-1, 1)$.

The proof of 11.19 is based on the following lemma.

Lemma 11.20. *The radii of convergence of* $\sum_{n=1}^{\infty} nc_n x^{n-1}$ *and* $\sum_{n=1}^{\infty} c_{n-1}x^n/n$ *are the same as that of* $\sum_{n=0}^{\infty} c_n x^n$.

Proof. The case that $c_n = 1$ for all $n \geq 0$ is clear from the ratio test. For the general case, we assume that the radius of convergence of $\sum_{n=0}^{\infty} c_n x^n$ is R. Fix $x \in (-R, R)$ and set $r = (|x| + R)/2$. Since $\sum_{n=0}^{\infty} c_n r^n$ converges, we may select $M > 0$ such that $|c_n|r^n < M$ for all $n \geq 0$. As $|x| < r < R$, this implies

$$\sum_{n=1}^{\infty} |nc_n x^{n-1}| \leq \frac{M}{r} \sum_{n=1}^{\infty} n \left(\frac{|x|}{r}\right)^{n-1} < \infty, \quad \sum_{n=1}^{\infty} \left|\frac{c_{n-1}x^n}{n}\right| \leq Mr \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{|x|}{r}\right)^n < \infty.$$

Hence, we may conclude that the radii of convergence of $\sum_{n=1}^{\infty} nc_n x^{n-1}$ and $\sum_{n=1}^{\infty} c_{n-1}x^n/n$ are not less than that of $\sum_{n=0}^{\infty} c_n x^n$ for any sequence $\{c_n\}_{n=0}^{\infty}$.

Next, let R' be the radius of convergence of $\sum_{n=1}^{\infty} nc_n x^{n-1}$. Assume the inverse that $R' > R$ and let $s \in (R, R')$. Since $\sum_{n=1}^{\infty} nc_n s^{n-1}$ converges absolutely, one has

$$\sum_{n=0}^{\infty} |c_n| s^n = |c_0| + s \sum_{n=1}^{\infty} \frac{n|c_n|s^{n-1}}{n} \leq |c_0| + s \sum_{n=1}^{\infty} n|c_n|s^{n-1} < \infty,$$

which makes a confliction. Thus, $R' = R$. Similarly, let R'' be the radius of convergence of $\sum_{n=1}^{\infty} c_{n-1}x^n/n$. Assume that $R'' > R$ and choose $R < t_1 < t_2 < R''$. As $\sum_{n=1}^{\infty} c_{n-1}t_2^n/n$

converges absolutely, we may select $M > 0$ such that $|c_{n-1}|t_2^n/n < M$ for all $n \geq 1$. This implies

$$\sum_{n=0}^{\infty} |c_n| t_1^n \leq \frac{M}{t_1} \sum_{n=0}^{\infty} (n+1) \left(\frac{t_1}{t_2}\right)^{n+1} < \infty,$$

which leads to another contradiction. Thus, $R'' = R$. □

Proof of Theorem 11.19. Without loss of generality, we may assume that $a = 0$. First, we prove the differentiability of f. By Lemma 11.20, $\sum_{n=1}^{\infty} nc_n x^n$ also has radius of convergence R. Let $x \in (-R, R)$, $(|x| - R)/2 < h < (R - |x|)/2$ and set $r = (R + |x|)/2$. It is easy to check that $|x| + |h| < r < R$ and $-R < -r < x + h < r < R$. In some computations, one has

$$\frac{f(x+h) - f(x)}{h} - \sum_{n=1}^{\infty} nc_n x^{n-1} = \sum_{n=1}^{\infty} \sum_{i=0}^{n-1} c_n [(x+h)^i - x^i] x^{n-1-i}.$$

Let $\epsilon > 0$. As $\sum_{n=1}^{\infty} nc_n r^{n-1}$ converges absolutely, we may choose $N > 0$ such that $\sum_{n=N+1}^{\infty} n|c_n| r^{n-1} < \epsilon/3$. This implies

$$\sum_{n=N+1}^{\infty} \sum_{i=0}^{n-1} |c_n| \times |[(x+h)^i - x^i] x^{n-1-i}| \leq 2 \sum_{n=N+1}^{\infty} n|c_n| r^{n-1} < \frac{2\epsilon}{3},$$

and then

$$\left| \sum_{n=N+1}^{\infty} \sum_{i=0}^{n-1} c_n [(x+h)^i - x^i] x^{n-1-i} \right| \leq \frac{2\epsilon}{3}.$$

By the continuity of polynomials, one may select $\delta > 0$ such that $|\sum_{n=1}^{N} \sum_{i=0}^{n-1} c_n [(x+h)^i - x^i] x^{n-1-i}| \leq \epsilon/3$ for $|h| < \delta$. As a consequence, we obtain $|[f(x+h) - f(x)]/h - \sum_{n=1}^{\infty} nc_n x^{n-1}| < \epsilon$ for $|h| < \min\{\delta, (R - |x|)/2\}$. This proves the differentiability of f on $(-R, R)$.

For the indefinite integral of f, set $g(x) = \sum_{n=1}^{\infty} c_{n-1} x^n / n$. By Lemma 11.20, g has radius of convergence R and, for $|x| < R$, $g'(x) = \sum_{n=1}^{\infty} c_{n-1} x^{n-1} = f(x)$. Thus, $f(x) = g(x) + C$. □