13. Vector functions

13.1. Vector functions and space curves.

Definition 13.1. A vector-valued function or vector function, $r(t)$, is a function of which domain is a subset of $\mathbb{R}$ and range is a subset of a vector space V. When $V = \mathbb{R}^3$, we may write $r(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where f, g, h are called the component functions of r.

Remark 13.1. If $r(t) = \langle f(t), g(t), h(t)\rangle$ and $v = \langle a, b, c\rangle$, then $|r(t) - v|^2 = |f(t) - a|^2 + |g(t) - b|^2 + |h(t) - c|^2$. By the squeeze theorem, we may conclude that $r(t) \to v$ if and only if $f(t) \to a$, $g(t) \to b$ and $h(t) \to c$.

Definition 13.2. Consider the vector function $r(t) = \langle f(t), g(t), h(t)\rangle$.

(1) $r(t)$ converges as $t \to T$ if $f(t)$, $g(t)$ and $h(t)$ converge as $t \to T$ and, also,

$$\lim_{t\to T} r(t) = \left\langle \lim_{t\to T} f(t), \lim_{t\to T} g(t), \lim_{t\to T} h(t) \right\rangle.$$

(2) $r(t)$ is continuous at T if $\lim_{t\to T} r(t) = r(T)$.

Remark 13.2. It follows immediately from the above definition that $r(t)$ is continuous at $t = T$ if and only if f, g, h are continuous at $t = T$.

Remark 13.3. If the domain of $r(t)$ is an interval I, then the trajectory of $r(t)$ as t moves along I is called the space curve and the following three equations,

$$x = f(t), \quad y = g(t), \quad z = h(t),$$

are called the parametric equations, where t is called the parameter.

Example 13.1. Consider the following space curves.

$$(1)\, r(t) = \langle 1 + t, 2 + 5t, -1 + 6t\rangle, \quad (2)\, r(t) = \langle t, \sin t, 2\cos t\rangle.$$

For (1), the trajectory forms a line passing through $(1, 2, -1)$ and is parallel to $\langle 1, 5, 6\rangle$. For (2), observe that the parametric equations are

$$x = t, \quad y = \sin t, \quad z = 2\cos t.$$

This implies $y^2 + z^2/4 = 1$ and, thus, the trajectory lies on the elliptic cylinder $y^2 + z^2/4 = 1$.

Example 13.2. Let C be the intersection of the elliptic cylinder $x^2/4 + y^2/9 = 1$ and the plane $y + z = 2$. Using the polar coordinate on the xy-plane, one may write the elliptic cylinder as

$$x = 2\cos\theta, \quad y = 3\sin\theta \quad \theta \in [0, 2\pi].$$

Then, the plane can be expressed as $3\sin\theta + z = 2$. As a result, C turns out the space curve with parametric equations,

$$r(t) = \langle 2\cos t, 3\sin t, 2 - 3\sin t\rangle \quad t \in [0, 2\pi].$$