13.2. **Derivatives and integrals of vector functions.**

Definition 13.3. The derivative of a vector function r is defined by

$$r'(t) = \lim_{\Delta t \to 0} \frac{r(t+\Delta t) - r(t)}{\Delta t},$$

provided the limit exists.

Remark 13.4. $r'(t)$ is called the tangent vector to the curve at $r(t)$ and $T(t) := r'(t)/|r'(t)|$ is called the unit tangent vector.

Theorem 13.1. *Let $r(t) = \langle f(t), g(t), h(t)\rangle$ be a vector function. Then, r is differentiable at t if and only if f, g, h are differentiable at t. In particular, $r'(t) = \langle f'(t), g'(t), h'(t)\rangle$.*

Example 13.3. For the helix $r(t) = \langle 2\cos t, y = \sin t, z = t\rangle$, one has $r'(t) = \langle -2\sin t, \cos t, 1\rangle$ and $r'(\pi) = \langle 0, -1, 1\rangle$.

Theorem 13.2 (Differentiation rules)**.** *Let u, v be differentiable vector functions and F be a differentiable function. Then,*

$$(1)\ \frac{d}{dt}(u(t)+v(t)) = u'(t)+v'(t), \quad (4)\ \frac{d}{dt}[u(t)\times v(t)] = u'(t)\times v(t) + u(t)\times v'(t),$$

$$(2)\ \frac{d}{dt}(F(t)u(t)) = F'(t)u(t) + F(t)u'(t), \quad (5)\ \frac{d}{dt}u(F(t)) = F'(t)u'(F(t)).$$

$$(3)\ \frac{d}{dt}[u(t)\cdot v(t)] = u'(t)\cdot v(t) + u(t)\cdot v'(t),$$

Remark 13.5. Let $r(t) = \langle t, t^2, t^3\rangle$. It is easy to see from the computation of $r'(t) = \langle 1, 2t, 3t^2\rangle$ that there is no $t \in [0,1]$ such that $r'(t) = r(1/2)$. This means that the mean value theorem does not apply for vector functions.

Definition 13.4. Let $r(t) = \langle f(t), g(t), h(t)\rangle$ and assume that f, g, h are continuous. The definite integral of $r(t)$ is defined by

$$\int_a^b r(t)dt = \left\langle \int_a^b f(t)dt, \int_a^b g(t)dt, \int_a^b h(t)dt \right\rangle,$$

and the indefinite integral of $r(t)$ is defined by

$$\int r(t)dt = \left\langle \int f(t)dt, \int g(t)dt, \int h(t)dt \right\rangle.$$

Theorem 13.3 (Integration rules)**.** *Let u, v be differentiable vector functions and $F(t)$ be a differentiable function. Then,*

$$(1)\ \int [u(t)+v(t)]dt = \int u(t)dt + \int v(t)dt, \quad (2)\ \int F(t)u'(t)dt = F(t)u(t) - \int F'(t)u(t)dt,$$

$$(3)\ \int u(t)\cdot v'(t)dt = u(t)\cdot v(t) - \int u'(t)\cdot v(t)dt,$$

$$(4)\ \int u(t)\times v'(t)dt = u(t)\times v(t) - \int u'(t)\times v(t)dt, \quad (5)\ \int F'(t)u'(F(t))dt = u(F(t)) + C.$$