13.3. **Arc length and curvature.** Recall that a plane curve with parametric equations $x = f(t)$ and $y = g(t)$ for $t \in [a, b]$ has length $L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2} dt$.

Theorem 13.4. *Consider a vector function* $r(t) = \langle f(t), g(t), h(t) \rangle$. *If* r *is continuously differentiable, then the length* L *of the curve* $\{r(t) | a \le t \le b\}$ *is given by* $L = \int_a^b |r'(t)| dt = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2} dt$.

Example 13.4. Consider the circular helix $r(t) = \langle \cos t, \sin t, t \rangle$ with $t \in [0, 2\pi]$. Then, the length of r is

$$L = \int_0^{2\pi} |r'(t)| dt = \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t + 1^2} dt = 2\sqrt{2}\pi.$$

Example 13.5. Consider the vector functions $r_1(t) = \langle t, t^2, t^3 \rangle$ with $t \in [1, 2]$ and $r_2(s) = \langle e^s, e^{2s}, e^{3s} \rangle$ with $s \in [0, \ln 2]$. It is easy to see that both equations denote the same curve and, theoretically, should have the same length. In detail, if L_i is the length of r_i, then

$$L_1 = \int_1^2 \sqrt{1^2 + (2t)^2 + (3t^2)^2} dt, \quad L_2 = \int_0^{\ln 2} e^s \sqrt{1 + 4e^{2s} + 9e^{4s}} ds.$$

By the substitute $t = e^s$, one may derive $L_1 = L_2$.

Remark 13.6. Let $r(t)$ be a continuously differentiable vector function with $t \in [a, b]$. As in the case of plane curves, the length function of this curve between $r(a)$ and $r(t)$ is defined by $s(t) = \int_a^t |r'(u)| du$. By the fundamental theorem of calculus, we have $s'(t) = |r'(t)|$.

Example 13.6. The length function of the helix $r(t) = \langle \cos t, \sin t, t \rangle$ starting from $r(0)$ is $s(t) = \int_0^t \sqrt{\sin^2 u + \cos^2 u + 1} du = \sqrt{2} t$.

Definition 13.5. A parametrization of a curve is its vector equation, say $r(t)$. If $r(t)$ is continuously differentiable on an interval I, then the parametrization is called smooth on I. A curve is called smooth if it has a smooth parametrization.

Definition 13.6. Let $r(t)$ be a parametrization of a curve C with unit tangent vector $T(t)$. Then, the curvature of C at $r(t)$ is defined by

$$\kappa = \left| \frac{dT}{ds} \right|. \quad \text{or precisely} \quad \kappa(t) = \left| \frac{T'(t)}{s'(t)} \right| = \frac{|T'(t)|}{|r'(t)|}.$$

Remark 13.7. The definition of curvature is independent of the choice of parametrization.

Example 13.7. Consider the ellipse $x^2/a^2 + y^2/b^2 = 1$ and the smooth parametrization $r(t) = \langle a \cos t, b \sin t \rangle$ with $t \in [0, 2\pi]$. Note that

$$r'(t) = \langle -a \sin t, b \cos t \rangle, \quad T'(t) = \frac{d}{dt} \left(\frac{r'(t)}{|r'(t)|} \right) = \frac{r''(t)|r'(t)| - r'(t)|r'(t)|'}{|r'(t)|^2},$$

and

$$|r'(t)|^2 = a^2 \sin^2 t + b^2 \cos^2 t, \quad |r'(t)|' = \frac{(a^2 - b^2) \sin t \cos t}{|r'(t)|}.$$

This implies

$$T'(t) = \frac{\langle -ab^2 \cos t, -a^2 b \sin t \rangle}{|r'(t)|^3}, \quad |T'(t)| = \frac{ab}{a^2 \sin^2 t + b^2 \cos^2 t}.$$

Hence, we have

$$\kappa(t) = \frac{|T'(t)|}{|r'(t)|} = \frac{ab}{(a^2 \sin^2 t + b^2 \cos^2 t)^{3/2}}.$$

When $a > b$, $\kappa(0) = a/b^2 > b/a^2 = \kappa(\pi/2)$. When $a = b$, $\kappa(t) = 1/a$ for all t.

Theorem 13.5. *If $r(t)$ is a secondly differentiable parametrization of a curve C, then $\kappa(t) = |r'(t) \times r''(t)|/|r'(t)|^3$.*

Proof. By the identities $T(t) = r'(t)/|r'(t)|$ and $s'(t) = |r'(t)|$, one has $r'(t) = s'(t)T(t)$. This implies $r''(t) = s''(t)T(t)+s'(t)T'(t)$. Since $|T(t)| = 1$ for all t, $T(t)$ and $T'(t)$ are perpendicular for all t, which leads to $r'(t)\times r''(t) = s'(t)^2(T(t)\times T'(t))$ and then $|r'(t)\times r''(t)| = s'(t)^2|T(t)\times T'(t)| = s'(t)^2|T(t)||T'(t)|$. As a result, $\kappa(t) = |T'(t)|/|r'(t)| = |r'(t) \times r''(t)|/|r'(t)|^3$. □

Example 13.8. Consider the space curve $r(t) = \langle t^2, 2t, \ln t\rangle$. In some computations, one can show that $r'(t) = \langle 2t, 2, 1/t\rangle$, $r''(t) = \langle 2, 0, -1/t^2\rangle$ and $r'(t) \times r''(t) = \langle -2/t^2, 4/t, -4\rangle$. Consequently, we obtain

$$\kappa(t) = \frac{|\langle -2/t^2, 4/t, -4\rangle|}{|\langle 2t, 2, 1/t\rangle|^3} = \frac{2(1/t^4 + 4/t^2 + 4)^{1/2})}{(4t^2 + 4 + 1/t^2)^{3/2}}.$$

At the point $(1, 2, 0)$, the curvature is $2/9$.

Example 13.9. To see the curvature of the plane curve $y = f(x)$, we embed it in $\mathbb{R}^3$ as the space curve $r(x) = \langle x, f(x), 0\rangle$. Immediately, this implies $r'(x) = \langle 1, f'(x), 0\rangle$, $r''(x) = \langle 0, f''(x), 0\rangle$ and $r'(x) \times r''(x) = \langle 0, 0, f''(x)\rangle$. Hence, the curvature $\kappa(x)$ at point $(x, f(x), 0)$ is given by $\kappa(x) = \frac{|f''(x)|}{[1+(f'(x))^2]^{3/2}}$. When $f(x) = \ln x$, $\kappa(x) = \frac{1/x^2}{(1+1/x^2)^{3/2}} = \frac{x}{(x^2+1)^{3/2}} \to 0$ as $x \to 0$ or $x \to \infty$.

Definition 13.7. Given a smooth curve $r(t)$ with unit tangent vector $T(t)$, the principal unit normal vector and the binormal vector are respectively defined by

$$N(t) = \frac{T'(t)}{|T'(t)|}, \quad B(t) = T(t) \times N(t).$$

Remark 13.8. Since $T(t)$ and $N(t)$ are perpendicular and of length 1, $|B(t)| = |T(t)||N(t)| = 1$. The plane determined by N and B is called the normal plane and the plane determined by N and T is called the osculating plane. The osculating circle of C at P is the circle lying on the osculating plane of C and on the concave side of C, which is the side that N points toward, with the same tangent as C at P and radius $1/\kappa$.

Example 13.10. Consider the helix $r(t) = \langle \cos t, \sin t, t\rangle$ and $P = (-1, 0, \pi)$. The normal plane at P has normal vector $r'(\pi) = \langle 0, -1, 1\rangle$ and the osculating plane at P has normal vector $B(\pi)$. A direct computation yields $B(t) = 2^{-1/2}\langle \sin t, -\cos t, 1\rangle$ and then $B(\pi) = 2^{-1/2}\langle 0, 1, 1\rangle$. Consequently, the normal and osculating planes at P are respectively $-y+z = \pi$ and $y+z = \pi$.

Example 13.11. For the curve $y = \ln x$, recall that $\kappa(x) = x/(x^2 + 1)^{3/2}$. By identifying this curve with $\langle x, \ln x, 0\rangle$, we may compute

$$T(t) = \frac{x}{\sqrt{x^2+1}}\langle 1, 1/x, 0\rangle, \quad N(t) = \frac{1}{\sqrt{x^2+1}}\langle 1, -x, 0\rangle.$$

Then, the osculating circle of the curve $y = \ln x$ at the point $(x, \ln x)$ is centered at $(2x + 1/x, \ln x - x^2 - 1)$ with radius $(x^2 + 1)^{3/2}/x$.