14. Partial Derivatives

14.1. **Functions of several variables.**

Definition 14.1. A function of n variables is a rule that assigns to each n-vector $(x_1, ..., x_n)$ a unique real number denoted by $f(x_1, ..., x_n)$. The set D where f is defined is called the domain of f and the range is the set of values where f takes, that is, $\{f(x_1, ..., x_n) | (x_1, ..., x_n) \in D\}$.

Example 14.1. Let $f(x, y) = \sqrt{4 - x^2 - y^2}$. One can see without difficulty that the domain of f is $D = \{(x, y) | 4 - x^2 - y^2 \geq 0\}$ and the range is $R = \{z | 0 \leq z \leq 2\}$.

Definition 14.2. Let f be a function of n variables with domain D. The graph of f is the set $\{(x_1, ..., x_n, z) \in \mathbb{R}^{n+1} | z = f(x_1, ..., x_n), (x_1, ..., x_n) \in D\}$.

Example 14.2. Let $f(x, y) = 6 - 3x - 2y$, $g(x, y) = \sqrt{4 - x^2 - y^2}$ and $h(x, y) = x^2 + y^2/4$. Then, the graphs of f, g, h are respectively a plane, a sphere and an elliptic paraboloid.

Definition 14.3. The level curve of a function f of two variables are the curves determined by the equation $f(x, y) = k$ where k is any constant. A level surface of a function g of three variables is the surface determined by $g(x, y, z) = k$.

Example 14.3. For the functions $f(x, y) = 6 - 3x - 2y$, $g(x, y) = \sqrt{4 - x^2 - y^2}$ and $h(x, y) = x^2 + y^2/4$, the level curves are lines, circles, ellipses and their degenerate forms.

Example 14.4. For the function $f(x, y, z) = x^2 - y^2 + z^2$, the domain is $\mathbb{R}^3$ and the level surface $f(x, y, z) = k$ is a hyperboloid of one sheet if $k > 0$ and a hyperboloid of two sheet if $k < 0$, whereas the level surface $f(x, y, z) = 0$ is a cone.