14.2. **Limits and continuity.** Consider the following functions.

$$f(x,y) = \frac{\sin(x^2+y^2)}{x^2+y^2}, \quad g(x,y) = \frac{x^2-y^2}{x^2+y^2}, \quad \forall (x,y) \neq (0,0).$$

Note that (x,y) approaches $(0,0)$ if and only if x^2+y^2 approaches 0. By L'Hôpital's rule, $f(x,y) \to 1$ as (x,y) approaches $(0,0)$. For g, it is clear that $g(x,0) = 1$ for $x \neq 0$ and $g(0,y) = -1$ for $y \neq 0$. Intuitively, $g(x,y)$ has no limit as (x,y) approaches $(0,0)$.

Definition 14.4. Let f be a function of two variables whose domain includes points arbitrarily close to (a,b). The value of $f(x,y)$ can be arbitrarily close to L as (x,y) approaches (a,b) if, for any $\epsilon > 0$, there is $\delta > 0$ such that

$$|f(x,y) - L| < \epsilon \quad \text{for } (x,y) \in D,\ 0 < \sqrt{(x-a)^2+(y-b)^2} < \delta.$$

Briefly, we write $f(x,y) \to L$ as $(x,y) \to (a,b)$ or

$$\lim_{(x,y)\to(a,b)} f(x,y) = L.$$

Remark 14.1. Let $f(x,y) = x$ and $g(x,y) = y$. For any $(a,b) \in \mathbb{R}^2$, $f(x,y) \to a = f(a,b)$ and $g(x,y) \to b = g(a,b)$ as $(x,y) \to (a,b)$.

Theorem 14.1. *Let f be a function with domain $D \subset \mathbb{R}^2$. Assume that f has limit L as (x,y) approaches (a,b). Then, by setting $f(a,b) = L$,*

$$\lim_{t\to T} f(a(t), b(t)) = L,$$

for any functions $a(t), b(t)$ satisfying $a(t) \to a, b(t) \to b$ as $t \to T$.

Remark 14.2. By Theorem 14.1, if f has no limit at (a,b) along one curve or f has two different limits at (a,b) along two curves, then f has no limit at (a,b).

Example 14.5. Consider the following functions.

$$f(x,y) = \frac{xy}{x^2+y^2}, \quad g(x,y) = \frac{xy^2}{x^2+y^2}, \quad \forall (x,y) \neq (0,0).$$

As $t \to 0$, $f(t,0) \to 0$ and $f(t,t) \to 1/2$. This implies that f has no limit at $(0,0)$. For g, let $\epsilon > 0$ and choose $\delta = \epsilon$. If $0 < \sqrt{x^2+y^2} < \delta$, then $|x| < \delta = \epsilon$, which implies

$$|g(x,y) - 0| = \frac{|x|y^2}{x^2+y^2} \leq \epsilon.$$

This proves $\lim_{(x,y)\to(0,0)} g(x,y) = 0$.

Limit laws Let f, g be functions of two variables with limits L, M at (a,b) and $c \in \mathbb{R}$. Then, as $(x,y) \to (a,b)$,

(1) $f(x,y) + g(x,y) \to L + M$, (2) $cf(x,y) \to cL$, (3) $f(x,y)g(x,y) \to LM$,
(4) $f(x,y)/g(x,y) \to L/M$ provided $M \neq 0$.

Remark 14.3. A polynomial of two variables is a finite sum of terms of the form cx^my^n, where c is a constant and m, n are non-negative integers. A rational function of two variables is a ratio of two polynomials of two variables. By the limit laws, if $f(x,y)$ is a polynomial or rational function with domain D, then $f(x,y) \to f(a,b)$ for all $(a,b) \in D$.

Squeeze theorem If $f \leq g \leq h$ and f, h have limits L at (a,b), then g has limit L at (a,b).

Definition 14.5. A function f of two variables is continuous at (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b).$$

f is continuous on D if f is continuous at (a, b) for all $(a, b) \in D$.

Remark 14.4. Polynomials and rational functions are continuous on their domains.

Example 14.6. Consider the following functions.

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{for } (x, y) \neq (0, 0) \\ 0 & \text{for } (x, y) = (0, 0) \end{cases}, \quad g(x, y) = \begin{cases} \frac{xy^4}{x^2+y^4} & \text{for } (x, y) \neq (0, 0) \\ 0 & \text{for } (x, y) = (0, 0) \end{cases}.$$

As f, g are rational functions on $\mathbb{R}^2 \setminus \{(0, 0)\}$, they must be continuous on $\mathbb{R}^2 \setminus \{(0, 0)\}$. It has been proved before that f has no limit at $(0, 0)$, this implies that f is not continuous at $(0, 0)$. For g, note that $-|x| \leq g(x, y) \leq |x|$ for $(x, y) \neq (0, 0)$. By the squeeze theorem, $g(x, y) \to 0 = g(0, 0)$ as $(x, y) \to (0, 0)$. This proves that g is continuous at $(0, 0)$.

Theorem 14.2. *If f has limit L at (a, b) and g is continuous at L, then $(g \circ f)(x, y)$ has limit $g(L)$ at (a, b). In particular, if f is continuous at (a, b) and g is continuous at $f(a, b)$, then $g \circ f$ is continuous at (a, b).*

Proof. Assume that f has limit L at (a, b) and g is continuous at L. Let $\epsilon > 0$. By the continuity of g, we may choose $\eta > 0$ such that $|g(z) - g(L)| < \epsilon$ for $|z - L| < \eta$. As f has limit L at (a, b), we may select $\delta > 0$ such that $|f(x, y) - L| < \eta$ for $0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta$. Replacing z with $f(x, y)$ implies that $|g \circ f(x, y) - g(L)| < \epsilon$ for $0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta$. □

Example 14.7. Let $g(t) = t + \ln t$ and $f(x, y) = (1 - xy)/(1 + x^2y^2)$. Note that f is continuous on $\mathbb{R}^2$ and g is continuous on $(0, \infty)$. This implies that $g \circ f$ is continuous on its domain, which is $\{(x, y)|xy < 1\}$.

Remark 14.5. For functions of n variables, their limits and continuities are defined in the same way. In detail, f has limit L at $a = (a_1, ..., a_n)$ if, for any $\epsilon > 0$, there is $\delta > 0$ such that

$$|f(x) - L| < \epsilon \quad \text{for } |x - a| = \sqrt{(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2} < \delta,$$

where $x = (x_1, x_2, ..., x_n)$. Also, f is said to be continuous at a if $f(x) \to f(a)$ as $x \to a$. In a similar reasoning, the limit laws, squeeze theorem and all above theorems are applicable for functions of n variables.