14.3. **Partial derivatives.**

Definition 14.6. Let $f(x, y)$ be a function of two variables. The partial derivative of f w.r.t. x at (a, b) is defined by

$$f_x(a,b) = \lim_{s\to 0} \frac{f(a+s,b) - f(a,b)}{s} = g'(a),$$

where $g(x) = f(x, b)$, whereas the partial derivative of f w.r.t. y at (a, b) is defined by

$$f_y(a,b) = \lim_{t\to 0} \frac{f(a,b+t) - f(a,b)}{t} = h'(b),$$

where $h(y) = g(a, y)$.

Example 14.8. Let $f(x, y) = (e^x + e^y)/(x + y)$ and set $g(x) = f(x, 0) = (e^x + 1)/x$ and $h(y) = f(1, y) = (e + e^y)/(1 + y)$. Note that $g'(x) = [(x-1)e^x - 1]/x^2$ and $h'(y) = (ye^y - e)/(1 + y)^2$. This implies $f_x(1, 0) = g'(1) = -1$ and $f_y(1, 0) = h'(0) = -e$.

Definition 14.7. Let $f(x, y)$ be a function of two variables. The partial derivatives of f refer to functions, f_x and f_y, defined by

$$f_x(x,y) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x,y) - f(x,y)}{\Delta x}, \quad f_y(x,y) = \lim_{\Delta y\to 0} \frac{f(x,y+\Delta y) - f(x,y)}{\Delta y}.$$

Remark 14.6 (Notations for partial derivatives). When writing $z = f(x, y)$, we also use the following notations to express the partial derivatives.

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x} = D_x f$$

and

$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = \frac{\partial z}{\partial y} = D_y f.$$

Example 14.9. For $f(x, y) = x^3 + x^2y^3 - 2y^2$, $f_x(x, y) = 3x^2 + 2xy^3$ and $f_y(x, y) = 3x^2y^2 - 4y$.

Remark 14.7 (Interpretation of partial derivatives). Let $f(x, y)$ be a function of two variables and assume that $f_x(a, b)$ and $f_y(a, b)$ exist. By setting $g(x) = f(x, b)$ and $h(y) = f(a, y)$, one can see that $f_x(a, b) = g'(a)$ and $f_y(a, b) = h'(b)$. Note that the graph of $z = g(x)$ is the intersection of graphs of $z = f(x, y)$ and $y = b$. Clearly, $g'(a)$ denotes the slope of the tangent line to the intersecting curve of $z = f(x, y)$ and $y = b$ at $(a, f(a, b))$. Similar, $h'(b)$ refers to the slope of the tangent line to the intersecting curve of $z = f(x, y)$ and $x = a$ at $(b, f(a, b))$.

Example 14.10. Let $f(x, y) = xy/(x^2 + y^2)$ for $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$. It has be proved before that f has no limit at $(0, 0)$, so f is not continuous at $(0, 0)$. But, the partial derivatives of f at $(0, 0)$ exist and equal to 0.

Example 14.11 (Chain rule). Let $f(x, y) = h(g(x, y))$. If f_x, f_y, h' exist, then $f_x(x, y) = h'(g(x, y))g_x(x, y)$ and $f_y(x, y) = h'(g(x, y))g_y(x, y)$. In particular, if $g(x, y) = xy$, then $\frac{\partial}{\partial x} h(xy) = yh'(xy)$ and $\frac{\partial}{\partial y} h(xy) = xh'(xy)$.

Example 14.12 (Implicit partial derivatives). Assume that z can be implicitly expressed as a function of x, y through equation $x^3 + y^2z^2 + xyz = 1$. By regarding z as a function of x, y, say $z = f(x, y)$, and treating x, y as constants respectively, one may derive

$$0 = \frac{\partial(x^3 + y^2 f^2(x,y) + xyf(x,y))}{\partial x} = 3x^2 + 2y^2 f(x,y) f_x(x,y) + yf(x,y) + xyf_x(x,y),$$

and

$$0 = \frac{\partial(x^3 + y^2 f(x,y)^2 + xyf(x,y))}{\partial y} = 2yf(x,y)^2 + 2y^2 f(x,y) f_y(x,y) + xf(x,y) + xyf_y(x,y).$$

This implies

$$\frac{\partial z}{\partial x} = f_x(x,y) = -\frac{3x^2 + yz}{2y^2 z + xy}, \quad \frac{\partial z}{\partial y} = f_y(x,y) = -\frac{2yz^2 + xz}{2y^2 z + xy}.$$

Remark 14.8. For functions of n variables, say $f(x_1, ..., x_n)$, the partial derivatives are defined by

$$f_{x_i}(x_1, ..., x_n) = \lim_{h \to 0} \frac{f(x_1, ..., x_{i-1}, x_i + h, x_{i+1}, ..., x_n) - f(x_1, ..., x_n)}{h}.$$

Example 14.13. Let $f(x,y,z) = z \ln(x+y)$. Then,

$$f_x(x,y,z) = f_y(x,y,z) = \frac{z}{x+y}, \quad f_z(x,y,z) = \ln(x+y).$$

Remark 14.9. For higher derivatives, we write

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}, \quad f_{xyy} = (f_{xy})_y = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y \partial x}\right) = \frac{\partial^3 f}{\partial y^2 \partial x}.$$

Example 14.14. Let $f(x,y) = x^3 - x^2 y + xy^2 - y^3$. Then, $f_x(x,y) = 3x^2 - 2xy + y^2$, $f_y(x,y) = -x^2 + 2xy - 3y^2$ and

$$f_{xx} = 6x - 2y, \quad f_{xy} = -2x + 2y, \quad f_{yx} = -2x + 2y, \quad f_{yy} = 2x - 6y.$$

Theorem 14.3 (Clairaut's theorem)**.** *Let f be a function with domain $D \subset \mathbb{R}^2$. If f_{xy} and f_{yx} are continuous on D, then $f_{xy} = f_{yx}$.*

Example 14.15. Consider the following function,

$$f(x,y) = \frac{x^3 y - xy^3}{x^2 + y^2}, \; \forall (x,y) \neq (0,0), \quad f(0,0) = 0.$$

In some computations, we obtain $f_x(0,0) = 0$, $f_y(0,0) = 0$ and, for $(x,y) \neq (0,0)$,

$$f_x(x,y) = \frac{x^4 y + 4x^2 y^3 - y^5}{(x^2 + y^2)^2}, \quad f_y(x,y) = \frac{-xy^4 - 4x^3 y^2 + x^5}{(x^2 + y^2)^2}$$

This implies

$$f_{xy}(0,0) = \lim_{h \to 0} \frac{f_x(0,h) - f_x(0,0)}{h} = -1, \quad f_{yx}(0,0) = \lim_{h \to 0} \frac{f_y(h,0) - f_y(0,0)}{h} = 1.$$

Theorem 14.4. *Let $f(x)$ be a function of n variables, where $x = (x_1, ..., x_n)$, with domain $D \subset \mathbb{R}^n$. For $i \neq j$, if $f_{x_i x_j}$ and $f_{x_j x_i}$ are continuous on D, then $f_{x_i x_j} = f_{x_j x_i}$ on D.*