14.4. **Tangent planes and linear approximations.** Let $f(x, y)$ be a function of two variables and $P = (x_0, y_0, z_0)$ be a point on the graph of $z = f(x, y)$. Suppose the graph is smooth at P in the sense that the surface of the graph gradually looks like a plane when the graph zooms in. The plane serves as a tangent plane to the surface at P and we express it through equation $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$. When $c \neq 0$, we may rewrite the equation as $z - z_0 = a'(x - x_0) + b'(y - y_0)$, where $a' = -a/c$ and $b' = -b/c$. Intuitively, the tangent lines to the curves of $z = f(x, y_0)$ and $z = f(x_0, y)$ should sit on the plane. As the tangent lines are respectively $z - z_0 = f_x(x_0, y_0)(x - x_0)$ and $z - z_0 = f_y(x_0, y_0)(y - y_0)$, we obtain $a' = f_x(x_0, y_0)$ and $b' = f_y(x_0, y_0)$.

Definition 14.8. Let $f(x, y)$ be a function of two variables with continuous partial derivatives. The tangent plane to the surface $z = f(x, y)$ at (x_0, y_0, z_0) with $z_0 = f(x_0, y_0)$ is defined to be the plane of $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$.

Example 14.16. Consider the function $f(x, y) = x^2 + 2y^2$. The graph of $z = f(x, y)$ is an elliptic paraboloid and the tangent plane to the surface at $(1, 1, 3)$ is $z - 3 = f_x(1, 1)(x - 1) + f_y(1, 1)(y - 1) = 2(x - 1) + 4(y - 1)$.

Remark 14.10. As a plane has parallel lines as level curves, to see whether a zooming-in surface looks like a plane, it suffices to examine the zooming-in level curves of the surface and check whether they are parallel lines.

Note that the tangent plane to the graph $z = f(x, y)$ at $(x_0, y_0, f(x_0, y_0))$ is the graph of the following function

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Thus, $L(x, y)$ approximates $f(x, y)$ as (x, y) approaches (x_0, y_0). We call $L(x, y)$ the linearization of f at (x_0, y_0). A natural question arises:

"Whether the linearization of a function is a good approximation to the function itself?"

The following example says that the existence of f_x and f_y is not enough.

Example 14.17. Let $f(x, y) = xy/(x^2 + y^2)$ for $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$. It has been proved before that f is continuous on $\mathbb{R}^2$ except at $(0, 0)$. As $f_x(0, 0) = f_y(0, 0) = 0$, the linearization of f at $(0, 0)$ is $L(x, y) = 0$ for all x, y. However, along the line $x = y$, one has $f(h, h) \to 1/2$ as $h \to 0$. It's worthwhile to remark that f_x, f_y are not continuous at $(0, 0)$ since $f_x(x, y) = y(y^2 - x^2)/(x^2 + y^2)^2$, $f_y(x, y) = x(x^2 - y^2)/(x^2 - y^2)^2$ and $f_x(0, h) \to \infty$ and $f_y(h, 0) \to \infty$ as $h \to 0^+$.

In the following, we use $\Delta x, \Delta y$ to denote the increments of x, y and set $\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$.

Definition 14.9. A function of two variables $f(x, y)$ is differentiable at (x_0, y_0) if the corresponding increment Δz can be expressed as

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y, \tag{14.1}$$

where ϵ_1, ϵ_2 are functions of $\Delta x, \Delta y$ and $(\epsilon_1, \epsilon_2) \to (0, 0)$ as $(\Delta x, \Delta y) \to (0, 0)$.

Remark 14.11. The requirement of (14.1) is the same as

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon\sqrt{(\Delta x)^2 + (\Delta y)^2},$$

where $\epsilon \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

Lemma 14.5. *If f is differentiable at (x_0, y_0), then f is continuous at (x_0, y_0).*

Theorem 14.6. *Let f be a function of two variables. If f_x and f_y are continuous at (x_0, y_0), then f is differentiable at (x_0, y_0).*

Example 14.18. Let $f(x, y) = \sin(x^2 + y^2)$. Note that $f_x(x, y) = 2x\cos(x^2 + y^2)$ and $f_y(x, y) = 2y\cos(x^2 + y^2)$. As f_x, f_y are continuous on $\mathbb{R}^2$, f is differentiable everywhere.

Definition 14.10. For any differentiable function $f(x, y)$, the total differential or simply the differential is defined by

$$dz = df = f_x(x, y)dx + f_y(x, y)dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

Remark 14.12. By taking $dx = \Delta x = x - x_0$ and $dy = \Delta y = y - y_0$, one has

$$\Delta z = f(x, y) - f(x_0, y_0) \approx dz = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Example 14.19. The total differential of $f(x, y) = x^2 + 3xy - y^2$ is $dz = (2x + 3y)dx + (3x - 2y)dy$. When $x_0 = 2$, $y_0 = 3$, $dx = \Delta x = 0.05$ and $dy = \Delta y = -0.04$, we have

$$\Delta z = f(2.05, 2.96) - f(2, 3) = 0.6449, \quad dz = 13dx = 0.65.$$

Practically, dz is easier to compute out than Δz.

Remark 14.13. For functions of n variables, the definitions of linearization, differentiability and total differentials are similar.