14.5. **The chain rules.**

Theorem 14.7. *Let $z = f(x, y)$ be differentiable in x, y and $x = g(t), y = h(t)$ be differentiable in t. Then, $z = f(g(t), h(t))$ is differentiable in t and*

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = f_x(g(t), h(t))g'(t) + f_y(g(t), h(t))h'(t).$$

Proof. Since f is differentiable, we have

$$\Delta z = f_x \Delta x + f_y \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

where $(\epsilon_1, \epsilon_2) \to (0, 0)$ as $(\Delta x, \Delta y) \to (0, 0)$. Dividing both sides with Δt gives

$$\frac{\Delta z}{\Delta t} = f_x \frac{\Delta x}{\Delta t} + f_y \frac{\Delta y}{\Delta t} + \epsilon_1 \frac{\Delta x}{\Delta t} + \epsilon_2 \frac{\Delta y}{\Delta t}.$$

Letting $\Delta t \to 0$ implies $(\Delta x, \Delta y) \to (0, 0)$, which leads to

$$\frac{dz}{dt} = \lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt}.$$

□

Remark 14.14. In Leibnitz's notation, the chain rule can be expressed as $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$.

Example 14.20. Let $z = x^2y + xy^2$, $x = e^{2t}$ and $y = \cos t$. Then,

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (2xy + y^2)(2e^{2t}) + (x^2 + 2xy)(-\sin t)\\ &= [2e^{2t}\cos t + \cos^2 t]2e^{2t} + [e^{4t} + 2e^{2t}\cos t](-\sin t)\end{aligned}$$

Theorem 14.8. *Let $z = f(x, y)$, $x = g(s, t)$ and $y = h(s, t)$ be differentiable functions. Then, $f(g(s, t), h(s, t))$ is differentiable in s, t and*

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}, \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$

Proof. Since f, g and h are differentiable, we have

$$\Delta z = f_x \Delta x + f_y \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y,$$

where $\epsilon_1 \to 0$, $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$, and

$$\Delta x = g_s \Delta s + g_t \Delta t + \epsilon_3 \Delta s + \epsilon_4 \Delta t, \quad \Delta y = h_s \Delta s + h_t \Delta t + \epsilon_5 \Delta s + \epsilon_6 \Delta t,$$

where $(\epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6) \to (0, 0, 0, 0)$ as $(\Delta s, \Delta t) \to (0, 0)$. This implies

$$\Delta z = (f_x g_s + f_y h_s)\Delta s + (f_x g_t + f_y h_t)\Delta t + \epsilon_7 \Delta s + \epsilon_8 \Delta t,$$

where

$$\epsilon_7 = \epsilon_1(g_s + \epsilon_3) + \epsilon_2(h_s + \epsilon_5), \quad \epsilon_8 = \epsilon_1(g_t + \epsilon_4) + \epsilon_2(h_t + \epsilon_6).$$

It is easy to see that, as $(\Delta s, \Delta t) \to (0, 0)$, $(\Delta x, \Delta y) \to (0, 0)$ and then $(\epsilon_7, \epsilon_8) \to (0, 0)$. This proves that z is differentiable in s, t and $\frac{\partial z}{\partial s} = f_x g_s + f_y h_s$, $\frac{\partial z}{\partial t} = f_x g_t + f_y h_t$. □

Example 14.21. Let $z = x^2y + xy^2$, $x = r\cos\theta$ and $y = r\sin\theta$. Then,

$$\begin{aligned}\frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = (2xy + y^2)\cos\theta + (x^2 + 2xy)\sin\theta\\ &= (2r^2\sin\theta\cos\theta + r^2\sin^2\theta)\cos\theta + (r^2\cos^2\theta + 2r^2\sin\theta\cos\theta)\sin\theta\end{aligned}$$

and

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} = (2xy + y^2)r(-\sin\theta) + (x^2 + 2xy)r\cos\theta$$
$$= (2r^2\sin\theta\cos\theta + r^2\sin^2\theta)r(-\sin\theta) + (r^2\cos^2\theta + 2r^2\sin\theta\cos\theta)r\cos\theta$$

Remark 14.15. In the chain rule, s, t are called independent variables and z is called the dependent variable, whereas x, y are called the indeterminate variables.

Theorem 14.9. *Let z be differentiable in $y_1, y_2, ..., y_n$ and y_i be differentiable in $x_1, ..., x_m$ for all $1 \leq i \leq n$. Then, z is differentiable in $x_1, ..., x_m$ and*

$$\frac{\partial z}{\partial x_i} = \sum_{j=1}^{n} \frac{\partial z}{\partial y_j}\frac{\partial y_j}{\partial x_i}.$$

Example 14.22. Let $w = x^2yz + xy^2z + xyz^2$, $x = r\cos\theta\cos\phi$, $y = r\cos\theta\sin\phi$ and $z = r\sin\theta$. Then,

$$\frac{\partial w}{\partial \phi} = (2xyz + y^2z + yz^2)\frac{\partial x}{\partial \phi} + (x^2z + 2xyz + xz^2)\frac{\partial y}{\partial \phi} + (x^2y + xy^2 + 2xyz)\frac{\partial z}{\partial \phi}$$
$$= (2xyz + y^2z + yz^2)(-r\cos\theta\sin\phi) + (x^2z + 2xyz + xz^2)r\cos\theta\cos\phi$$

To see partial derivatives of implicit functions, let F be a function of variables x, y, z. Assume that the solution of $F(x, y, z) = 0$ has the implicit function $z = f(x, y)$, that is, $F(x, y, f(x, y)) = 0$. Differentiating both sides partially with respect to x yields

$$0 = \frac{\partial F(x, y, f(x, y))}{\partial x} = F_x(x, y, f(x, y)) + F_z(x, y, f(x, y))f_x(x, y),$$

which implies

$$\frac{\partial z}{\partial x} = f_x(x, y) = -\frac{F_x(x, y, f(x, y))}{F_z(x, y, f(x, y))}.$$

Similarly, one can show that

$$\frac{\partial z}{\partial y} = f_y(x, y) = -\frac{F_y(x, y, f(x, y))}{F_z(x, y, f(x, y))}.$$

The theorem supporting the above computations is the implicit function theorem.

Theorem 14.10 (The implicit function theorem)**.** *Let $F(x, y, z)$ be a function defined on an open set $D \subset \mathbb{R}^3$ and $(x_0, y_0, z_0) \in D$. Assume that F_x, F_y, F_z are continuous on D and $F_z(x_0, y_0, z_0) \neq 0$. Then, in a neighborhood of (x_0, y_0, z_0), the solution of the equation, $F(x, y, z) = F(x_0, y_0, z_0)$, can be expressed as a function $z = f(x, y)$. Moreover, the function f is continuously differentiable at (x_0, y_0) and*

$$f_x(x_0, y_0) = -\frac{F_x(x_0, y_0, z_0)}{F_z(x_0, y_0, z_0)}, \quad f_y(x_0, y_0) = -\frac{F_y(x_0, y_0, z_0)}{F_z(x_0, y_0, z_0)}.$$

Example 14.23. For equation $x^3 + y^3 + z^3 + 6xyz = 1$, one may use the above formula to derive

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{x^2 + 2yz}{z^2 + 2xy}, \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{y^2 + 2xz}{z^2 + 2xy}.$$