14.7. **Maximum and minimum values.**

Definition 14.13. A function f of n variables is said to have a local maximum (resp. local minimum) at $\mathbf{a} = (a_1, ..., a_n)$ if there is $\epsilon > 0$ such that

$$f(\mathbf{x}) \le f(\mathbf{a}) \quad (\text{resp. } f(\mathbf{x}) \ge f(\mathbf{a})) \quad \forall |\mathbf{x} - \mathbf{a}| = \left(\sum_{i=1}^{n} |x_i - a_i|^2 \right)^{1/2} < \epsilon. \tag{14.2}$$

Here, $f(\mathbf{a})$ is called a local maximum (resp. minimum) value. When (14.2) holds for all $\mathbf{x}$ in the domain of f, we say that f has an absolute maximum (resp. minimum) at $\mathbf{a}$.

Theorem 14.13. *If $f(\mathbf{x})$ has a local maximum or minimum at $\mathbf{a}$ and f has first-order partial derivatives at $\mathbf{a}$, then $\nabla f(\mathbf{a}) = \mathbf{0} := (0, ..., 0)$.*

Remark 14.20. The proof follows directly from Fermat's theorem. As in the one-dimensional case, we call $\mathbf{x}$ a critical point of f if either $\nabla f(\mathbf{x})$ does not exist or $\nabla f(\mathbf{x}) = \mathbf{0}$

Example 14.31. For $f(x, y) = x^2 + y^2 + x - 2y$, note that $\nabla f = \langle 2x + 1, 2y - 2 \rangle$. Clearly, $\nabla f(x, y) = \langle 0, 0 \rangle$ if and only if $(x, y) = (-1/2, 1)$.

Remark 14.21. Note that $\nabla f = 0$ is sufficient for the existence of local extrema. For example, if $f(x, y) = x^2 - y^2$, then $(0, 0)$ is the unique critical point but $(0, 0)$ is neither a local maximum nor a local minimum. Such a point is called a saddle point, a critical point which is not a local maximum or minimum.

Theorem 14.14. *Let f be a function of two variables with continuous second order partial derivatives. Assume $\nabla f(x_0, y_0) = \mathbf{0}$ and set $D(x_0, y_0) = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - [f_{xy}(x_0, y_0)]^2$.*

1. *If $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) > 0$, then $f(x_0, y_0)$ is a local minimum.*
2. *If $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) < 0$, then $f(x_0, y_0)$ is a local maximum.*
3. *If $D(x_0, y_0) < 0$, then (x_0, y_0) is a saddle point.*
4. *If $D(x_0, y_0) = 0$, no conclusion.*

Remark 14.22. Let $f(x, y) = x^2 y^2$ and $g(x, y) = xy^2$. Note that $(0, 0)$ is a critical point of f and g, while f has a local minimum at $(0, 0)$ and g has a saddle point at $(0, 0)$.

Example 14.32. Consider the function $f(x, y) = x^4 + y^4 - 4xy + 1$. Note that

$$f_x = 4x^3 - 4y, \quad f_y = 4y^3 - 4x, \quad f_{xx} = 12x^2, \quad f_{yy} = 12y^2, \quad f_{xy} = f_{yx} = -4.$$

Clearly, the critical points (x, y) of f satisfy $x^3 = y$ and $y^3 = x$. This implies $x^9 = x$ or equivalently $x(x-1)(x+1)(x^2+1)(x^4+1) = 0$. Hence, the critical points of f contain $(0, 0)$, $(1, 1)$, $(-1, -1)$ and $D(0, 0) = -16$, $D(1, 1) = D(-1, -1) = 128$. By Theorem 14.14, $(0, 0)$ is a saddle point, whereas $(1, 1)$ and $(-1, -1)$ are local minima.

Example 14.33. A rectangle box without a lid is to be made from a 12 m^2 cardboard. Let x, y, z be the length, width and height of the box. Then, the volume V is given by $V = xyz$. Along with the restriction of $xy + 2yz + 2xz = 12$, we may rewrite the volume as

$$V(x, y) = \frac{xy(12 - xy)}{2(x + y)} \quad \forall (x, y) \in E = \{(x, y) | x > 0,\ y > 0,\ xy < 12\}.$$

Clearly, $V(x, y) = 0$ on the boundary of E and $V(x, y) \le 18/(x + y) \to 0$ as $x \to \infty$ or $y \to \infty$. This implies that the maximum of f must exist and in the interior of E. In some computations, one has

$$V_x = \frac{y^2(-x^2 - 2xy + 12)}{2(x + y)^2}, \quad V_y = \frac{x^2(-y^2 - 2xy + 12)}{2(x + y)^2}.$$

This implies that $(2,2)$ is the unique critical point. Hence, the dimensions of the box with maximum volume are $(x,y,z) = (2,2,1)$ and the volume is $4m^3$.

Theorem 14.15. *If f is continuous on a closed and bounded set $D \subset \mathbb{R}^n$, then f attains its absolute maximum and minimum.*

Strategy to find the absolute extremum Let f be a continuous function on a closed and bounded set $D \subset \mathbb{R}^n$.

(1) Find the values of f at the critical points of f in D.
(2) Find the extremum values of f on the boundary of D.
(3) The largest and smallest values in Steps 1 and 2 are the absolute maximum and minimum.

Example 14.34. Let $f(x,y) = x^2 - 2xy + 2y$ and $D = [0,3] \times [0,2]$. Note that $\nabla f(x,y) = \langle 2x - 2y, -2x + 2\rangle$. This implies that $(1,1)$ is the unique critical point and $(1,1) \in D$. On the boundary, one has

$$f(x,0) = x^2, \quad f(x,2) = x^2 - 4x + 4, \quad \forall x \in [0,3],$$

and

$$f(0,y) = 2y, \quad f(3,y) = 9 - 4y, \quad \forall y \in [0,2].$$

Thus, the maximum and minimum values of f on the boundary of D are 9 and 0. As $f(1,1) = 1$, the maximum and minimum values of f on D are 9 and 0.

Partial proof of Theorem 14.14. Here, we consider the case $f_{xx}(x_0,y_0) > 0$ and $D(x_0,y_0) > 0$. Let $u = \langle a,b\rangle$ be a unit vector. Note that

$$D_u f = af_x + bf_y, \quad D_u^2 f = D_u(D_u f) = a^2 f_{xx} + 2ab f_{xy} + b^2 f_{yy},$$

where the second directional derivative also uses Clairaut's theorem. Write

$$D_u^2 f = f_{xx}\left(a + \frac{f_{xy}}{f_{xx}} b\right)^2 + \frac{b^2}{f_{xx}}(f_{xx}f_{yy} - f_{xy}^2).$$

Since f_{xx} and D are continuous and positive at (x_0,y_0), we may select $\delta > 0$ such that f_{xx} and D are positive on $B = \{(x,y) : |(x - x_0, y - y_0)| < \delta\}$. This implies $D_u^2 f > 0$ on B for all u. To show (x_0,y_0) is a local minimum of f, it suffices to show that $f(x_0,y_0) \le f(x,y)$ for $(x,y) \in B$. Set $F(h) = f(x_0 + ah, y_0 + bh)$. Note that

$$F'(0) = \nabla f(x_0,y_0) \cdot u = 0, \quad F''(h) = D_u^2 f(x_0 + ah, y_0 + bh) > 0, \quad \forall |h| < \delta.$$

By the second derivative test, F is concave upward on $(-\delta,\delta)$ and 0 is a local minimum, as desired □