14.8. **Lagrange multipliers.** In this section, we consider the extreme value of a function with constraint given by another function. As a beginning, let f, g be functions of two variables. Here, the aim is to find the extremum values of $f(x,y)$ with constraint $g(x,y)=k$, where k is a constant. By regarding $g(x,y)=k$ as a geometric curve on the xy-plane, one can imagine that, if m is the extreme value of f over $g=k$, then the level curve $f(x,y)=m$ will intersect $g(x,y)=k$. More precisely, if $r(t)=\langle x(t),y(t)\rangle$ is the parametric equation for $g=k$ and $r(t_0)$ is the maximum of f over $g=k$, then

$$g(x(t),y(t))=k,\quad f(x(t),y(t))\le f(x(t_0),y(t_0)),\quad \forall t.$$

This implies

$$\nabla g\cdot r'=0,\quad \nabla f(x(t_0),y(t_0))\cdot r'(t_0)=0.$$

Hence, if $\nabla g(x(t_0),y(t_0))\neq \mathbf{0}$, then $\nabla f(x(t_0),y(t_0))=\lambda\nabla g(x(t_0),y(t_0))$ for some constant λ.

Next, let f, g be functions of three variables and k be a constraint. If $r(t)=\langle x(t),y(t),z(t)\rangle$ is a curve on the level surface $g=k$ and, along $r(t)$, f reaches its extremum at $t=t_0$, then

$$\nabla g\cdot r'=0,\quad \nabla f(x(t_0),y(t_0),z(t_0))\cdot r'(t_0)=0.$$

By choosing another curve with non-parallel tangent vector at $r(t_0)$, we may conclude that

$$\nabla f(x(t_0),y(t_0),z(t_0))=\lambda\nabla g(x(t_0),y(t_0),z(t_0)),$$

for some constant λ, provided $\nabla g(r(t_0))\neq\mathbf{0}$.

In either case of dimensions two and three, λ is called a Lagrange multiplier.

Method of Lagrange multipliers: One constraint Let f, g be functions of two or three variables. Assume that the extremum values of f under the constraint $g=k$ exist and $\nabla g\neq 0$ on $g=k$.

(1) Solve the system of equations, $\nabla f=\lambda\nabla g$ and $g=k$.
(2) Determine the values of f for all solutions in (1). The largest and smallest are the maximum and minimum values of f under the constraint $g=k$.

Example 14.35. Let $f(x,y)=x^2+2y^2$ and $g(x,y)=2x^2+y^2$. Note that $\nabla f=\langle 2x,4y\rangle$ and $\nabla g=\langle 4x,2y\rangle$. Clearly, $\nabla g\neq\mathbf{0}$ on $g=1$. By the Lagrange multiplier method, we need to solve the following system,

$$2x=4\lambda x,\quad 4y=2\lambda y,\quad 2x^2+y^2=1.$$

The first equation leads to $x=0$ or $\lambda=1/2$. When $x=0$, $y=\pm1$. When $\lambda=1/2$, $y=0$ and $x=\pm1/\sqrt{2}$. Consequently, $(0,\pm1)$ and $(\pm1/\sqrt{2},0)$ are all solutions. As $f(0,\pm1)=2$ and $f(\pm1/\sqrt{2},0)=1/2$, the maximum and minimum values of f on $g=1$ are 2 and 1/2.

Example 14.36. Consider the extremum values of $f(x,y)=x^2+2y^2$ on $D=\{g\le 1\}$, where $g(x,y)=2x^2+y^2$. First, one may solve the system $\nabla f=\mathbf{0}$ on D to conclude that $(0,0)$ is the unique critical point, which has value $f(0,0)=0$. Along with the extremum values of f on $g=1$ (see the previous example), the extremum values of f on D are respectively 2 (the maximum value) and 0 (the minimum value).

Next, let's consider the extremum value problem of $f(x,y,z)$ with constraints, $g_1(x,y,z)=k_1$ and $g_2(x,y,z)=k_2$. Geometrically, the set $\gamma=\{(x,y,z)|g_1(x,y,z)=k_1, g_2(x,y,z)=k_2\}$ is the intersection of two (level) surfaces. Assuming that ∇g_1 and ∇g_2 are not parallel on γ, one may regard γ as a curve, say $r(t)=\langle x(t),y(t),z(t)\rangle$. As before, if $r(t_0)$ is an (local or global) extremum point of f, then

$$\nabla f(r(t_0))\cdot r'(t_0)=\nabla g_1(r(t_0))\cdot r'(t_0)=\nabla g_2(r(t_0))\cdot r'(t_0)=0.$$

When $r'(t_0) \neq \mathbf{0}$, we obtain

$$\nabla f(r(t_0)) = \lambda \nabla g_1(r(t_0)) + \mu \nabla g_2(r(t_0)),$$

where λ and μ are two constants and called Lagrange multipliers. It is worthwhile to note that the assumption of $r'(t_0) \neq \mathbf{0}$ can be easily satisfied.

Method of Lagrange multipliers: Two constraints Let f, g_1, g_2 be functions of three variables. Assume that the extremum values of f under constraints $g_1 = k_1$ and $g_2 = k_2$ exist, and further that ∇g_1 and ∇g_2 are not parallel on $\{g_1 = k_1, g_2 = k_2\}$.

(1) Solve the system of equations, $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$, $g_1 = k_1$ and $g_2 = k_2$.

(2) Determine the values of f for all solutions in (1). The largest and smallest are the maximum and minimum values of f under the constraint $\{g_1 = k_1, g_2 = k_2\}$.

Example 14.37. Let $f(x, y, z) = x+2y+3z$ and γ be the intersecting curve of plane $g(x, y, z) = x - y + z = 1$ and cylinder $h(x, y, z) = x^2 + y^2 = 1$. It is easy to examine that ∇g and ∇h are not parallel on γ. First, let's solve the following system

$$\nabla f = \lambda \nabla g + \mu \nabla h, \quad g = h = 1,$$

or equivalently

$$1 = \lambda + 2\mu x, \quad 2 = -\lambda + 2\mu y, \quad 3 = \lambda, \quad x - y + z = 1, \quad x^2 + y^2 = 1.$$

The first three equations give $x = -1/\mu$, $y = 5/(2\mu)$, while the combination of the last equality implies $1/\mu^2 + 25/(4\mu^2) = 1$. Hence, we have $\mu = \pm\sqrt{29}/2$ and then

$$x = \mp 2/\sqrt{29}, \quad y = \pm 5/\sqrt{29}, \quad z = 1 \pm 7/\sqrt{29}.$$

As the corresponding values of the above solutions in f are

$$\mp 2/\sqrt{29} + 2(\pm 5/\sqrt{29}) + 3(1 \pm 7/\sqrt{29}) = 3 \pm \sqrt{29},$$

we may conclude that the maximum and minimum values of f are $3 + \sqrt{29}$ and $3 - \sqrt{29}$.

Example 14.38. Consider the extremum value of the function $f(x, y) = x$ with constraint $g(x, y) = y^2 + x^4 - x^3 = 0$. Note that $g = 0$ implies $x \in [0, 1]$ and $y = \pm\sqrt{x^3(1 - x)}$. Let r be the parametric curve of the equation $g = 0$. Then,

$$r(t) = \begin{cases} \langle -t, t\sqrt{-t(1+t)} \rangle & \text{for } t \in [-1, 0] \\ \langle t, t\sqrt{t(1-t)} \rangle & \text{for } t \in [0, 1] \end{cases}$$

It is clear that $f(r(t))$ attains its minimum at $t = 0$ with $f(r(0)) = f(0, 0) = 0$. Back to the method of Lagrange multiplier, one has to solve the system of $\nabla f = \lambda \nabla g$ and $g = 0$, i.e.

$$1 = \lambda(4x^3 - 3x^2), \quad 0 = 2\lambda y, \quad y^2 = x^3(x - 1).$$

As it is easy to check that this system has only one solution $(1, 0)$, the method of Lagrange multiplier fails in this example. It is remarkable that, at the minimum point $(0, 0)$, $\nabla f(0, 0) = \langle 1, 0 \rangle$ and $\nabla g(0, 0) = \langle 0, 0 \rangle$.