15.1. **Double integrals over rectangles.** Let $z = f(x, y) \geq 0$ for $(x, y) \in R = [a, b] \times [c, d]$ and consider the solid $S = \{(x, y, z)|0 \leq z \leq f(x, y), (x, y) \in R\}$. To estimate the volume of $S$, we partition $S$ into sub-rectangles of form

$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] = \{(x, y)|x_{i-1} \leq x \leq x_i, y_{j-1} \leq y \leq y_j\}$$

where $x_i = a + i\Delta x$, $y_j = c + j\Delta y$, $\Delta x = (b - a)/m$ and $\Delta y = (d - c)/n$. Then, the area of $R_{ij}$ equals $\Delta A = \Delta x \Delta y$. By choosing a sample point $(x_{ij}^*, y_{ij}^*)$ in $R_{i,j}$, the volume of the solid with basement $R_{ij}$ is approximately $f(x_{ij}^*, y_{ij}^*)\Delta A$ and the volume of $S$ is roughly $\sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*)\Delta A$. When $f$ is "smooth" enough, we may expect the volume equals the following limit

$$\lim_{m,n\to\infty} \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*)\Delta A.$$

**Definition 15.1.** The double integral of $f$ over the rectangle $R = [a, b] \times [c, d]$ is defined by

$$\iint_R f(x, y)dA = \lim_{m,n\to\infty} \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*)\Delta A, \tag{15.1}$$

provided the limit exists and is independent of the choice of $(x_{ij}^*, y_{ij}^*)$. The summation at the right-hand side is called a double Riemann sum.

*Remark* 15.1. Precisely, the limit in (15.1) is $L$ if, for any $\epsilon > 0$, there is $N = N(\epsilon)$ such that

$$n, m \geq N, \quad \Rightarrow \quad \left|\sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*)\Delta A - L\right| < \epsilon.$$

*Remark* 15.2. If $f \geq 0$ on $R$, then the volume of the solid under $z = f(x, y)$ and above $R$ is defined to be $\iint_R f(x, y)dA$.

**Properties of double integrals** Let $f, g$ be integrable functions on the region $R$. Then, for any constants $\alpha, \beta$, $\alpha f + \beta g$ is integrable on $R$ and

$$\iint_R [\alpha f(x, y) + \beta g(x, y)]dA = \alpha \iint_R f(x, y)dA + \beta \iint_R g(x, y)dA.$$

If $f \geq g$ on $R$, then

$$\iint_R f(x, y)dA \geq \iint_R g(x, y)dA.$$

*Example* 15.1. Consider function $f(x, y) = 16 - x^2 - y^2$ with domain $R = [0, 2] \times [0, 2]$. Set $\Delta x = 2/m$ and $\Delta y = 2/n$. Note that, for $(x, y) \in R_{ij} = [(i - 1)\Delta x, i\Delta x] \times [(j - 1)\Delta y, j\Delta y]$,

$$\begin{aligned}|f(x, y) - f(i\Delta x, j\Delta y)| &= |x^2 - (i\Delta x)^2 + y^2 - (j\Delta y)^2| \\ &= (x + i\Delta x)|x - i\Delta x| + (y + j\Delta y)|y - j\Delta y| \leq \frac{8}{m} + \frac{8}{n}\end{aligned}$$

This implies, for any $(x_{ij}^*, y_{ij}^*) \in R_{ij}$,

$$\left|\sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*)\Delta A - \sum_{i=1}^m \sum_{j=1}^n f(i\Delta x, j\Delta y)\Delta A\right| \leq \frac{32}{m} + \frac{32}{n}.$$

Note that

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(i\Delta x, j\Delta y)\Delta A = 64 - \sum_{i=1}^{m}\sum_{j=1}^{n}\left(\frac{4i^2}{m^2}+\frac{4j^2}{n^2}\right)\frac{4}{mn}$$
$$= 64 - 16\sum_{i=1}^{m}\left(\frac{i}{m}\right)^2\frac{1}{m} - 16\sum_{j=1}^{n}\left(\frac{j}{n}\right)^2\frac{1}{n}.$$

Consequently, we obtain

$$\iint_R f(x,y)dA = 64 - 16\int_0^1 x^2dx - 16\int_0^1 y^2dy = \frac{160}{3}.$$

*Remark* 15.3. The midpoint rule for double integrals uses the midpoints as the sample points, i.e. $x_{ij}^* = (x_i + x_{i-1})/2$ and $y_{ij}^* = (y_i + y_{j-1})/2$.

**Theorem 15.1** (Fubini's theorem)**.** *Let $f(x,y)$ be a function defined on $R = [a,b]\times[c,d]$. If $f$ is continuous on $R$, then $\iint_R f(x,y)dA$ exists and*

$$\iint_R f(x,y)dA = \int_c^d\int_a^b f(x,y)dxdy = \int_a^b\int_c^d f(x,y)dydx.$$

*The integrals on the right-hand side are called iterated integrals. In particular, if $f(x,y) = g(x)h(y)$ and $g,h$ are continuous, then*

$$\iint_R f(x,y)dA = \int_a^b g(x)dx \times \int_c^d h(y)dy.$$

*Remark* 15.4. Fubini's theorem in fact holds for bounded functions of which discontinuity appears in a finite union of smooth curves.

*Example* 15.2. Let $f(x,y) = y\cos xy$ and $R = [1,2]\times[0,\pi]$. Then,

$$\iint_R f(x,y)dA = \int_0^\pi\int_1^2 f(x,y)dxdy = \int_0^\pi\left(\sin xy\Big|_{x=1}^{x=2}\right)dy = \int_0^\pi[\sin 2y - \sin y]dy$$
$$= \left(\cos y - \frac{\cos 2y}{2}\right)\Bigg|_{y=0}^{y=\pi} = -2.$$

For $R\subset\mathbb{R}^2$, the average value of $f(x,y)$ over $R$ is defined to be

$$f_{\text{ave}} = \frac{1}{A(R)}\iint_R f(x,y)dA,$$

where $A(R)$ is the area of $R$.

*Example* 15.3. The average value of $f(x,y) = 16 - x^2 - y^2$ over $R = [0,2]\times[0,2]$ is

$$\frac{1}{A(R)}\iint_R f(x,y)dA = \frac{40}{3}.$$