15.2. **Double integrals over general regions.**

Definition 15.2. Let $D \subset \mathbb{R}^2$ be a bounded region (not necessarily a rectangle) and R be a rectangle containing D. The integral of a function f over D is defined to be

$$\iint_D f(x,y)dA = \iint_R F(x,y)dA, \text{ where } F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D \\ 0 & \text{if } (x,y) \in R \setminus D \end{cases}, \tag{15.2}$$

provided the integral $\iint_R F(x,y)dA$ is independent of R that contains D.

In the following, we consider two specific types of region D.

Type I region $D = \{(x,y) | a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$, where g_1, g_2 are continuous on $[a,b]$. Let $R = [a,b] \times [c,d]$ be a rectangle containing D and F be the function in (15.2). By Fubini's theorem, if F is continuous on D, then F is integrable and

$$\iint_D f(x,y)dA = \iint_R F(x,y)dA = \int_a^b \int_c^d F(x,y)dydx \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx.$$

Type II region $D = \{(x,y) | c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$, where h_1, h_2 are continuous functions. Then,

$$\iint_D f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy.$$

Example 15.4. Let $f(x,y) = xy$ and D be the region enclosed by $y = x$ and $y = x^2$. Note that $D = \{(x,y) | 0 \le x \le 1,\ x^2 \le y \le x\}$. Then,

$$\iint_D f(x,y)dA = \int_0^1 \int_{x^2}^x xydydx = \frac{1}{2}\int_0^1 x(x^2 - x^4)dx = \frac{1}{24}.$$

Example 15.5. Let S be the solid enclosed by the parabolic cylinders $y = x^2 - 1$, $y = 1 - x^2$ and the planes $x + y + z = 2$ and $2x + 2y - z + 10 = 0$. Let $D = \{(x,y) | -1 \le x \le 1,\ x^2 - 1 \le y \le 1 - x^2\}$. Note that the height of S at $(x,y) \in D$ is given by $h(x,y) = |(2x + 2y + 10) - (2 - x - y)| = |3x + 3y + 8| = 3x + 3y + 8$. Then, the volume of S is

$$V = \int_{-1}^1 \int_{x^2-1}^{1-x^2} (3x + 3y + 8)dydx = \int_{-1}^1 2(3x + 8)(1 - x^2)dx = 16\int_{-1}^1 (1 - x^2)dx = \frac{64}{3}.$$

Example 15.6. Consider the iterated integral $\int_0^1 \int_x^1 \sin y^2 dydx$. Set

$$D = \{(x,y) | 0 \le x \le 1,\ x \le y \le 1\} = \{(x,y) | 0 \le y \le 1,\ 0 \le x \le y\}.$$

By Fubini's theorem, one may rewrite the above integral as

$$\int_0^1 \int_x^1 \sin y^2 dydx = \int_0^1 \int_0^y \sin y^2 dxdy = \int_0^1 y \sin y^2 dy = -\frac{\cos y^2}{2}\bigg|_0^1 = \frac{1 - \cos 1}{2}.$$

Properties of double integrals Let f, g be integrable functions over $D \subset \mathbb{R}^2$.

(1) For $\alpha, \beta \in \mathbb{R}$, $\iint_D [\alpha f(x,y) + \beta g(x,y)]dA = \alpha \iint_D f(x,y)dA + \beta \iint_D g(x,y)dA$.
(2) If $A(D)$ is the area of D, then $\iint_D 1dA = A(D)$.
(3) If $f \ge g$ on D, then $\iint_D f(x,y)dA \ge \iint_D g(x,y)dA$.
(4) If $m \le f(x,y) \le M$ on D, then $mA(D) \le \iint_D f(x,y)dA \le MA(D)$.
(5) Assume that f is integrable on D_1 and D_2, where $D = D_1 \cup D_2$ and D_1, D_2 do not overlap except on their boundaries. Then, $\iint_D f(x,y)dA = \iint_{D_1} f(x,y)dA + \iint_{D_2} f(x,y)dA$.