15.3. **Double integrals in polar coordinates.** Consider the following region

$$D = \{(x, y)|y \geq 0, 1 \leq x^2 + y^2 \leq 4\}.$$

Through the polar coordinate $x = r\cos\theta$ and $y = r\sin\theta$, the region can be expressed as

$$R = \{(r, \theta)|0 \leq \theta \leq \pi,\ 1 \leq r \leq 2\}.$$

Here, we call the set D a polar rectangle. In the following, let's treat a more general case. Let $D \subset \mathbb{R}^2$ be the polar rectangle $R = \{(r, \theta)|a \leq r \leq b,\ \alpha \leq \theta \leq \beta\}$ with $\beta - \alpha \leq 2\pi$. Set

$$\Delta r = \frac{b-a}{m}, \quad \Delta\theta = \frac{\beta - \alpha}{n}, \quad r_i = a + i\Delta r, \quad \theta_j = \alpha + j\Delta\theta, \quad S_{ij} = [r_{i-1}, r_i] \times [\theta_{j-1}, \theta_j].$$

Use a similar reasoning for the double integral, if f is continuous on D, then

$$\iint_D f(x, y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}^* \cos\theta_{ij}^*, r_{ij}^* \sin\theta_{ij}^*)\Delta A_{ij} \tag{15.3}$$

where $(r_{ij}^*, \theta_{ij}^*) \in S_{ij}$ and ΔA_{ij} is the area of S_{ij}. By the midpoint rule, (15.3) also holds for the specific case of $r_{ij}^* = r_i^* = \frac{1}{2}(r_{i-1} + r_i)$ and $\theta_{ij}^* = \theta_j^* = \frac{1}{2}(\theta_{j-1} + \theta_j)$. In addition with these notations, one may derive $\Delta A_{ij} = \frac{1}{2}(r_i^2 - r_{i-1}^2)\Delta\theta = r_i^*\Delta r\Delta\theta$ and then obtain

$$\iint_D f(x, y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)r_i^*\Delta r\Delta\theta = \iint_R g(r, \theta)dA,$$

where $g(r, \theta) = rf(r\cos\theta, r\sin\theta)$. We summarize the above discussion in the following theorem.

Theorem 15.2. *Let $f(x, y)$ be a continuous function defined on a polar rectangle D given by $R = \{(r, \theta)|a \leq r \leq b, \alpha \leq \theta \leq \beta\}$ with $\beta - \alpha \leq 2\pi$. Then,*

$$\iint_D f(x, y)dA = \iint_R rf(r\cos\theta, r\sin\theta)dA = \int_\alpha^\beta \int_a^b rf(r\cos\theta, r\sin\theta)drd\theta.$$

Remark 15.5. Theorem 15.2 also holds for types I and II polar regions.

Example 15.7. Consider the integral $I = \int_0^a \int_{-\sqrt{a^2-y^2}}^0 x^2ydxdy$. Set $f(x, y) = x^2y$ and D be the polar rectangle with $0 \leq r \leq a$ and $\pi/2 \leq \theta \leq \pi$. Then,

$$I = \iint_D f(x, y)dA = \iint_R rf(r\cos\theta, r\sin\theta)dA = \int_{\pi/2}^{\pi} \int_0^a r^4 \sin\theta\cos^2\theta drd\theta = \frac{a^5}{15}.$$

Example 15.8. Let S be the solid below the cone $z = \sqrt{x^2 + y^2}$ and above the disk D : $x^2 + y^2 \leq 4$ on the xy-plane. Set $f(x, y) = \sqrt{x^2 + y^2}$ and $R = \{(r, \theta)|0 \leq r \leq 2,\ 0 \leq \theta \leq 2\pi\}$. Then, the volume of S is given by

$$V = \iint_D f(x, y)dA = \iint_R rf(r\cos\theta, r\sin\theta)dA = \int_0^{2\pi} \int_0^2 r^2drd\theta = \frac{16\pi}{3}.$$

Example 15.9. Consider the solid below the paraboloid $z = x^2 + y^2$, above the xy-plane and inside the cylinder $x^2 + y^2 = 2x$. Let $R = \{(r, \theta)|0 \leq r \leq 2\cos\theta,\ -\pi/2 \leq \theta \leq \pi/2\}$. Then,

$$\begin{aligned} V &= \int_R r^3drd\theta = \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^3drd\theta = \int_{-\pi/2}^{\pi/2} 4\cos^4\theta d\theta \\ &= 2\int_0^{\pi/2}(1 + \cos 2\theta)^2 d\theta = \int_0^{\pi/2} [2 + 4\cos 2\theta + 1 + \cos 4\theta]d\theta = \frac{3\pi}{2} \end{aligned}$$