15.4. **Application of double integrals.** A random variable X is said to have a normal distribution with mean 0 and variance 1 if the probability density function of X is given by

$$f(x) = ce^{-x^2/2}, \quad \forall x \in \mathbb{R},$$

where c is a normalizing constant, which must be positive. In this setting, the probability of the event $\{X \le x\}$ equals the improper integral $\int_{-\infty}^{x} f(t)dt$ for any $x \in \mathbb{R}$. To determine the precise value of the integral, one needs the value of c. Note that the probability of $\{X \in (-\infty, \infty)\}$ is 1 and this leads to $c\int_{-\infty}^{\infty} e^{-x^2/2}dx = 1$. By Fubini's theorem, we have

$$c^{-2} = \int_{-\infty}^{\infty} e^{-x^2/2}dx \cdot \int_{-\infty}^{\infty} e^{-y^2/2}dy = \iint_{\mathbb{R}^2} e^{-(x^2+y^2)/2}dA.$$

Through the polar coordinate, the region $\mathbb{R}^2$ is the polar rectangle $R = \{(r,\theta)|0 \le r < \infty,\ 0 \le \theta \le 2\pi\}$. As a result, this implies

$$c^{-2} = \iint_R re^{-r^2/2}dr = 2\pi \int_0^{\infty} re^{-r^2/2}dr = -2\pi e^{-r^2/2}\Big|_0^{\infty} = 2\pi.$$

15.5. **Surface area.** Let $z = f(x,y)$ be a function defined on $D = [a,b] \times [c,d]$ and S be the surface $\{(x,y,f(x,y))|(x,y) \in D\}$. To compute the area of S, we partition D into sub-rectangles $[x_{i-1},x_i] \times [y_{j-1},y_j]$ as before, i.e. $\Delta x = \frac{b-a}{m}$, $\Delta y = \frac{d-c}{n}$, $x_i = a + i\Delta x$ and $y_j = c + j\Delta y$. Note that S is a union of $S_{ij} = \{(x,y,f(x,y))|x_{i-1} \le x \le x_i, y_{j-1} \le y \le y_j\}$. Let $A(S)$ and $A(S_{ij})$ be the surface areas of S and S_{ij}. Clearly, $A(S) = \sum_{i=1}^{m}\sum_{j=1}^{n} A(S_{ij})$. To compute $A(S_{ij})$, observe that the tangent plane to S at (x_i,y_j) is given by

$$L(x,y) = f(x_i,y_j) + f_x(x_i,y_j)(x - x_i) + f_y(x_i,y_j)(y - y_j).$$

Let $P = (x_i,y_j,f(x_i,y_j))$, $Q = (x_{i-1},y_j,L(x_{i-1},y_j))$ and $R = (x_i,y_{j-1},L(x_i,y_{j-1}))$. If f is differentiable on D, then S_{ij} is approximately the parallelogram spanned by vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ of which area is given by

$$\|\overrightarrow{PQ} \times \overrightarrow{PR}\| = \|\langle -f_x(x_i,y_j), -f_y(x_i,y_j), 1\rangle\|\Delta x\Delta y = \sqrt{f_x^2(x_i,y_j) + f_y^2(x_i,y_j) + 1}\Delta x\Delta y.$$

Letting n,m tend to infinity leads to the following theorem.

Theorem 15.3. *Let $z = f(x,y)$ be a function defined on a region D and $A(S)$ be the area of the surface $\{(x,y,f(x,y))|(x,y) \in D\}$. Suppose f_x, f_y are continuous on D. Then,*

$$A(S) = \iint_D \sqrt{f_x^2 + f_y^2 + 1}dA.$$

Example 15.10. Consider the paraboloid $z = x^2 + y^2$ with domain $D = \{(x,y)|x^2 + y^2 \le 1\}$. Let $R = \{(r,\theta)|0 \le r \le 1, 0 \le \theta \le 2\pi\}$. By Theorem 15.3, the surface area equals

$$\iint_D \sqrt{4x^2 + 4y^2 + 1}dA = \iint_R \sqrt{4r^2 + 1}rdA = 2\pi \times \frac{(4r^2+1)^{3/2}}{12}\Big|_0^1 = \frac{(5\sqrt{5} - 1)\pi}{6}.$$