15.6. **Triple integrals.**

Definition 15.3. Let $f(x, y, z)$ be a function defined on $B = [a, b] \times [c, d] \times [r, s]$. For $m, n, l > 0$, set $\Delta x = \frac{b-a}{l}$, $\Delta y = \frac{d-c}{m}$ and $\Delta z = \frac{s-r}{n}$. The integral of f over B is defined by

$$\iiint_B f(x, y, z)dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\Delta V,$$

provided the limit exists and is independent of the selection of sample points $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \in B_{ijk}$, where $\Delta V = \Delta x \Delta y \Delta z$, $B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$ and $x_i = a + i\Delta x$, $y_j = c + j\Delta y$, $z_k = r + k\Delta z$.

Theorem 15.4 (Fubini's theorem)**.** *Let f be a continuous function on $B = [a, b] \times [c, d] \times [r, s]$. Then, f is integrable over B and*

$$\iiint_B f(x, y, z)dV = \int_r^s \int_c^d \int_a^b f(x, y, z)dxdydz.$$

In particular, if F, G, H are continuous and $f(x, y, z) = F(x)G(y)H(z)$, then

$$\iiint_B f(x, y, z)dV = \int_a^b F(x)dx \times \int_c^d G(y)dy \times \int_r^s H(z)dz.$$

Remark 15.6. For general bounded regions, E, let B be a box containing E and F is an extension of f obtained by defined $F = 0$ on $B \setminus E$. Then, the integrability of f over E is defined to be the integrability of F over B. Further, Fubini's theorem applies in this case when the discontinuity of F is contained in a finite union of smmoth surfaces.

In the following, we consider three specific types of E.

Type I $E = \{(x, y, z)|u_1(x, y) \le z \le u_2(x, y),\ (x, y) \in D\}$, where D is the projection of E onto the xy-plane. In this case, we have

$$\iiint_E f(x, y, z)dV = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)dz \right) dA.$$

In particular, if $D = \{(x, y)|g_1(x) \le y \le g_2(x),\ a \le x \le b\}$, then

$$\iiint_E f(x, y, z)dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)dzdydx.$$

Similarly, if $D = \{(x, y)|h_1(y) \le x \le h_2(y),\ c \le y \le d\}$, then

$$\iiint_E f(x, y, z)dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)dzdxdy.$$

Example 15.11. Let E be a tetrahedron bounded by $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$. Find $\iiint_E zdV$. Note that

$$E = \{(x, y, z)|0 \le z \le 1 - x - y,\ 0 \le y \le 1 - x,\ 0 \le x \le 1\}.$$

Then,

$$\iiint_E zdV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} zdzdydx = \frac{1}{24}.$$

Types II and III A region E is of type II if it is of the form $\{(x,y,z)|u_1(y,z) \le x \le u_2(y,z),\ (y,z) \in D\}$ and of type III if it is of the form $\{(x,y,z)|u_1(x,z) \le y \le u_2(x,z),\ (x,z) \in D\}$. Respectively, their integrals are given by

$$\iiint_E f(x,y,z)dV = \iint_D \left(\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)dx \right) dA, \quad \text{(Type II)},$$

and

$$\iiint_E f(x,y,z)dV = \iint_D \left(\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)dy \right) dA, \quad \text{(Type III)}.$$

Example 15.12. Let E be the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$. It is clear that E is of Types I, II and III. Then, for any continuous function f,

$$\begin{aligned}
&\iiint_E f(x,y,z)dV \\
&= \int_{-2}^{2}\int_{x^2}^{4}\int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} f(x,y,z)dzdydx = \int_0^4\int_{-\sqrt{y}}^{\sqrt{y}}\int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} f(x,y,z)dzdxdy \\
&= \int_{-2}^{2}\int_{z^2}^{4}\int_{-\sqrt{y-z^2}}^{\sqrt{y-z^2}} f(x,y,z)dxdydz = \int_0^4\int_{-\sqrt{y}}^{\sqrt{y}}\int_{-\sqrt{y-z^2}}^{\sqrt{y-z^2}} f(x,y,z)dxdzdy \\
&= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{x^2+z^2}^{4} f(x,y,z)dydzdx = \int_{-2}^{2}\int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}}\int_{x^2+z^2}^{4} f(x,y,z)dydxdz
\end{aligned}$$

In particular, if $f(x,y,z) = x^2 + z^2$, then

$$\iiint_E f(x,y,z)dV = \iint_D (4 - x^2 - z^2)(x^2 + z^2)dA,$$

where $D = \{(x,z)|x^2 + z^2 \le 4\}$. By using the polar coordinate, the above integral becomes

$$\iiint_E f(x,y,z)dV = \int_0^{2\pi}\int_0^2 (4 - r^2)r^3 drd\theta = \frac{32\pi}{3}.$$

Remark 15.7. For any bounded region E, the volume is given by $V(E) = \iiint_E 1dV$.

Example 15.13. Let E be the pyramid of height h of which edge of the base is d. Let T be the tetrahedron bounded by $x = 0$, $y = 0$, $z = 0$ and $\sqrt{2}(x + y)/d + z/h = 1$. Note that $V(E) = 4V(T)$ and

$$\begin{aligned}
V(T) = \iiint_T 1dV &= \int_0^{d/\sqrt{2}}\int_0^{d/\sqrt{2}-x}\int_0^{h(1-\sqrt{2}(x+y)/d)} 1dzdydx \\
&= \frac{h}{\sqrt{2}d}\int_0^{d/\sqrt{2}} \left(\frac{d}{\sqrt{2}} - x\right)^2 dx = \frac{d^2h}{12}
\end{aligned}$$

This implies $V(E) = d^2h/3$.