15.7. **Triple integrals in cylindrical coordinates.** The cylindrical coordinate system uses the polar coordinate to present one plane, while its normal line (through the pole) is one of the axis. For instance, the point (x, y, z) in Cartesian coordinate has cylindrical coordinate (r, θ, z), where $x = r\cos\theta$ and $y = r\sin\theta$.

Example 15.14. The solid $S = \{(x, y, z)|x^2 + y^2 \le 2, x \ge 0, 0 \le z \le 1\}$ can be expressed as $\{(r, \theta, z)|0 \le r \le 2,\ -\pi/2 \le \theta \le \pi/2,\ 0 \le z \le 1\}$ in the cylindrical coordinate.

Let $E = \{(x, y, z)|(x, y) \in D,\ u_1(x, y) \le z \le u_2(x, y)\}$. If $D = \{(r, \theta)|\alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$ with $\beta - \alpha < 2\pi$, then

$$\begin{aligned}\iiint_E f(x, y, z)dV &= \iint_D \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)dzdA \\ &= \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} rf(r\cos\theta, r\sin\theta, z)dzdrd\theta.\end{aligned}$$

Example 15.15. Consider the integral $\iiint_E \sqrt{x^2 + y^2}dV$, where E is the region lying inside the cylinder $x^2 + y^2 = 16$ and between the planes $z = 4$ and $z = -5$. In the cylindrical coordinate, $E = \{(r, \theta, z)| -5 \le z \le 4,\ 0 \le \theta \le 2\pi,\ 0 \le r \le 4\}$. This implies

$$\iiint_E \sqrt{x^2 + y^2}dV = \int_0^{2\pi} \int_0^4 \int_{-5}^4 r^2 dzdrd\theta = 6\pi r^3\Big|_0^4 = 384\pi.$$

Example 15.16. Consider the integral $\iiint_E x^2 dV$, where E is the solid above $z = 0$, below $z = \sqrt{4x^2 + 4y^2}$ and inside $x^2 + y^2 = 1$. In the cylindrical coordinate, $E = \{(r, \theta, z)|0 \le z \le 2r,\ 0 \le r \le 2,\ 0 \le \theta \le 2\pi\}$. As a result, we have

$$\iiint_E x^2 dV = \int_0^{2\pi} \int_0^2 \int_0^{2r} r^3 \cos^2\theta dzdrd\theta = \int_0^2 2r^4 dr \int_0^{2\pi} \cos^2\theta = \frac{64\pi}{5}.$$

Example 15.17. If E is a solid of which volume equals $\int_0^4 \int_0^{2\pi} \int_r^4 rdzd\theta dr$, then E can be $\{(x, y, z)|\sqrt{x^2 + y^2} \le z \le 4,\ x^2 + y^2 \le 4\}$.

Example 15.18. To compute the following integral using the cylindrical coordinate,

$$I = \int_{-3}^3 \int_0^{\sqrt{9-x^2}} \int_0^{9-x^2-y^2} \sqrt{x^2 + y^2}dzdydx,$$

we set $f(x, y, z) = \sqrt{x^2 + y^2}$ and let E be the solid enclosed by $z = 9 - x^2 - y^2$, $z = 0$, $y = 0$ and $y = \sqrt{9 - x^2}$. Then, $I = \iiint_E f(x, y, z)dV$. In the cylindrical coordinate, $E = \{(r, \theta, z)|0 \le z \le 9 - r^2,\ 0 \le r \le 3,\ 0 \le \theta \le \pi\}$ and, hence,

$$I = \int_0^\pi \int_0^3 \int_0^{9-r^2} r^2 dzdrd\theta = \frac{162\pi}{5}.$$