15.8. **Triple integrals in spherical coordinates.** The spherical coordinate is given as follows. For $(x, y, z) \in \mathbb{R}^3$, set $\rho = \sqrt{x^2 + y^2 + z^2}$ and let ϕ be the angle between $\langle x, y, z\rangle$ and the $\langle 0, 0, 1\rangle$ and let (r, θ) be the polar coordinate of (x, y) on the xy-plane. As (x, y, z) is the projection of (x, y, z) onto the xy-plane, one has $r = \rho \sin\phi$ and this implies

$$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi \tag{15.4}$$

with $\rho \geq 0$, $0 \leq \theta < 2\pi$ and $0 \leq \phi \leq \pi$. The triple (ρ, θ, ϕ) is called the spherical coordinate of (x, y, z).

Example 15.19. The spherical coordinates of points $(0, 0, 1)$, $(\sqrt{2}/2, \sqrt{6}/2, \sqrt{2})$, $(1, 1, \sqrt{2})$ and $(-\sqrt{3}, -3, -2)$ are respectively $(1, 0, 0)$, $(2, \pi/3, \pi/4)$, $(2, \pi/4, \pi/4)$ and $(4, 2\pi/3, 2\pi/3)$.

Example 15.20. Consider the surfaces, $\rho = \sin\theta\sin\phi$ and $\rho^2(\sin^2\phi\sin^2\theta + \cos^2\phi) = 9$. In the former equation, one has $x^2+y^2+z^2 = \rho^2 = \rho\sin\theta\sin\rho = y$, which is the sphere of radius $1/2$ centered at $(0, 1/2, 0)$. In the latter equation, we have $9 = (\rho\sin\theta\sin\phi)^2 + (\rho\cos\phi)^2 = y^2 + z^2$, which is a cylinder.

Example 15.21. The solid of which spherical coordinate (ρ, θ, ϕ) satisfies $2 \leq \rho \leq 3$ and $\pi/2 \leq \phi \leq \pi$ is enclosed by spheres $x^2 + y^2 + z^2 = 4$, $x^2 + y^2 + z^2 = 9$ and below the plane $z = 0$.

Example 15.22. Let S be the solid lying above the cone $z = \sqrt{x^2 + y^2}$ and inside the sphere $x^2 + y^2 + z^2 = z$. Clearly, $S = \{(x, y, z) | z \geq \sqrt{x^2 + y^2}, z \geq x^2 + y^2 + z^2\}$. By (15.4), the inequalities describing S become

$$\rho\cos\phi \geq \rho\sin\phi, \quad \rho\cos\phi \geq \rho^2, \quad 0 \leq \theta \leq 2\pi,$$

or equivalently

$$0 \leq \phi \leq \pi/4, \quad \rho \leq \cos\phi, \quad 0 \leq \theta \leq 2\pi.$$

Let E be the spherical box $\{(\rho, \theta, \phi) | a \leq \rho \leq b, \alpha \leq \theta \leq \beta, c \leq \phi \leq d\}$. Set $\Delta\rho = (b - a)/l$, $\Delta\phi = (\beta - \alpha)/m$, $\Delta\theta = (d - c)/n$ and $E_{ijk} = \{(\rho, \theta, \phi) | \rho \in [\rho_{i-1}, \rho_i], \theta \in [\theta_{j-1}, \theta_j], \phi \in [\phi_{k-1}, \phi_k]\}$, where $\rho_i = a + i\Delta\rho$, $\theta_j = \alpha + j\Delta\theta$ and $\phi_k = c + k\Delta\phi$. Then, the the volume ΔV_{ijk} of E_{ijk} is approximately

$$\Delta V_{ijk} \approx (\Delta\rho)(\rho_i\Delta\phi)(\rho_i\sin\phi_k\Delta\theta) = \rho_i^2\sin\phi_k\Delta\rho\Delta\theta\Delta\phi.$$

As a result, for any continuous function f on E,

$$\iiint_E f(x, y, z)dV = \lim_{l,m,n\to\infty}\sum_{i,j,k} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\Delta V_{ijk}$$

$$= \lim_{l,m,n\to\infty}\sum_{i,j,k} F(\rho_{ijk}^*, \theta_{ijk}^*, \phi_{ijk}^*)\Delta\rho\Delta\theta\Delta\phi = \int_c^d\int_\alpha^\beta\int_a^b F(\rho, \theta, \phi)d\rho d\theta d\phi,$$

where $F(\rho, \theta, \phi) = \rho^2\sin\phi f(\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)$.

Example 15.23. Consider the following integrals

$$\text{(a)} \iiint_B (x^2 + y^2 + z^2)^2 dV, \quad \text{(b)} \iiint_E x^2 dV,$$

where B is the ball centered at the origin with radius 5 and E is the region enclosed by the xz-plane and hemispheres $y = \sqrt{9 - x^2 - z^2}$ and $y = \sqrt{16 - x^2 - z^2}$. In the spherical coordinates, B and E can be expressed as

$$\{(\rho, \theta, \phi) | \rho \in [0, 5], \theta \in [0, 2\pi], \phi \in [0, \pi]\}, \quad \{(\rho, \theta, \phi) | \rho \in [3, 4], \theta \in [0, \pi], \phi \in [0, \pi]\}.$$

This implies

$$\iiint_B (x^2+y^2+z^2)^2 dV = \int_0^{\pi}\int_0^{2\pi}\int_0^5 \rho^6 \sin\phi d\rho d\theta d\phi = \frac{312500\pi}{7}$$

and

$$\iiint_E x^2 dV = \int_0^{\pi}\int_0^{\pi}\int_3^4 \rho^4 \cos^2\theta \sin^3\phi d\rho d\theta d\phi = \frac{1562\pi}{15}.$$

Example 15.24. Let E be the solid lying above the cone $z=\sqrt{x^2+y^2}$ and inside the sphere $x^2+y^2+z^2-2z=0$ and V be the volume of E. Note that, in the spherical coordinate, E can be expressed as

$$\{(\rho,\theta,\phi)|0\le\rho\le 2\cos\phi,\ 0\le\phi\le\pi/4,\ 0\le\theta\le 2\pi\}$$

and then

$$\begin{aligned} V &= \iiint_E 1dV = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{2\cos\phi} \rho^2 \sin\phi d\rho d\phi d\theta \\ &= \frac{8}{3}\int_0^{2\pi}\int_0^{\pi/4} \sin\phi\cos^3\phi d\phi d\theta = \pi \end{aligned}$$

Example 15.25. Consider the following integral

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} xydzdydx.$$

By writing $I=\iiint_D xydV$, one has $D=\{(x,y,z)|x^2+y^2\le 1, x^2+y^2\le z^2\le 2-x^2-y^2, x\ge 0, y\ge 0, z\ge 0\}$. In the spherical coordinate, D can be expressed as

$$\{(\rho,\theta,\phi)|0\le\theta\le\pi/2,\ 0\le\phi\le\pi/4,\ 0\le\rho\le\sqrt{2}\},$$

which implies

$$I = \int_0^{\pi/2}\int_0^{\pi/4}\int_0^{\sqrt{2}} \rho^4\sin^3\phi\sin\theta\cos\theta d\rho d\phi d\theta = \frac{4\sqrt{2}-5}{15}.$$